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ON INTEGRATION IN BANACH SPACES, VI

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INTRODUCTION

This part contains, as main result, a mean value theorem for our theory of integration of vector valued functions with respect to an operator valued measure, which is countably additive in the strong operator topology. For this we apply the convexity structure introduced by Price [37]. The organization of the paper is as follows: Section 1 reviews the notion of a convexity structure $\mathscr{C}$ on an abstract set X; we make here some historical remarks about this notion and we describe Price's $\mathscr{C}$-hull operator, when X is a Banach space. Section 2 establishes the mean value theorem which generalizes the mean value theorem for elements of $\mathscr{L}_1(\boldsymbol{m})$ from [34] to all integrable functions. Section 3 gives some applications for Fréchet differentiable normed space valued functions. These generalize the classical differentiation mean value theorem, a result of McLeod [33] and a beautiful theorem of L. Schwartz [39] (in normed space setting). Our proof of the mean value theorem requires the notion of the S-integral of Kolmogoroff [29] (see also [26] and [43]). In final section 4 we investigate relations between integrability and S-integrability.

We shall use the notations and concepts of the previous parts of our theory. Particularly, $\mathscr{P}$ will be a δ-ring of subsets of a non empty set T. $\sigma(\mathscr{P})$ denotes the smallest σ-ring which contains $\mathscr{P}$. $\boldsymbol{X}$ and $\boldsymbol{Y}$ are Banach spaces over the same scalar field with norms denoted by $|\cdot|$. $\boldsymbol{m}: \mathscr{P} \to L(\boldsymbol{X}, \boldsymbol{Y})$ is an operator valued measure countably additive in the strong operator topology with finite semivariation $\hat{\boldsymbol{m}}$ on $\mathscr{P}$.

1. CONVEXITY STRUCTURES

The applicability and intuitive appeal of convexity have led to a wide range of notions of generalized convexity. The usual definition of convexity in a linear space can be generalized as follows: A family $\mathscr{C}$ of subsets of a set X is called a *convexity structure* for X if the following conditions are satisfied:

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(a) $\emptyset$ and X belong to $\mathscr{C}$.

(b) If $\mathscr{C}_0$ is a subfamily of $\mathscr{C}$, then $\bigcap\{C: C \in \mathscr{C}_0\} \in \mathscr{C}$.

(c) $\{x\} \in \mathscr{C}$ for all $x \in X$.

For $S \subseteq X$ define $\mathscr{C}(S) = \bigcap\{C \in \mathscr{C}: S \subseteq C\}$. The set $\mathscr{C}(S)$ is called the *$\mathscr{C}$-hull* of S, and S is called *$\mathscr{C}$-convex* if $\mathscr{C}(S) = S$. It is clear that the operator $\mathscr{C}(\cdot): 2^X \to 2^X$ enjoys the properties:

(i) $S \subseteq \mathscr{C}(S)$.

(ii) If $S_1 \subseteq S_2$, then $\mathscr{C}(S_1) \subseteq \mathscr{C}(S_2)$.

(iii) $\mathscr{C}(\mathscr{C}(S)) = \mathscr{C}(S)$.

(iv) $S \in \mathscr{C}$ if and only if $\mathscr{C}(S) = S$.

This general notion appeared for the first time in a paper of Levi [30], where he assumes further the axiom: If $y_1, y_2, \ldots, y_m \in \mathscr{C}(\{x_1, x_2, \ldots, x_n\})$ with $m > n$, then there exist a positive integer $p < m$ and a permutation σ of $\{1, 2, \ldots, m\}$ such that $\mathscr{C}(\{y_{\sigma(1)}, \ldots, y_{\sigma(p)}\}) \cap \mathscr{C}(\{y_{\sigma(p+1)}, \ldots, y_{\sigma(m)}\}) \neq \emptyset$. His main result contains as corollaries the Helly theorem on convex sets in $\mathbb{R}^n$, its extension to n-dimensional geometries satisfying Hilbert's axioms of incidence and order, and an intersection theorem for free abelian groups with n generators. Another specific example of a convexity structure is given by the notion of quasiconvexity introduced in an earlier paper of Green and Gustin [23], where they used this convexity to study measure and topological properties of the graphs of the discontinous solutions of the classical functional equation $\phi(x + y) = \phi(x) + \phi(y)$. Later developments are due to Ellis [22], Prenowitz ([35], [36]) with his notion of joint space that he applies specially to projective and euclidean geometries, Bryant ([4], [5]), and Bryant and Webster ([6], [7], [8]). Among their contributions are the axiomatization of the notion of convexity space, a generalization of the Kakutani-Tukey separation theorem, the existence of a support hyperplane, a notion of dimension, generalizations of the classical results of Carathéodory, Helly, Radon and Steinitz, a notion of topological convexity space and a generalization of the Krein-Milman theorem. More abstract developments are due to Eckhoff ([20], [21]), Reay [38], Kay and Womble [27], Guay and Naimpally [24], Mah, Naimpally and Whitfield [31], and Szafron and Weston [42]. The subjects treated in the above-mentioned papers are related to product of convexity structures, definitions and relations between the Carathéodory, Helly and Radon numbets, and some solutions to the linearization problem.

The fundamental importance of the usual convex sets in vector integration was emphasized, for the first time, by Birkhoff [2]. In order to develop a Lebesgue theory in the bilinear case along the lines of the Birkhoff integral, Price [37] introduced a suitable convexity structure for a Banach space. Since this convexity structure is essential for our purposes, we indicate briefly its general construction: Let X be a Banach space, and let $L(X)$ be the Banach space of all bounded linear operators from X to X. Consider the set $\mathscr{F}_0 = \{(T_i)_{i=1}^n: T_i \in L(X), \sum_{i=1}^n T_i = I$ and $n = 1, 2, 3, \ldots\}$, where I denotes the identity operator. Let $\mathscr{F}$ be any non-empty set of finite sequences in $L(X)$ with the following property: If $(T_i)_{i=1}^n$ belongs to $\mathscr{F}$ and $\sum_{i=1}^n T_i \neq 0$, then the operator $T = \sum_{i=1}^n T_i$ is bijective. Given a such element of $\mathscr{F}$, consider the subset $\mathscr{F}'$ of $\mathscr{F}_0$ consisting of finite sequences of the form: $(T^{-1}T_i)_{i=1}^n$ or $(T_i T^{-1})_{i=1}^n$. Let $\mathscr{F}^*$ be the smallest subset of $\mathscr{F}_0$ containing $\mathscr{F}'$ and having the following multiplicative property: If $(T_i)_{i=1}^m$ and $(T_j')_{j=1}^n$ are two elements of $\mathscr{F}^*$, then $(T_1T_1', \ldots, T_1T_n', \ldots, T_mT_1', \ldots, T_mT_n')$ belongs to $\mathscr{F}^*$. If $\mathscr{C}$ is the family of all subsets C of X such that $\{\sum_{i=1}^n T_i x_i: (T_i)_{i=1}^n \in \mathscr{F}^*,\ x_i \in C \text{ and } n = 1, 2, \ldots\} \subseteq C$, then C is a convexity structure for X. If we identify each $\lambda \in \mathbb{R}$ with the operator $T_\lambda: x \to \lambda x$ of $L(X)$, then for $\mathscr{F} = \{(T_{\lambda_i})_{i=1}^n: \lambda_i \geqq 0 \text{ and } n = 1, 2, 3, \ldots\}$ the $\mathscr{C}$-hull of every subset S of X is reduced to the classical convex hull $\operatorname{co}(S)$.

