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EXT AND VON NEUMANN REGULAR RINGS

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INTRODUCTION

Torsion theories, introduced in [4], have become a widespread and effective tool of contemporary theory of rings and modules. In [8], a generalization of the notion of a torsion theory to an orthogonal theory of an arbitrary set-valued bifunctor was carried out. Moreover, characterizations of orthogonal theories of basic homological bifunctors over abelian groups were given there.

Now, a natural question arises, namely to classify rings via the number of orthogonal theories of the basic homological bifunctors over the corresponding module categories. For example, in [6], a full description of rings with only trivial torsion theories was given. Similarly, commutative rings with only trivial orthogonal theories of the tensor product bifunctor were characterized in [9]. In [2, Appendix A], rings with only trivial orthogonal theories of the bifunctor Ext were studied and divided into five types. In [11], the rings of types 3, 4 and 5 were fully described.

The main purpose of the present paper is to continue the work of [11]. Our paper is divided into three sections. In Section I, we study relations between orthogonal theories of the bifunctors Ext and Tor and apply them to the rings of type 1. Section II is devoted to the investigation of the rings of type 2, the main result being Theorem II.4. In Section III, we deal with its converse and characterize orthogonal theories of the bifunctor Ext over modules over simple countable von Neumann regular rings.

PRELIMINARIES

In this paper, except for III.5 and III.6, we work in the Zermelo-Frankel set theory plus the Axiom of Choice (ZFC). In III.5 and III.6 we work in ZFC plus the Axiom of Constructibility ($V = L$). We shall always identify an ordinal α with the set of ordinals less than α, i.e. $\alpha = \{\beta \mid \beta < \alpha\}$. A cardinal is identified with an initial ordinal. If κ is a cardinal, $\mathrm{cf}(\kappa)$ denotes its cofinality. A cardinal κ is regular or singular if $\mathrm{cf}(\kappa) = \kappa$ or $\mathrm{cf}(\kappa) < \kappa$, respectively. For a set A, the cardinality of A is denoted by card (A).

In the whole paper, all rings are associative with unit. If S and T are rings, then $S \boxplus T$ is the ring direct sum of S and T. Let R be a ring. Then R-mod and mod-R denotes the category of unitary left and right R-modules, respectively. Unitary left R-modules are simply called *modules*.

Let $\alpha \geqq 1$ be an ordinal and for each $\beta < \alpha$ let M_β be a module. Then $\dot{\Sigma} M_\beta$, $\beta < \alpha$ and $\dot{\Pi} M_\beta$, $\beta < \alpha$ denotes the direct sum and the direct product, respectively, of the modules M_β. If $M_\beta = M$ for all $\beta < \alpha$, we write $M^{(\alpha)}$ and M^α instead of $\dot{\Sigma} M_\beta$, $\beta < \alpha$ and $\dot{\Pi} M_\beta$, $\beta < \alpha$, respectively. For $\alpha = 2$, we write $M_0 \dotplus M_1$ instead of $\dot{\Sigma} M_\beta$, $\beta < 2$. If M is a module and N and P are submodules of M, then $N + P$ denotes the submodule of M generated by N and P.

Let R be a ring. If n is a non-negative integer, then Ext_R^n and Tor_R^n denotes the n-th derived bifunctor of the bifunctor Hom_R and of the tensor product functor, respectively. Further, R is said to be *completely reducible* if the module R is a direct sum of simple modules.

Recall that by [2, Appendix A], a ring R is called a *left T-ring* if there are only trivial orthogonal theories of the bifunctor Ext for R-mod. Hence R is a left T-ring if and only if $\mathrm{Ext}_R(M, N) \neq 0$ for each non-projective module M and each non-injective module N. Clearly, every completely reducible ring is a left T-ring. In [2, Appendix A, p. 216], non-completely reducible left T-rings were divided into five types. Recall that by [11, 2.2, 2.5, 4.4 and 6.1], a left T-ring is of type 1 iff it is singular. Similarly, a left T-ring is of type 2 iff it is (von Neumann) regular and non-completely reducible.

For further details and terminology concerning orthogonal theories, rings, modules and set theory, the reader is referred to [8], [2], [1], [3] and [10].

I. ORTHOGONAL THEORIES OF EXT AND TOR

I.1. Proposition. *Let R be a ring and n a non-negative integer. Suppose there are only trivial orthogonal theories of the bifunctor* Ext_R^n *for* R-mod. *Then there are only trivial orthogonal theories of the bifunctor* Tor_R^n.

Proof. Let Z be the ring of integers and C an injective cogenerator in Z-mod. Suppose there exist $A \in R$-mod and $B \in$ mod-R such that the weak dimension of A is at least n and $\mathrm{Tor}_R^n(B, A) = 0$ (see [3, Ch. VI., Exercise 3]). Using [3, Ch. VI., 5.1], we get

$$\mathrm{Ext}_R^n(A, \mathrm{Hom}_Z(B, C)) \simeq \mathrm{Hom}_Z(\mathrm{Tor}_R^n(B, A), C) = 0 .$$

Since the projective dimension of A is at least n and there are only trivial orthogonal theories of Ext_R^n, the module $\mathrm{Hom}_Z(B, C)$ has the injective left dimension at most $n - 1$. Hence the assertion is clear for $n = 0$. If $n \geqq 1$, then, by [3, Ch. VI., 2.1a], the functor $\mathrm{Ext}_R^{n-1}(-, \mathrm{Hom}_Z(B, C))$ is exact. Further, by [3, Ch. VI., 5.1] and [1, 18.14], the functor $\mathrm{Tor}_R^{n-1}(B, -)$ is exact and the weak right dimension of B is at most $n - 1$, q.e.d.

I.2. Remark. The converse of I.1 holds for $n = 0$ for commutative rings (compare [6] and [9]). For $n = 1$, it does not hold even for commutative rings. Namely, for every regular ring there are only trivial orthogonal theories of the bifunctor Tor, but by [11, 3.1] every commutative regular T-ring is completely reducible.

I.3. Lemma. *Let R be a QF-ring. Then the following conditions are equivalent:*

(i) $\mathrm{Ext}_R(A, B) \neq 0$ *for all modules A and B such that A is non-projective and finitely generated and B is non-injective.*

(ii) $\mathrm{Tor}_R(A, B) \neq 0$ *for each non-flat $A \in$ mod-R and each non-flat finitely generated module B.*

Proof. (i) implies (ii). Use the proof of I.1.

(ii) implies (i). Suppose (i) does not hold for modules A and B with the required properties. Let Z be the ring of integers, let C be an injective cogenerator in mod-Z and put $D = \mathrm{Hom}_Z(B, C)$. By [3, Ch. VI., 5.3]

$$\mathrm{Tor}_R(D, A) \simeq \mathrm{Hom}_Z(\mathrm{Ext}_R(A, B), C) = 0\,,$$

whence D is a flat right R-module. Since R is a QF-ring, D is also an injective right R-module, hence B is a flat module and B is injective, a contradiction.

I.4. Lemma. *Let R be a local left and right perfect ring. Denote by J the Jacobson radical of R. Then the following conditions are equivalent:*

(i) $\mathrm{Tor}_R(A, B) \neq 0$ *for each non-flat $A \in$ mod-R and each non-flat cyclic module B.*

(ii) $K . I \neq K \cap I^{(\kappa)}$ *for each proper left ideal I of R, each cardinal κ and each non-zero right R-submodule K of the bimodule $J^{(\kappa)}$.*

Proof. Since R is local left perfect, B is a non-flat cyclic module iff $B \simeq R/I$ for a proper left ideal I of R. As R is right perfect, A is a non-flat right R-module iff $A \simeq R^{(\kappa)}/K$ for a cardinal κ and a non-zero superfluous right R-submodule K of $R^{(\kappa)}$. Since J is right T-nilpotent, K is a superfluous right submodule of $R^{(\kappa)}$ iff K is a right R-submodule of the bimodule $J^{(\kappa)}$. Finally, $\mathrm{Tor}^1_R(A, B) = 0$ iff the canonical mapping $\mathrm{Tor}^0_R(A, I) \to A . I$ is an abelian group isomorphism iff $K . I = K \cap I^{(\kappa)}$, q.e.d.