We note the following useful result:

Lemma 1. *Let $S \subseteq X$. Then*:

(a) $\sum_{i=1}^n T_i(\mathscr{C}(S)) = \mathscr{C}(S)$ *for all* $(T_i)_{i=1}^n \in \mathscr{F}^*$.

(b) *If* $(T_i)_{i=1}^n$ *is an element of* $\mathscr{F}$ *such that* $T = \sum_{i=1}^n T_i \neq 0$, *then* $\sum_{i=1}^n T_i(\mathscr{C}(S)) = T(\mathscr{C}(S))$.

Proof. (a) It suffices to prove the inclusion $\sum_{i=1}^n T_i(\mathscr{C}(S)) \subseteq \mathscr{C}(S)$. But an element of the left member is of the form $\sum_{i=1}^n T_i x_i$ with $x_i \in \mathscr{C}(S)$. Since $\mathscr{C}(S) \in \mathscr{C}$, $\sum_{i=1}^n T_i x_i \in \mathscr{C}(S)$.

(b) The hypotheses imply that T is bijective. Then, by (a), we have $\mathscr{C}(S) = \sum_{i=1}^n (T^{-1}T_i)(\mathscr{C}(S)) = T^{-1}(\sum_{i=1}^n T_i(\mathscr{C}(S)))$.

2. MEAN VALUE THEOREM

First let us remind the notion of the S-integral of Kolmogoroff. Let $E \in \mathscr{P}$. By a partition of E we mean a finite sequence $(E_i)_{i=1}^n$ of pairwise disjoint non empty sets from $\mathscr{P}$ whose union is E. If $\pi_1(E) = (E_i)_{i=1}^n$ and $\pi_2(E) = (F_j)_{j=1}^m$ are two partitions of E, we say that $\pi_2(E)$ is a *refinement* of $\pi_1(E)$, denoted $\pi_1(E) \leqq \pi_2(E)$, if each F_j is contained in some E_i. The set $\Pi(E)$ of all partitions of E with the partial ordering $\leqq$ form a directed set. Let now $f: T \to X$. If $\pi(E) = (E_i)_{i=1}^n$ is a partition of E, choose points $t_i \in E_i$, $i = 1, \ldots, n$, and write

$$S_{\pi(E)}(m, f, t_1, \ldots, t_n) = \sum_{i=1}^n m(E_i) f(t_i) .$$

The multivalued function $\underset{\bullet}{S}(m, f, \ldots)$ is a net on Y. We say that $\underset{\bullet}{S}(m, f, \ldots)$ *converges* to a vector $y \in Y$ if, for every $\varepsilon > 0$, there exists a partition $\pi_\varepsilon(E)$ of E such that

$$|S_{\pi(E)}(m, f, t_1, \ldots, t_m) - y| < \varepsilon$$

whenever $\pi(E) \in \Pi(E)$ and $\pi_\varepsilon(E) \leqq \pi(E)$. In this case we say that $\boldsymbol{f}$ is *S-integrable* on E with respect to $\boldsymbol{m}$. We define its *S-integral* on E, $S_E \boldsymbol{f} \, \mathrm{d}\boldsymbol{m} = \boldsymbol{y}$ and write

$$\lim_{\Pi(E)} S_{\pi(E)}(\boldsymbol{m}, \boldsymbol{f}, t_1, \ldots, t_n) = S_E \boldsymbol{f} \, \mathrm{d}\boldsymbol{m} .$$

The following facts are immediate from definitions. If $N \in \mathscr{P}$ is an $\boldsymbol{m}$-null set, then each function $\boldsymbol{f}$: $T \to \boldsymbol{X}$ is S-integrable on N and $S_N \boldsymbol{f} \, \mathrm{d}\boldsymbol{m} = 0$.

If $\boldsymbol{f}$: $T \to \boldsymbol{X}$, $E, F \in \mathscr{P}$, $F \subseteq E$ and $\boldsymbol{f}$ is S-integrable on E, then $\boldsymbol{f}$ is S-integrable on F.

If $\boldsymbol{f}$ is S-integrable on both $E, F \in \mathscr{P}$, then $\boldsymbol{f}$ is S-integrable on $E \cup F$, and if moreover $E \cap F = \emptyset$, then

$$S_{E \cup F} \boldsymbol{f} \, \mathrm{d}\boldsymbol{m} = S_E \boldsymbol{f} \, \mathrm{d}\boldsymbol{m} + S_F \boldsymbol{f} \, \mathrm{d}\boldsymbol{m} .$$

If both $\boldsymbol{f}, \boldsymbol{g}$: $T \to \boldsymbol{X}$ are S-integrable on $E \in \mathscr{P}$, a, b are scalars, then $a\boldsymbol{f} + b\boldsymbol{g}$ is S-integrable on E, and

$$S_E(a\boldsymbol{f} + b\boldsymbol{g}) \, \mathrm{d}\boldsymbol{m} = a \, . \, S_E \boldsymbol{f} \, \mathrm{d}\boldsymbol{m} + b \, . \, S_E \boldsymbol{g} \, \mathrm{d}\boldsymbol{m} .$$

The following lemma is also immediate.

Lemma 2. *Simple integrable functions, and their uniform limits, are S-integrable on each set $E \in \mathscr{P}$, and the values of both integrals are equal.*

If $\boldsymbol{f}$: $T \to \boldsymbol{X}$, $E \in \mathscr{P}$, $\pi(E) = (E_i)_{i=1}^n$ is a partition of E and $t_i \in E_i$, $i = 1, 2, \ldots, n$, then clearly

$$|S_{\pi(E)}(\boldsymbol{m}, \boldsymbol{f}, t_1, \ldots, t_n)| = \Big|\sum_{i=1}^n \boldsymbol{m}(E_i) \boldsymbol{f}(t_i)\Big| \leqq \|\boldsymbol{f}\|_E \, . \, \hat{\boldsymbol{m}}(E) .$$

To establish our mean value theorem we need the following seting: $\boldsymbol{Y} = \boldsymbol{X}$ and $\boldsymbol{m}$: $\mathscr{P} \to L(\boldsymbol{X})$ satisfies the axiom of Price [37, p. 20]: For every $E \in \mathscr{P}$, either $\boldsymbol{m}(E) = 0$ or $\boldsymbol{m}(E)$ is bijective. Taking $\mathscr{F} = \{(\boldsymbol{m}(E_i))_{i=1}^n : E_i \in \mathscr{P}$ and $E_i \cap E_k = \emptyset$ for $i \neq k$, $i, k = 1, 2, \ldots, n$, $n = 1, 2, \ldots\}$ we can define the Price's $\mathscr{C}$-hull for every subset of $\boldsymbol{X}$.