I.5. Proposition. *Let R be a left T-ring of type 1. Then either $R = D$ or $R = C \boxplus D$, where C is a completely reducible ring and D is isomorphic to a full matrix ring over a local left artinian ring S such that $K . I \neq K \cap I$ for each proper right ideal K of S and each proper left ideal I of S.*

Proof. Use [2, A.3.1-2], I.1 and I.4.

II. REGULAR T-RINGS

II.1. Lemma. *Let R be a regular ring. Then each projective module is a direct sum of cyclic modules, and each countably generated left ideal of R is generated by a set of pairwise orthogonal idempotents of R.*

Proof. By [1, 26.2] and [7, 2.6 and 2.14].

II.2. Lemma. *Let R be a regular ring, let N be a module and λ a cardinal with $\mathrm{cf}(\lambda) = \aleph_0$. Denote by π_ν the ν-th natural projection of N^λ onto N, $\nu < \lambda$. Take a cofinal subset $\{\lambda_i \mid i < \aleph_0\}$ of λ and put*

$$N_i = \{n \in N^\lambda \mid n\pi_\nu = 0 \quad \text{for all} \quad \nu \geqq \lambda_i\}$$

and $N_\lambda = \bigcup N_i$, $i < \aleph_0$. Suppose I is a countably generated left ideal of R. Then $\mathrm{Ext}_R(R/I, N^\lambda/N_\lambda) = 0$.

Proof. If I is finitely generated, then R/I is projective and the assertion is clear. Hence, in view of II.1, we may assume that there exist pairwise orthogonal idempotents $e_i \in R$, $i < \aleph_0$ such that $I = \dot{\Sigma} Re_i$, $i < \aleph_0$. Let f be an R-homomorphism of I to N^λ/N_λ, i.e. $e_i f = e_i n_i + N_\lambda$, where $n_i \in N^\lambda$ for all $i < \aleph_0$. Define $m \in N^\lambda$ by $m\pi_\nu = 0$ for $\nu < \lambda_0$ and $m\pi_\nu = e_0 n_0 \pi_\nu + \ldots + e_i n_i \pi_\nu$ for $\lambda_i \leqq \nu < \lambda_{i+1}$. Then, for all $i < \aleph_0$, $e_i f = e_i m + N_\lambda$ and consequently $\mathrm{Ext}_R(R/I, N^\lambda/N_\lambda) = 0$, q.e.d.

II.3. Theorem. *Let R be a regular left T-ring. Then each left ideal of R is countably generated.*

Proof. By [11, 2.4, 2.5, 4.4 and 6.1], R is left hereditary. Suppose there is a left ideal L of R which is not countably generated. By II.1, there are a cardinal $\kappa > \aleph_0$ and idempotents $e_\alpha \in R$, $\alpha < \kappa$ such that $L = \dot{\Sigma}\, Re_\alpha$, $\alpha < \kappa$. Let N be a module and define an abelian group G by $G = \dot{\Pi} e_\alpha N$, $\alpha < \kappa$. Denote by ϱ_α the α-th natural projection of G onto $e_\alpha N$. Let P be an infinite set containing G and β an ordinal such that $\mathrm{card}(P) = \aleph_\beta$, i.e. $P = \{p_\gamma \mid \gamma < \aleph_\beta\}$. Put $\lambda = \aleph_{\beta+\aleph_0}$ and $\lambda_i = \aleph_{\beta+i}$ for $i < \aleph_0$. Then, by II.2, the module N^λ/N_λ is injective. Define an R-homomorphism f of L to N^λ/N_λ by $e_\alpha f = n_\alpha + N_\lambda$, $\alpha < \kappa$, where $n_\alpha \pi_\nu = p_\nu \varrho_\alpha$ if $\nu < \aleph_\beta$ and $p_\nu \in G$, $n_\alpha \pi_\nu = p_\mu \varrho_\alpha$ if $\nu = \aleph_{\beta+i} + \mu$, $i < \aleph_0$, $\mu < \aleph_\beta$ and $p_\mu \in G$, $n_\alpha \pi_\nu = 0$ otherwise.