If μ: $\mathscr{P} \to [0, +\infty)$ is a countably additive measure and if we put $\boldsymbol{m}(E) \boldsymbol{x} = \mu(E) \, . \, \boldsymbol{x}$, $E \in \mathscr{P}$, $\boldsymbol{x} \in \boldsymbol{X}$, then clearly $\boldsymbol{m}$ satisfies the axiom of Price. Let us give another less trivial example. Let $T = \{1, 2, \ldots\}$, let $\mathscr{P} = 2^T$ and let $\boldsymbol{X} = l_2$. For every $k = 1, 2, \ldots$ and $\boldsymbol{x} = (x_1, \ldots, x_k, \ldots) \in l_2$ define $\boldsymbol{m}(\{k\}) \boldsymbol{x} = k^{-2}(\boldsymbol{x} + x_k \boldsymbol{e}_k)$, where $\boldsymbol{e}_k$ is the k-th unit vector in l_2, and $\boldsymbol{m}(E) \boldsymbol{x} = \sum_{k \in E} \boldsymbol{m}(\{k\}) \boldsymbol{x}$ for $E \in \mathscr{P}$. Then clearly $v(\boldsymbol{m}, T) < +\infty$ and $\boldsymbol{m}(E)$ is bijective for each $E \neq \emptyset$.

Lemma 3. *If $\boldsymbol{m}(E) \boldsymbol{x} = \mu(E) \, . \, \boldsymbol{x}$ where μ is a finite positive countably additive measure on $\mathscr{P}$, then $\mathscr{C}(S) \subseteq \mathrm{co}(S)$ for every subset S of $\boldsymbol{X}$.*

Proof. Let $\mathscr{F}_1 = \{(T_{\lambda_i})_{i=1}^n : \lambda_i \geqq 0, \ \sum_{i=1}^n \lambda_i = 1, \ n = 1, 2, \ldots\}$. Then $\mathscr{F}_1$ satisfies multiplicative property and, since μ is finite and positive, it contains $\mathscr{F}'$. So $\mathscr{F}_1 \supseteq \mathscr{F}^*$. Hence every convex subset of $\boldsymbol{X}$ is $\mathscr{C}$-convex, and therefore $\mathscr{C}(S) \subseteq \mathrm{co}(S)$ for every $S \subseteq \boldsymbol{X}$.

Remark. Let $T = \{1, 2, \ldots\}$, let $\mathscr{P}$ be the δ-ring of all finite subsets of T and let μ be the counting measure on $\mathscr{P}$. Put $\boldsymbol{m}(E)\,\boldsymbol{x} = \mu(E)\,.\,\boldsymbol{x}$ for $E \in \mathscr{P}$ and $\boldsymbol{x} \in X$. In this case, $\mathscr{F}'$ satisfies the multiplicative property, and therefore $\mathscr{F}^* = \mathscr{F}'$. It follows that, if $\boldsymbol{x}, \boldsymbol{y} \in X - \{0\}$ and $\boldsymbol{x} \neq \boldsymbol{y}$, then $\mathscr{C}(\{x, y\}) = \{\boldsymbol{T}_1\boldsymbol{x} + \boldsymbol{T}_2\boldsymbol{y}: (\boldsymbol{T}_1, \boldsymbol{T}_2) \in \boldsymbol{F}^*\} \subset \operatorname{co}(\{\boldsymbol{x}, \boldsymbol{y}\})$ and $\mathscr{C}(\{x, y\}) = \{\boldsymbol{T}_1\boldsymbol{x} + \boldsymbol{T}_2\boldsymbol{y}: (\boldsymbol{T}_1, \boldsymbol{T}_2) \in \boldsymbol{F}^*\} \neq \operatorname{co}(\{\boldsymbol{x}, \boldsymbol{y}\})$.

Theorem 1. (Mean value theorem). *Suppose that $\boldsymbol{m}: \mathscr{P} \to L(X)$ satisfies the axiom of Price: For every $E \in \mathscr{P}$ either $\boldsymbol{m}(E) = 0$ or $\boldsymbol{m}(E)$ is bijective. Let $\boldsymbol{f}: T \to X$ be an integrable function and let $E \in \mathscr{P}$ be such that $\boldsymbol{m}(E) \neq 0$. Then there exists a unique vector $\boldsymbol{x} \in \overline{\mathscr{C}(\boldsymbol{f}(E))}$ with $\int_E \boldsymbol{f}\,\mathrm{d}\boldsymbol{m} = \boldsymbol{m}(E)\,\boldsymbol{x}$.*

Proof. The uniqueness of $\boldsymbol{x}$, if it exists, follows from the axiom of Price.

To prove the existence of the required $\boldsymbol{x}$, suppose first that $\boldsymbol{f}\,.\,\chi_E$ is a uniform limit of a sequence of simple integrable functions. Then, since $\hat{\boldsymbol{m}}(E) < +\infty$, $\boldsymbol{f}\,.\,\chi_E$ is integrable as well as S-integrable on E and $\int_E \boldsymbol{f}\,\mathrm{d}\boldsymbol{m} = \int_E \boldsymbol{f}\,.\,\chi_E\,\mathrm{d}\boldsymbol{m} = S_E \boldsymbol{f}\,\mathrm{d}\boldsymbol{m}$, see Lemma 2. Let $\pi(E) = (E_i)_{i=1}^n$ be a partition of E and choose points $t_i \in E_i$. It follows from the axiom of Price and Lemma 1(b) that

$$\sum_{i=1}^{n} \boldsymbol{m}(E_i)\,\boldsymbol{f}(t_i) \in \boldsymbol{m}(E)\,(\mathscr{C}(\boldsymbol{f}(E)))\,.$$

Hence $S_E \boldsymbol{f}\,\mathrm{d}\boldsymbol{m} \in \overline{\boldsymbol{m}(E)\,(\mathscr{C}(\boldsymbol{f}(E)))} = \boldsymbol{m}(E)\,(\overline{\mathscr{C}(\boldsymbol{f}(E))})$, since $\boldsymbol{x}' \to \boldsymbol{m}(E)\,\boldsymbol{x}'$, $\boldsymbol{x}' \in X$, is a homeomorphism. So there exists a vector $\boldsymbol{x} \in \overline{\mathscr{C}(\boldsymbol{f}(E))}$ such that $\int_E \boldsymbol{f}\,\mathrm{d}\boldsymbol{m} = S_E \boldsymbol{f}\,\mathrm{d}\boldsymbol{m} = \boldsymbol{m}(E)\,\boldsymbol{x}$.

Let now $\boldsymbol{f}: T \to X$ be an integrable function. Take a sequence of simple integrable functions $\boldsymbol{f}_n: T \to X$, $n = 1, 2, .f.$, such that $\boldsymbol{f}_n(t) \to \boldsymbol{f}(t)$ for each $t \in T$. Let $X_1 \subseteq X$ be a closed linear subspace which is separable and contains the ranges of all $\boldsymbol{f}_n$, $n = 1, 2, \ldots$, hence also the range of $\boldsymbol{f}$. Then in our consideration concerning the function $\boldsymbol{f}$ we may suppose that $\boldsymbol{m}: \mathscr{P} \to L(X_1, X)$. Now by Theorem 13-1) in part III (see [15]) there is a countably additive measure $\lambda_E: E \cap \mathscr{P} \to [0, 1]$ such that $N \in E \cap \mathscr{P}$ and $\lambda_E(N) = 0$ implies $\hat{\boldsymbol{m}}_1(E \cap N) = 0$, where $\hat{\boldsymbol{m}}_1$ is the semivariation of $\boldsymbol{m}: \mathscr{P} \to L(X_1, X)$. Owing to Egoroff-Lusin theorem, see section 1.4 in part I ([13]), there is a set $N \in E \cap \mathscr{P}$ and a non decreasing sequence of sets $F_k \in E \cap \mathscr{P}$, $k = 1, 2, \ldots$, such that $\lambda_E(N) = 0$ (hence also $\hat{\boldsymbol{m}}_1(N) = 0$), $F_k \uparrow E - N$ and on each F_k, $k = 1, 2, \ldots$, the sequence $\boldsymbol{f}_n$, $n = 1, 2, \ldots$, converges uniformly to $\boldsymbol{f}$. But then $\int_N \boldsymbol{f}\,\mathrm{d}\boldsymbol{m} = 0$, and for any $\boldsymbol{x}_0 \in X_1$ the function $\boldsymbol{x}_0\,.\,\chi_N$ is integrable with respect to $\boldsymbol{m}: \mathscr{P} \to L(X)$ and $\int_E \boldsymbol{x}_0\,.\,\chi_N\,\mathrm{d}\boldsymbol{m} = 0$.

Take any $\boldsymbol{x}_0 \in \boldsymbol{f}(E)$ and define the sequence $\boldsymbol{f}'_k$, $k = 1, 2, \ldots$, by equalities

$$\boldsymbol{f}'_k = \boldsymbol{f}\,.\,\chi_{F_k} + \boldsymbol{x}_0\,.\,\chi_{E-N-F_k} + \boldsymbol{x}_0\,.\,\chi_N\,.$$

Since

$$\int_E \boldsymbol{f}\,\mathrm{d}\boldsymbol{m} = \int_{E-N} \boldsymbol{f}\,\mathrm{d}\boldsymbol{m} = \int_{F_k} \boldsymbol{f}\,\mathrm{d}\boldsymbol{m} + \int_{E-N-F_k} \boldsymbol{f}\,\mathrm{d}\boldsymbol{m}\,,$$

we have

$$\int_E f\,\mathrm{d}\boldsymbol{m} = \lim_{k\to\infty} \int_{F_k} f\,\mathrm{d}\boldsymbol{m} = \lim_{k\to\infty} \int_E f \cdot \chi_{F_k}\,\mathrm{d}\boldsymbol{m}$$

by the countable additivity of the integral.

Obviously each f_k', $k = 1, 2, \ldots$, is integrable and

$$\int_E f_k'\,\mathrm{d}\boldsymbol{m} = \int_E f \cdot \chi_{F_k}\,\mathrm{d}\boldsymbol{m} + \boldsymbol{m}(E - N - F_k)\,\boldsymbol{x}_0 .$$

But $\boldsymbol{m}(E - N - F_k)\,\boldsymbol{x}_0 \to 0$ as $k \to \infty$ by the countable additivity of $\boldsymbol{m}$ in the strong operator topology. Hence $\lim_{k\to\infty} \int_E f_k'\,\mathrm{d}\boldsymbol{m}$ exists and is equal to $\int_E f\,\mathrm{d}\boldsymbol{m}$.

Since clearly each f_k', $k = 1, 2, \ldots$, is a uniform limit of a sequence of simple integrable functions, by the first part of the proof for each $k = 1, 2, \ldots$, there is an $x_k \in \overline{\mathscr{C}(f_k'(E))} \subseteq \overline{\mathscr{C}(f(E))}$ such that $\int_E f_k'\,\mathrm{d}\boldsymbol{m} = \boldsymbol{m}(E)\,x_k$. Hence $\boldsymbol{m}(E)\,x_k \to \int_E f\,\mathrm{d}\boldsymbol{m}$ as $k \to \infty$. But $\boldsymbol{m}(E) \neq 0$, so by the axiom of Price $\boldsymbol{m}(E)$ is bijective. Thus $\lim_{k\to\infty} x_k = (\boldsymbol{m}(E))^{-1} \int_E f\,\mathrm{d}\boldsymbol{m} = x$. Since $x_k \in \overline{\mathscr{C}(f(E))}$ for each $k = 1, 2, \ldots$ $x \in \overline{\mathscr{C}(f(E))}$. The theorem is proved.

The theorem together with Lemma 3 yield the following result, see [10, p. 48] and [28].

Corollary 1. *Let $\mu: \mathscr{P} \to [0, +\infty)$ be a countably additive measure and let $f: T \to X$ be integrable with respect to μ (= strongly measurable and Pettis integrable). Then for each $E \in \mathscr{P}$ with $\mu(E) > 0$ one has $(\mu(E))^{-1} \int_E f\,\mathrm{d}\mu \in \overline{\mathrm{co}}(f(E))$.*

3. APPLICATIONS TO FRÉCHET DIFFERENTIABILITY

If X and Y are normed spaces (both real or both complex), the symbol $L(X, Y)$ denote now the normed space of all linear continuous operators from X to Y. Let U be a non-empty subset of X and let $f: U \to Y$. The function f is said to be *Fréchet differentiable* at $a \in U$ if there exists $u \in L(X, Y)$ such that

$$\lim_{\substack{x\to 0\\ x\neq 0}} \frac{f(a + x) - f(a) - u(x)}{|x|} = 0 .$$

The linear mapping u is uniquely determined and is denoted by $f'(a)$, which is called the *Fréchet differential* of f at a. It is clear that if f is Fréchet differentiable at a, it is continuous at this point. The function f is said to be *Fréchet differentiable* on U if it is Fréchet differentiable at each point of U. In this case, the function $f': U \to L(X, Y)$ is well-defined, and if it is continuous, the function f is said to be *continuously Fréchet differentiable* on U. For more details concerning the Fréchet differentiability we will refer to [41] and [44].