Since $\mathrm{Ext}_R(R/L, N^\lambda/N_\lambda) = 0$, there exists $n \in N^\lambda$ with $e_\alpha n - n_\alpha \in N_\lambda$ for all $\alpha < \kappa$. Hence, there are a natural number j and a subset $A \subseteq \kappa$ such that $\mathrm{card}(A) = \kappa$ and $e_\alpha n - n_\alpha \in N_j$ for all $\alpha \in A$. Therefore, $e_\alpha n \pi_\nu = n_\alpha \pi_\nu$ for all $\alpha \in A$ and $\aleph_{\beta+j} \leqq \nu < \lambda$. Now, put $K = \dot{\Sigma} Re_\alpha$, $\alpha \in A$. We shall prove that $\mathrm{Ext}_R(R/K, N) = 0$. To this purpose take any R-homomorphism g of K to N and let $p \in G$ be such that $p\varrho_\alpha = e_\alpha g$ for all $\alpha \in A$. Clearly, $p = p_\gamma$ for some $\gamma < \aleph_\beta$. But $p_\gamma \varrho_\alpha = n_\alpha \pi_{\mu_0}$ for $\mu_0 = \aleph_{\beta+j} + \gamma$ and for all $\alpha < \kappa$. Hence $e_\alpha g = e_\alpha n \pi_{\mu_0}$ for all $\alpha \in A$. Now, define an R-homomorphism h of R to N by $1\,h = n\pi_{\mu_0}$. Then $h/K = g$ and consequently $\mathrm{Ext}_R(R/K, N) = 0$. Since K is not finitely generated, N is injective. Hence, if there is a left ideal of R which is not countably generated, then every module is injective and R is completely reducible, a contradiction.

II.4. Theorem. *Let R be a left T-ring of type 2. Then either $R = D$ or $R = C \boxplus D$, where C is a completely reducible ring and D is a simple regular ring such that each left ideal of D is countably generated.*

Proof. By [11, 2.4] and II.3.

III. EXT AND SIMPLE COUNTABLE REGULAR RINGS

In [7], various methods of construction of simple countable non-completely reducible regular rings were presented. Here, in III.4 and III.6, we show that for this class of rings there are only trivial orthogonal theories of the bifunctor Ext generated by "small" modules. Hence, we obtain a partial converse to II.4. Nevertheless, III.7 shows that for this class of rings the converse of II.4 does not hold in general.

III.1. Lemma. *Let R be a left hereditary ring. Suppose* $\mathrm{Ext}_R(A, B) = 0$ *for some modules A and B. Then, for any submodule C of A and any factormodule D of B,* $\mathrm{Ext}_R(C, D) = 0$.

Proof. Easy (see [3, Ch. VI.]).

III.2. Lemma. *Let R be a simple regular left hereditary ring. Suppose that the right dimension of every simple module over its endomorphism ring is $\aleph_0$. Then* $\mathrm{Ext}_R(M, N) \neq 0$ *for each pair (M, N) of finitely generated modules such that M is non-projective and N is non-injective.*

Proof. By [1, 10.5], II.1 and III.1, it suffices to prove that $\mathrm{Ext}_R(R^{(m)}/I, J) \neq 0$ for each simple module J, each $0 < m < \aleph_0$ and each countably generated submodule $I \subseteq R^{(m)}$. By II.1, there exist $0 \neq x_i \in I$ such that $I = \dot{\Sigma} R x_i$, $i < \aleph_0$. Let e_i be the idempotents of R with $R(1 - e_i) = \mathrm{Ann}_R(x_i)$ for all $i < \aleph_0$. Put $K = \mathrm{End}_R(J)$. By [1, 14.4], we can identify R with a dense subring of the endomorphism ring of the right K-module J. Since R is simple, each non-zero element of R is an endomorphism of rank equal to $\dim(J) = \aleph_0$. Hence, we have right K-module isomorphisms

$$\mathrm{Hom}_R(I, J) \simeq \dot{\Pi} e_i J \simeq J^{\aleph_0} \quad \text{and} \quad \mathrm{Hom}_R(R^{(m)}, J) \simeq J^{(m)}.$$

It is easy to see that $\dim(J^{\aleph_0}) \geqq 2^{\aleph_0}$ and hence $\mathrm{Ext}_R(R^{(m)}/I, J) \neq 0$, q.e.d.