If x and y are two distinct points of X we define the *segment* $[x, y] = \{x + \lambda(y - x): 0 \leqq \lambda \leqq 1\}$.

Theorem 2. *Let X be a real normed space, let x and y be two distinct points of X, and let U be an open subset of X such that $[x, y] \subseteq U$. If $f: U \to \mathbb{R}^n$ is continuously Fréchet differentiable on U, then there exist a finite sequence $(a_i)_{i=1}^n$ in $[x, y]$ and a finite sequence $(\lambda_i)_{i=1}^n$ in $\mathbb{R}$ such that $\lambda_i \geqq 0$, $\sum_{i=1}^n \lambda_i = 1$ and $f(y) - f(x) = \sum_{i=1}^n \lambda_i f'(a_i)(y - x)$.*

Proof. Let $h(t) = f(x + t(y - x))$ for $0 \leqq t \leqq 1$. Then h is continuously Fréchet differentiable on $[0, 1]$, and therefore h' is Bochner integrable in $[0, 1]$ with respect to Lebesgue measure and, moreover, $h(1) - h(0) = \int_0^1 h'(t)\, dt$.

By Corollary 1, $\int_0^1 h'(t)\, dt \in \overline{co}(h'[0, 1])$. Since $h'[0, 1]$ is a compact subset of $\mathbb{R}^n$, $co(h'[0, 1])$ is compact [9, p. 115], therefore closed, so that $\int_0^1 h'(t)\, dt \in co(h'[0, 1])$. Since $h'[0, 1]$ is connected, the Fenchel-Bunt theorem [25, p. 36] gives for the integral the expression: $\int_0^1 h'(t)\, dt = \sum_{i=1}^n \lambda_i h'(t_i)$, where $t_i \in [0, 1]$, $\lambda_i \geqq 0$ and $\sum_{i=1}^n \lambda_i = 1$.

But the chain rule for Fréchet differentiable functions implies that $h'(t) = f'(x + t(y - x))\,(y - x)$. Write $a_i = x + t_i(y - x)$ for $i = 1, 2, \dots, n$. Then $a_i \in [x, y]$ and $f(y) - f(x) = \sum_{i=1}^n \lambda_i f'(a_i)(y - x)$.

Corollary 2. *Let X be a complex normed space, let x and y be two distinct points of X, and let U be an open subset of X such that $[x, y] \subseteq U$. If $f: U \to \mathbb{C}^n$ is continuously Fréchet differentiable, then there exist a finite sequence $(a_i)_{i=1}^{2n}$ in $[x, y]$ and a finite sequence $(\lambda_i)_{i=1}^{2n}$ in $\mathbb{R}$ such that $\lambda_i \geqq 0$, $\sum_{i=1}^{2n} \lambda_i = 1$ and $f(y) - f(x) = \sum_{i=1}^{2n} \lambda_i f'(a_i)(y - x)$.*

Proof. It suffices to replace X by the underlying real normed space and $\mathbb{C}^n$ by $\mathbb{R}^{2n}$ [11, p. 145].

Let Y be a real vector space and let $|\cdot|_1$ and $|\cdot|_2$ two norms on Y which induce respectively the topologies τ_1 and τ_2 on Y. If $\eta > 0$ and $S \subseteq Y$ the symbols $\overline{B}_{\tau_1}(0, \eta)$ and $\overline{co}_{\tau_2}(S)$ denote respectively the closed ball of center 0 and radius η in the normed space $(Y, |\cdot|_1)$ and the closed convex hull of S in the normed space $(Y, |\cdot|_2)$. The topologies τ_1 and τ_2 are said to be (P)-*related* if τ_1 is finer than τ_2 and, for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if K is a τ_1-compact subset of $\overline{B}_{\tau_1}(0, \delta)$, then $\overline{co}_{\tau_2}(K) \subseteq \overline{B}_{\tau_1}(0, \varepsilon)$. For example, if τ_1 is finer than τ_2 and $(Y, |\cdot|_1)$ is a Banach space, then the Mazur theorem [19, p. 416] implies that τ_1 and τ_2 are (P)-related. The property (P) was introduced by L. Schwartz [39] (see also [44]) in a more general setting.

Theorem 3. *Let $(X, |\cdot|)$ be a real normed space, let Y be a real vector space and* *let $|\cdot|_1$ and $|\cdot|_2$ be two norms on Y such that $(Y, |\cdot|_2)$ is a Banach space and the respective induced topologies τ_1 and τ_2 are (P)-related. Let $f: X \to Y$ and suppose that $f: X \to (Y, |\cdot|_2)$ is Fréchet differentiable on X, $f'(a) \in L(X, (Y, |\cdot|_1))$ for every $a \in X$ and $f': X \to L(X, (Y, |\cdot|_1))$ is continuous on X. Then $f: X \to (Y, |\cdot|_1)$ is Fréchet differentiable on X and its Fréchet differential at every $a \in X$ is $f'(a)$.*

Proof. Let $a \in X$. It suffices to show that

$$\lim_{\substack{t \to 0 \\ t \in R - \{0\}}} \sup_{|x| \leqq 1} \left| \frac{f(a + tx) - f(a)}{t} - f'(a)\,x \right|_1 = 0 .$$

Let $\varepsilon > 0$. Let $\delta = \delta(\varepsilon)$ be a positive real number determined from the(P)-property. For every $x \in X$ such that $|x| \leqq 1$, we define the mapping h_x: $\mathbb{R} \to Y$ by the formula $h_x(t) = f(a + tx)$. Since τ_1 is finer than τ_2, the function $f': X \to L(X, (Y, |\cdot|_2))$ is continuous, and therefore h_x: $\mathbb{R} \to (Y, |\cdot|_2)$ is continuously Fréchet differentiable on $\mathbb{R}$ and $h'_x(t) = f'(a + tx)\,x$.

By the continuity of the function $f': X \to L(X, (Y, |\cdot|_1))$ at a, there exists a positive real number $\eta = \eta(\delta, a) = \eta(\varepsilon, a)$ such that $u \in X$ and $|u - a| < \eta$ implies

$$\sup_{\substack{y \in X \\ y \neq 0}} \frac{|(f'(u) - f'(a))\,y|_1}{|y|} < \delta .$$

In particular, $u \in X$ and $|u - a| < \eta$ implies $|f'(u)\,y - f'(a)\,y|_1 < \delta |y|$ for all $y \in X$.

But, for $t \neq 0$,

$$\frac{f(a + tx) - f(a)}{t} - f'(a)\,x = \frac{1}{t}\{h_x(t) - h_x(0)\} - h'_x(0) = \frac{1}{t}\int_0^t \{h'_x(s) - h'_x(0)\}\,\mathrm{d}s ,$$

because the function $s \to h'_x(s)$ is $|\cdot|_2$ – Bochner integrable on every compact interval $I \subseteq \mathbb{R}$ of extremities 0 and t with respect to Lebesgue measure. By Corollary 1,

$$\frac{1}{t}\int_0^t \{h'_x(s) - h'_x(0)\}\,\mathrm{d}s \in \overline{\mathrm{co}}_{\tau_2}((h'_x - h'_x(0))\,(I)) .$$

Let $K_x = (h'_x - h'_x(0))\,(I)$. It is clear that K_x is τ_1 – compact. We will show that, if $0 < |t| < \eta$, then $K_x \subseteq \overline{\mathrm{B}}_{\tau_1}(0, \delta)$. In fact, let $z \in K_x$. Then $z = h'_x(s) - h'_x(0)$ for some $s \in I$. Since $|a + sx - a| = |s|\,|x| \leqq |s| < \eta$, $|z|_1 = |f'(a + sx)\,x - f'(a)\,x|_1 < < \delta|x| \leqq \delta$, and therefore $z \in \overline{\mathrm{B}}_{\tau_1}(0, \delta)$. Since τ_1 and τ_2 are (P)-related, $\overline{\mathrm{co}}_{\tau_2}(K_x) \subseteq \overline{\mathrm{B}}_{\tau_1}(0, \varepsilon)$. Then $0 < |t| < \eta$ implies

$$\left| \frac{f(a + tx) - f(a)}{t} - f'(a)\,x \right|_1 < \varepsilon$$

for all $|x| \leqq 1$. This completes the proof.

Remarks. 1. The Theorem 2 generalizes the classical mean value theorem for continuously Fréchet differentiable functions from R^m to R.

2. The Corollary 2 generalizes the Theorem 10 of McLeod [33, p. 208].

3. The Theorem 3 generalizes, in the setting of normed spaces, a result of L. Schwartz [39].

4. It is possible to extend the Theorem 3 to n-times Fréchet differentiable functions, in order to generalize, in the setting of normed spaces, a more recent result of L. Schwartz [40].

4. INTEGRABILITY AND S-INTEGRABILITY

If $\mu: \mathscr{P} \to [0, +\infty)$ is a countably additive measure and the $\mathscr{P}$-measurable function $f: T \to [0, +\infty)$ is not μ-essentially bounded on a set $E \in \mathscr{P}$ with $\mu(E) > 0$, then clearly f is not S-integrable on E (of course f may be integrable on E). As the following simple example shows, in the vector case, the analogue is in general false.

Example 1. Let $T = \{1, 2, \ldots\}$, $\mathscr{P} = 2^T$, $X =$ the real l_2, $Y = R = (-\infty, +\infty)$. For $\boldsymbol{x} = (x_1, \ldots, x_k, \ldots) \in l_2$ put $\boldsymbol{m}(\{k\})\, \boldsymbol{x} = 2^{-k} x_{2k-1}$ and $\boldsymbol{m}(E)\, \boldsymbol{x} = \sum_{k \in E} \boldsymbol{m}(\{k\})\, \boldsymbol{x}$ for $E \in \mathscr{P}$. Then clearly $\boldsymbol{m}: 2^T \to L(l_2, R) = l_2$ is a countably additive vector measure with $v(\boldsymbol{m}, T) = 1$. Now evidently the function $\boldsymbol{f}: T \to l_2$ defined by equalities

$$
\begin{aligned}
\boldsymbol{f}(1) &= (0, 1, 0, 0, \ \ldots\ldots\ldots\ldots\ldots) \\
\boldsymbol{f}(2) &= (0, 1, 0, 1, 0, 0, \ \ldots\ldots\ldots\ldots) \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\boldsymbol{f}(k) &= (\underbrace{0, 1, 0, 1, \ldots 0, 1}_{2k}, 0, 0, \ldots)
\end{aligned}
$$

is unbounded on each infinite subset of the set of even integers. Nevertheless it is integrable and S-integrable on each set $E \in \mathscr{P}$ with both integrals equal to 0.

Theorem 4. *Let $\boldsymbol{f}: T \to X$ be an $\boldsymbol{m}$-essentially bounded measurable function and let the semivariation $\hat{\boldsymbol{m}}: \mathscr{P} \to [0, +\infty)$ be continuous. Then $\boldsymbol{f}$ is S-integrable on each set $E \in \mathscr{P}$ and $S_E \boldsymbol{f}\, \mathrm{d}\boldsymbol{m} = \int_E \boldsymbol{f}\, \mathrm{d}\boldsymbol{m}$ for each set $E \in \mathscr{P}$. ($\boldsymbol{f} \cdot \chi_E$ is integrable by Theorem 5 from part I).*

Proof. Without loss of generality we may suppose that $\boldsymbol{f}$ is bounded. Let $E \in \mathscr{P}$. Take a sequence $\boldsymbol{f}_n: T \to X$, $n = 1, 2, \ldots$, of simple integrable functions such that $\boldsymbol{f}_n(t) \to \boldsymbol{f}(t)$ for each $t \in T$. Since the set function $A \to \hat{\boldsymbol{m}}(A \cap E)$, $A \in \mathscr{P}$ is monotone, subadditive and continuous, we may apply the Egoroff-Lusin theorem, see section 1.4 in part I and Remark 3 in part IV, to it. Hence there are N, $F_k \in \mathscr{P} \cap E$, $k = 1, 2, \ldots$, such that $\hat{\boldsymbol{m}}(N) = 0$, $F_k \uparrow E - N$ and on each F_k, $k = 1, 2, \ldots$, the sequence $\boldsymbol{f}_n$, $n = 1, 2, \ldots$, converges uniformly to $\boldsymbol{f}$. Since $\boldsymbol{f}$ is bounded and $\hat{\boldsymbol{m}}$ is continuous on $\mathscr{P}$ we may suppose that $\|\boldsymbol{f}\|_E \cdot \hat{\boldsymbol{m}}(E - F_k) < 1/4k$ for each $k = 1, 2, \ldots$. Take a subsequence $\boldsymbol{f}_{n_k}$, $k = 1, 2, \ldots$, of the sequence $\boldsymbol{f}_n$, $n = 1, 2, \ldots$, such that $\|\boldsymbol{f} - \boldsymbol{f}_{n_k}\|_{F_k} \cdot \hat{\boldsymbol{m}}(E) < 1/4k$ for each $k = 1, 2, \ldots$. Put $\boldsymbol{y} = \int_E \boldsymbol{f}\, \mathrm{d}\boldsymbol{m}$. Then by the above inequalities we have

$$\left| y - \int_{F_k} f_{n_k} \, \mathrm{d}\boldsymbol{m} \right| < \frac{1}{2k}.$$