III.3. Lemma. *Let R be a regular left hereditary ring. Let N be a module such that* $\mathrm{Ext}_R(M, N) \neq 0$ *for every finitely generated non-projective module M. Then* $\mathrm{Ext}_R(M, N) \neq 0$ *for every countably generated non-projective module M.*

Proof. Let M be a countably infinitely generated module, $M \simeq R^{(\aleph_0)}/I$ for a submodule $I \subseteq R^{(\aleph_0)}$. Let $\{x_i \mid i < \aleph_0\}$ be a free basis of $R^{(\aleph_0)}$ and put $R_n = Rx_0 + \dots$

$\ldots \dot{+} Rx_n$ and $I_n = I \cap R_n$. $n < \aleph_0$. In view of III.1, it suffices to prove that M is projective iff I_n is finitely generated for all $n < \aleph_0$. But if M is projective, then there is a submodule $K \subseteq R^{(\aleph_0)}$ such that $I \dot{+} K = R^{(\aleph_0)}$ and $R_n/I_n \simeq (I + R_n) \cap \cap K \subseteq K$. Hence R_n/I_n is projective. On the other hand, if I_n is finitely generated for all $n < \aleph_0$, then by [7, 1.11], $I_{n+1} = I_n \dot{+} A_n$, $I_0 \dot{+} B_0 = R_0$ and $(I_{n+1} + R_n) \dot{+} \dot{+} B_{n+1} = R_{n+1}$ for some modules A_n, B_n, $n < \aleph_0$. Let $A = \dot{\Sigma} A_i$, $i < \aleph_0$ and $B = \dot{\Sigma} B_i$, $i < \aleph_0$. Then $I_0 \dot{+} A \dot{+} B = R^{(\aleph_0)}$ and M is projective, q.e.d.

III.4. Theorem. *Let R be a simple countable regular ring. Then* $\mathrm{Ext}_R(M, N) \neq 0$ *for each pair (M, N) of countably generated modules such that M is non-projective and N is non-injective.*

Proof. Clearly, the assertion holds for completely reducible rings. Hence, by III.2 and III.3, it suffices to prove that $\mathrm{Ext}_R(M, N) \neq 0$ for every finitely generated non-projective module M and every countably infinitely generated non-injective module N. We have $M \simeq R^{(m)}/I$, where $1 \leqq m < \aleph_0$, and there exist $0 \neq x_i \in I$ such that $I = \dot{\Sigma} Rx_i$, $i < \aleph_0$. Let e_i be the idempotents of R with $R(1 - e_i) = \mathrm{Ann}_R(x_i)$ for all $i < \aleph_0$. Since R is simple, e_iN contains a set of R-generators of N for all $i < \aleph_0$. Hence card $(e_iN) \geqq \aleph_0$ for all $i < \aleph_0$, and card $(\mathrm{Hom}_R(I, N)) = $ card $(\Pi \, \mathrm{Hom}_R(Rx_i, N)) \geqq 2^{\aleph_0}$. But $\mathrm{Hom}_R(R^{(m)}, N) \simeq N^{(m)}$ and hence $\mathrm{Ext}_R(M, N) \neq 0$, q.e.d.

III.5. Lemma. *Assume $V = L$. Let R be a countable regular ring. Let N be a module which is generated by at most $\aleph_1$ elements and let* $\mathrm{Ext}_R(M, N) \neq 0$ *for each countably generated non-projective module M. Then* $\mathrm{Ext}_R(M, N) \neq 0$ *for each non-projective module M.*

Proof. We shall prove the assertion by induction on $\alpha = $ card (M). By the premises, the assertion holds for $\alpha \leqq \aleph_0$. Suppose it holds for all $\alpha < \kappa$, κ being a regular uncountable cardinal. Let M be a module with card $(M) = \kappa$ and $\mathrm{Ext}_R(M, N) = 0$. Since R is left hereditary, M is projective by the induction hypothesis and by [5, 1.5]. Hence, the assertion holds for $\alpha = \kappa$. Suppose the assertion holds for all $\alpha < \lambda$, λ being a singular cardinal. Let M be a module with card $(M) = \lambda$ and $\mathrm{Ext}_R(M, N) = 0$. We shall use [10, 2.1] in the following setting: U is a module containing M as a submodule, $A = M$, $B = 0$, $\gamma_0 = \aleph_0$ (the operations on U are 0, $+$ and the left multiplications by elements of R), $(C, D) \in F$ iff both C and D are submodules of U and $(C + D)/D$ is a projective module. Put $\gamma_1 = \aleph_0$ and let $\aleph_0 \leqq \gamma_2 < \lambda$. Let T be an expansion of the model consisting of the set of sets which are hereditarily of cardinality less than γ_2 and of relations and operations $\in$, $=$, F, 0, $+$ and the left multiplications by elements of R such that T has Skolem functions. Then, by the induction hypothesis, A/B is λ-free (see [10, Definition 1.1]). We shall show that axioms Ax I*, Ax II, ..., Ax VII from [10, p. 325] are satisfied. Axiom Ax I* holds since R is left hereditary, axioms Ax II, Ax III and Ax IV are trivially satisfied. To prove Ax V, let S be an elementary submodel of T and let D, G, H and C_i,

$i < \beta$ be submodules of U such that $D \in S$, $C_i \in S$ for all $i < \beta$, C_i, $i < \beta$ is an increasing chain and $G \subseteq D$ and $H \subseteq D \subseteq C_0$. Suppose $(G, (C_0 \cap S) + H) \in F$ and $(C_i \cap S, (C_0 \cap S) + D) \in F$ for all $i < \beta$. Then $G + (C_0 \cap S) + H = ((C_0 \cap S) + H) \dotplus E$ for a projective module E. Clearly $E + (\bigcup C_i \cap S) + H = G + (\bigcup C_i \cap S) + H$. Suppose $E \cap ((\bigcup C_i \cap S) + H) \neq 0$. Then $E \cap ((C_i \cap S) + H) \neq 0$ for some $i < \beta$. Further, there is a projective module I with $((C_0 \cap S) + D) \dotplus I = (C_i \cap S) + D$. Let $\{u_j \mid j < \gamma\}$ be a set of R-generators of I. Clearly, for all $j < \gamma$, $u_j = c_j + d_j$ for some $c_j \in C_i \cap S$ and $d_j \in D$. Since $D \subseteq C_0$, we have $\Sigma Rc_j + (C_0 \cap S) = C_i \cap S$.

But $E \cap (\Sigma Rc_j + (C_0 \cap S) + H) \subseteq ((C_i \cap S) \cap ((C_0 \cap S) + D)) + (C_0 \cap S) + H$, the module on the right hand side of the inclusion being equal to $(C_0 \cap S) + H$. Thus the module on the left hand side is zero, whence $E \cap ((C_i \cap S) + H) = 0$, a contradiction. So $(G, (\bigcup C_i \cap S) + H) \in F$ and Ax V is proved. To prove Ax VI, let S be an elementary submodel of T, and let C, D and E be submodules of U such that $C, D, E \in S$ and $(C, D + E) \in F$. Since S is elementary, we have $C + D + E = (D + E) \dotplus G$ for a projective module $G \in S$. Moreover, $(C \cap S) + (D \cap S) + E$ is equal to $((C + D + E) \cap S) + (D \cap S) + E = (G \cap S) \dotplus ((D \cap S) + E)$. Since R is left hereditary, $G \cap S$ is a projective module, so $(C \cap S, (D \cap S) + E) \in F$ and Ax VI is proved. To prove Ax VII, let S be an elementary submodel of T and let C, D be submodules of U such that $C, D \in S$ and $(C, D) \in F$. Then $C + D = D \dotplus E$ for a projective module $E \in S$, and $(C \cap S) + D = ((C + D) \cap S) + D = (E \cap S) \dotplus D$. By II.1, there is a set $X = \{x_i \mid i < \beta\}$ such that $X \in S$, $X \subseteq E$ and $E = \dot{\Sigma} Rx_i$, $i < \beta$. Let G be a submodule of E generated by $X - (X \cap S)$. Since $E \cap S$ is generated by $X \cap S$, we have $G \cap ((C \cap S) + D) = 0$ and $G \dotplus ((C \cap S) + D) = C + D$. Hence $(C, (C \cap S) + D) \in F$ and Ax VII is proved. Finally, by [10, 2.1], $(A, B) \in F$, so M is projective and the assertion holds for $\alpha = \lambda$, q.e.d.