Since f_{n_k} is a $\mathscr{P}$-simple function, it is of the form $f_{n_k} = \sum_{i=1}^{i_k} x_{n_k,i} \cdot \chi_{E_{n_k,i}}$ with pairwise disjoint $E_{n_k,i} \in \mathscr{P}$. Put $E_{n_k,0} = F_k - \bigcup_{i=1}^{i_k} E_{n_k,i}$ if the right hand side is not euqal to $\emptyset$, and let $\pi_k(E) = \{(E_{n_k,i} \cap F_k)_{i=0}^{i_k}, E - F_k\}$. Then $\pi_k(E)$ is a partition of E, and for any partition $\pi(E) \geqq \pi_k(E)$, $\pi(E) = (A_j)_{j=1}^n$, and any points $t_j \in A_j$, $j = 1, 2, \ldots, n$, we have

$$|y - S_{\pi(E)}(\boldsymbol{m}, f, t_1, t_2, \ldots, t_n)| = |y - \sum_{j=1}^{n} \boldsymbol{m}(A_j) f(t_j)| \leqq$$

$$\leqq \left| y - \int_{F_k} f_{n_k} \, \mathrm{d}\boldsymbol{m} \right| + \left| \int_{F_k} f_{n_k} \, \mathrm{d}\boldsymbol{m} - \sum_{j=1}^{n} \boldsymbol{m}(A_j) f(t_j) \right| < \frac{1}{2k} +$$

$$+ \left| \int_{F_k} f_{n_k} \, \mathrm{d}\boldsymbol{m} - \sum_{\substack{j \\ A_j \subseteq F_k}} \boldsymbol{m}(A_j) f(t_j) \right| + \left| \sum_{\substack{j \\ A_j \subseteq E - F_k}} \boldsymbol{m}(A_j) f(t_j) \right| <$$

$$< \frac{1}{2k} + \|f - f_{n_k}\|_{F_k} \cdot \hat{\boldsymbol{m}}(E) + \|f\|_E \cdot \hat{\boldsymbol{m}}(E - F_k) < \frac{1}{k}.$$

Hence f is S-integrable on E and $S_E f \, \mathrm{d}\boldsymbol{m} = \int_E f \, \mathrm{d}\boldsymbol{m}$. The theorem is proved.

From here and from the $*$-Theorem from section 1.1 in part I, we immediately have the next

Corollary 3. *Let $c_0 \nsubseteq Y$, for example let Y be a weakly sequentially complete Banach space, and let $f\colon T \to X$ be an $\boldsymbol{m}$-essentially bounded measurable function. Then f is S-integrable on each set $E \in \mathscr{P}$ and $S_E f \, \mathrm{d}\boldsymbol{m} = \int_E f \, \mathrm{d}\boldsymbol{m}$ for each $E \in \mathscr{P}$.*

We now construct a measure on the σ-algebra of all subsets of $T = \{1, 2, \ldots\}$ and a bounded integrable function f which is not S-integrable on T. Hence Lemma 3.4 in [34] do not hold in general.

Example 2. Let $T = \{1, 2, \ldots\}$, $\mathscr{P} = 2^T$, and $X = Y =$ the real c_0. Define the measure $\boldsymbol{m}\colon \mathscr{P} \to L(c_0, c_0)$ in the following way: For $x = (x_1, \ldots, x_k, \ldots) \in c_0$ put

$$\boldsymbol{m}(\{1\})\, x = (0, x_2, 0, 0, \ldots\ldots\ldots)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\boldsymbol{m}(\{k\})\, x = (\underbrace{0, \ldots, 0, x_{k+1}}_{k+1}, 0, 0, .)$$

and $\boldsymbol{m}(E)\, x = \sum_{k \in E} \boldsymbol{m}(\{k\})\, x$ for $E \in \mathscr{P}$. Clearly $\boldsymbol{m}\colon \mathscr{P} \to L(c_0, c_0)$ is countably additive in the strong operator topology and $|\boldsymbol{m}(E)| = \hat{\boldsymbol{m}}(E) = 1$ for each non empty $E \in \mathscr{P}$.

Define now the function f: $T \to c_0$ as follows

$$f(1) = (1, 0, 0, \ldots\ldots\ldots)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$f(k) = (\underbrace{1, \ldots, 1}_{k}, 0, 0, \ldots)$$

Evidently f is a bounded integrable function with $\int_E f \, \mathrm{d}\boldsymbol{m} = 0$ for each $E \in \mathscr{P}$.

To show that f is not S-integrable on T let $\pi(T) = \{(F_i)_{i=1}^{i_0}, (G_j)_{j=1}^{j_0}\}$ be any partition of T with finite F_i, $i = 1, \ldots, i_0$, and infinite G_j, $j = 1, \ldots, j_0$. Take any points $s_i \in F_i$, $i = 1, \ldots, i_0$, and $t_j \in G_j$, $j = 1, \ldots, j_0$, and put

$$r = \max \{k: k \in \bigcup_{i=1}^{i_0} F_i \cup \{t_1, \ldots, t_{j_0}\}\} .$$

From the definitions of $\boldsymbol{m}$ and f is clear that

$$\boldsymbol{u} = S_{\pi(T)}(\boldsymbol{m}, f, s_1, \ldots, s_{i_0}, t_1, \ldots, t_{j_0}) = (y_1, \ldots, y_r, 0, 0, \ldots) \in c_0$$

with some real $y_1, \ldots, y_r$. Since $\pi(T)$ is a partition of T, there is a j_1, $1 \leqq j_1 \leqq j_0$ such that $r + 1 \in G_{j_1}$. Replace t_{j_1} by $r + 1 = t'_{j_1}$ and put $t'_j = t_j$ for $j \neq j_1$, $1 \leqq j \leqq j_0 \ldots$. Then

$$\boldsymbol{v} = S_{\pi(T)}(\boldsymbol{m}, f, s_1, \ldots, s_{i_0}, t'_1, \ldots, t'_{j_0}) = (y'_1, \ldots, y'_r, 1, 0, 0, \ldots) \in c_0$$

with some real $y'_1, \ldots, y'_r$. Hence $|\boldsymbol{u} - \boldsymbol{v}| \geqq 1$. Since $\pi(T)$ was an arbitrary partition of T, f is not S-integrable on T.

According to the definition, see Def. 4 part II, a measurable function f: $T \to X$ belongs to $\mathscr{L}_1(\boldsymbol{m})$ if its L_1-pseudonorm $\hat{\boldsymbol{m}}(f, \cdot)$ is continuous on $\sigma(\mathscr{P})$. Each $f \in \mathscr{L}_1(\boldsymbol{m})$ is integrable, see Lemma 1 in part II.