III.6. Theorem. *Assume $V = L$. Let R be a simple countable regular ring. Then* $\operatorname{Ext}_R(M, N) \neq 0$ *for each non-projective module M and each countably generated non-injective module N.*

Proof. By III.4 and III.5.

III.7. Theorem. *Let R be a direct limit of a countable directed system of simple countable completely reducible rings. Suppose R is not completely reducible. Then R is a simple countable regular ring and* $\operatorname{Ext}_R(M, N) = 0$ *for a non-projective cyclic module M and a non-injective uncountably generated module N.*

Proof. Let $(R_i, i \in I, \pi_i^j)$ be a countable directed system of simple countable completely reducible rings, i.e. let $(I, \leqq)$ be a countable directed set, R_i a simple countable completely reducible ring for each $i \in I$ and $\pi_i^j \colon R_i \to R_j$ a ring homomorphism for $i \leqq j$, $i, j \in I$ such that π_i^i is the identity for each $i \in I$ and $\pi_j^k \pi_i^j = \pi_i^k$ for each $i \leqq j \leqq k$, $i, j, k \in I$. Put $R = \varinjlim_I R_i$ and for each $i \in I$ let $\pi_i \colon R_i \to R$ be the canonical homomorphism. Since I has a cofinal subset K such that $(K, \leqq)$ is isomorphic to the ordered set of natural numbers and $\varinjlim_I R_i = \varinjlim_K R_i$, we can assume that $(I, \leqq) = (\aleph_0, \leqq)$. Moreover, since each π_i^j is a ring monomorphism, each π_i is a ring monomorphism and we can assume that each R_i is a subring of R_{i+1} and $R = \bigcup R_i$, $i < \aleph_0$. Clearly, R is a simple countable regular ring. Further, let F be a set of finite sequences containing only 0 or 1. For $x \in F$, $x = (x_0, \ldots, x_n)$ put $\ln(x) = n$. For $x_i \in \{0, 1\}$, let x_i' be the binary complement of x_i. We shall define a strictly increasing sequence $\{t_i \mid i < \aleph_0\}$ of natural numbers and for each $x \in F$ an idempotent $e_x \in R$ and a finite set of idempotents $A_x \subseteq R$ such that

1) $A_1 = \{e_0, e_1\}$ and $(A_0 \cup A_1)$ is a complete set of pairwise orthogonal primitive idempotents of R_{t_0},
2) if $x \in F$, $x = (x_0, \ldots, x_n)$ and $y = (x_0, \ldots, x_n, 0)$, $z = (x_0, \ldots, x_n, 1)$, then $(A_y \cup A_z)$ is a set of pairwise orthogonal primitive idempotents of $R_{t_{n+1}}$ and $A_z = \{e_y, e_z\}$ and $e_x = \Sigma a$, $a \in (A_y \cup A_z)$.

First, since R is not completely reducible, there is a $k < \aleph_0$ such that R_k is not a division ring. Let $A = \{f_0, \ldots, f_m\}$, $m \geqq 1$, be a complete set of pairwise orthogonal primitive idempotents of R_k. Put $t_0 = k$, $e_0 = f_0$, $e_1 = f_1$, $A_1 = \{f_0, f_1\}$ and $A_0 = A - A_1$. Then 1) is clear. Suppose $\{t_i \mid i \leqq n\}$, $\{A_x \mid x \in F,\ \ln(x) \leqq n\}$, $\{e_x \mid x \in F,\ \ln(x) \leqq n\}$ are defined and satisfy 1), and 2) for $x \in F$ with $\ln(x) < n$. Since R is not completely reducible, there is a $k > t_n$ such that no element of the set $\{e_x \mid x \in F,\ \ln(x) = n\}$ is a primitive idempotent of R_k. For each $x \in F$, $x = (x_0, \ldots, x_n)$ let A be the set of pairwise orthogonal primitive idempotents of R_k such that $e_x = \Sigma a$, $a \in A$. Let $y = (x_0, \ldots, x_n, 0)$ and $z = (x_0, \ldots, x_n, 1)$ and let e, f be two different elements of A. Put $t_{n+1} = k$, $A_z = \{e, f\}$, $A_y = A - A_z$, $e_y = e$ and $e_z = f$. Then 1) holds and 2) is satisfied for $x \in F$ with $\ln(x) \leqq n$. Now, we shall show that there are uncountably many maximal left ideals of R. For each $x \in 2^{\aleph_0}$, $x = (x_i \mid i < \aleph_0)$ let $u_0 = x_0'$, $v_0 = 0$, $w_0 = x_0$ and $u_i = (x_0, \ldots, x_{i-1}, x_i')$, $v_i = (x_0, \ldots, x_{i-1}, 0)$, $w_i = (x_0, \ldots, x_{i-1}, x_i)$ for $1 \leqq i < \aleph_0$. Define $E_x = \bigcup (A_{v_i} \cup \{e_{u_i}\})$, $i < \aleph_0$. Then E_x is a set of pairwise orthogonal idempotents of R. Let I_x be the direct sum of Re, $e \in E_x$. We shall show that I_x is a maximal left ideal of R. Let $r \in R$ with $r \notin I_x$. There is an $n < \aleph_0$ with $r \in R_{t_n}$. Put $s = \Sigma ra$, $a \in \bigcup (A_{v_i} \cup \{e_{u_i}\})$ $i \leqq n$. Then, by 1) and 2), $r = re_{w_n} + s$ and $re_{w_n} \neq 0$. Since e_{w_n} is a primitive idempotent of the completely reducible ring R_{t_n}, there is a $p \in R_{t_n}$ with $pre_{w_n} = e_{w_n}$. Hence e_{w_n} is an element of $Rr + I_x$ and, by 1) and 2), $Rr + I_x = R$ and I_x is maximal. Clearly, if $x, y \in 2^{\aleph_0}$ and $x \neq y$ and $i < \aleph_0$ is the smallest index with $x_i \neq y_i$, then $e_{u_i} \in (I_x - I_y)$. Hence we have constructed uncountably many maximal left ideals of R. Further, let I be an arbitrary maximal left ideal of R. Let J be a maximal left ideal of R such that the module R/J is isomorphic to the module R/I. Then there is an $r_J \in R$ with $J = \{r \in R \mid rr_J \in I\}$. Hence the mapping $J \mapsto r_J$ is injective and there are only countably many maximal left ideals J such that R/J is isomorphic to R/I. Consequently, there are uncountably many pairwise non-isomorphic simple modules. Finally, suppose that $\mathrm{Ext}_R(M, N) \neq 0$ for each cyclic non-projective module M and each non-injective module N. Let K be a simple module, let N be a non-injective module and H an injective hull of N. Since $\mathrm{Ext}_R(K, N) \neq 0$, we have $\mathrm{Hom}_R(K, H/N) \neq 0$. Thus the module H/N has a socle sequence with factors isomorphic to direct powers of K. Consequently, all simple modules are isomorphic, a contradiction. Hence $\mathrm{Ext}_R(M, N) = 0$ for a cyclic non-projective module M and a non-injective module N. By III.4, N is not countably generated, q.e.d.

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