Theorem 5. *Let $f \in \mathscr{L}_1(\boldsymbol{m})$ be $\boldsymbol{m}$-essentially bounded. Then f is S-integrable on each set $E \in \mathscr{P}$, and $S_E f \, \mathrm{d}\boldsymbol{m} = \int_E f \, \mathrm{d}\boldsymbol{m}$.*

Proof. Clearly it is enough to prove the theorem for bounded elements of $\mathscr{L}_1(\boldsymbol{m})$. Let $f \in \mathscr{L}_1(\boldsymbol{m})$ be bounded and let $E \in \mathscr{P}$. For each $a > 0$ put $E_a = \{t: t \in E, |f(t)| \geqq a\}$. Then by Tschebyscheff inequality, see Corollary of Theorem 1 in part II, $\hat{\boldsymbol{m}}(E_a) \leqq a^{-1} . \hat{\boldsymbol{m}}(f, E)$ for each $a > 0$. Since $\hat{\boldsymbol{m}}(f, \cdot)$ is continuous on $\sigma(\mathscr{P})$, $\hat{\boldsymbol{m}}$ is continuous on each E_a, $a > 0$, the function f is S-integrable (see Theorem 4) and $S_{E_a} f \, \mathrm{d}\boldsymbol{m} = \int_{E_a} f \, \mathrm{d}\boldsymbol{m}$. If $a_n \downarrow 0$, then

$$\lim_{n \to \infty} S_{E_{a_n}} f \, \mathrm{d}\boldsymbol{m} = \lim_{n \to \infty} \int_{E_{a_n}} f \, \mathrm{d}\boldsymbol{m} = \int_E f \, \mathrm{d}\boldsymbol{m}$$

by the countable additivity of the integral. Let $\varepsilon > 0$. Take $a_\varepsilon > 0$ so that $a_\varepsilon . \hat{\boldsymbol{m}}(E) < \frac{1}{3}\varepsilon$ and $|\int_E f \, \mathrm{d}\boldsymbol{m} - S_{E_{a_\varepsilon}} f \, \mathrm{d}\boldsymbol{m}| < \frac{1}{3}\varepsilon$. Take further a partition $\pi_\varepsilon(E_{a_\varepsilon})$ such that

$$|S_{\pi(E_{a_\varepsilon})}(\boldsymbol{m}, f, t_1, \ldots, t_n) - S_{E_a} f \, \mathrm{d}\boldsymbol{m}| < \frac{\varepsilon}{3}$$

for any partition $(E_i)_{i=1}^n = \pi(E_{a_\varepsilon}) \geqq \pi_\varepsilon(E_{a_\varepsilon})$ and any points $t_i \in E_i$, $i = 1, \ldots, n$. Put $\pi_\varepsilon(E) = \{\pi_\varepsilon(E_{a_\varepsilon}), E - E_{a_\varepsilon}\}$. Then $\pi_\varepsilon(E)$ is a partition of E and obviously for any partition $(F_j)_{j=1}^k = \pi(E) \geqq \pi_\varepsilon(E)$ and any points $t_j \in F_j$, $j = 1, \ldots, k$,

$$\left| S_{\pi(E)}(\boldsymbol{m}, \boldsymbol{f}, t_1, \ldots, t_k) - \int_E \boldsymbol{f} \, \mathrm{d}\boldsymbol{m} \right| < \varepsilon .$$

Hence $\boldsymbol{f}$ is S-integrable on E and $S_E \boldsymbol{f} \, \mathrm{d}\boldsymbol{m} = \int_E \boldsymbol{f} \, \mathrm{d}\boldsymbol{m}$. The theorem is proved.

Remark. In fact we proved that the assertions of the theorem are valid if the condition $\boldsymbol{f} \in \mathscr{L}_1(\boldsymbol{m})$ is replaced by the requirement $\{t: |\boldsymbol{f}(t)| \geqq a\} \in \tilde{\mathscr{P}}$ for each $a > 0$, where $\tilde{\mathscr{P}}$ denotes the δ-ring of all sets from $\mathscr{P}$ on which the semivariation $\hat{\boldsymbol{m}}$ is continuous.

Theorem 6. *Let $\boldsymbol{f}$: $T \to \boldsymbol{X}$ be measurable, let $E \in \mathscr{P}$, and let $\boldsymbol{f}$ be S-integrable on E. Then the function $\boldsymbol{f} . \chi_E$ is integrable, and $\int_A \boldsymbol{f} \, \mathrm{d}\boldsymbol{m} = S_A \boldsymbol{f} \, \mathrm{d}\boldsymbol{m}$ for each $A \in E \cap \mathscr{P}$.*

Proof. Let $\boldsymbol{f}_n'$: $T \to \boldsymbol{X}$, $n = 1, 2, \ldots$ be a sequence of simple integrable functions such that $\boldsymbol{f}_n'(t) \to \boldsymbol{f}(t)$ for each $t \in T$. Take a closed separable linear subspace $\boldsymbol{X}_1 \subseteq \boldsymbol{X}$ which contains the ranges of all $\boldsymbol{f}_n'$, $n = 1, 2, \ldots$, hence also of $\boldsymbol{f}$. Then in our consideration we may suppose that $\boldsymbol{m}$: $\mathscr{P} \to L(\boldsymbol{X}_1, \boldsymbol{Y})$. According to Theorem 13-1) in part III, there is a countably additive measure λ_E: $E \cap \mathscr{P} \to [0, 1]$ such that $N' \in E \cap \mathscr{P}$ and $\lambda_E(N') = 0 \Rightarrow \hat{\boldsymbol{m}}_1(N') = 0$, where $\hat{\boldsymbol{m}}_1$ is the semivariation of $\boldsymbol{m}$: $\mathscr{P} \to L(\boldsymbol{X}_1, \boldsymbol{Y})$. Now by Egoroff-Lusin theorem, see section 1.4 in part I, there is a set $N \in E \cap \mathscr{P}$ and a non decreasing sequence of sets $F_k \in E \cap \mathscr{P}$, $k = 1, 2, \ldots$, such that $\hat{\boldsymbol{m}}_1(N) = 0$, $F_k \uparrow E - N$ and on each F_k, $k = 1, 2, \ldots$, the sequence $\boldsymbol{f}_n'$, $n = 1, 2, \ldots$, converges uniformly to $\boldsymbol{f}$. Hence there is a subsequence $\boldsymbol{f}_{n_k}'$, $k = 1, 2, \ldots$, such that $\|\boldsymbol{f} - \boldsymbol{f}_{n_k}'\|_{F_k} . \hat{\boldsymbol{m}}(E) \leqq 1/k$ for each $k = 1, 2, \ldots$. Take a sequence of partitions $\pi_k(E) = (E_{k,i})_{i=1}^{i_k}$, $k = 1, 2, \ldots$, of E such that for any partitions $(E_i)_{i=1}^n = \pi(E) \geqq \pi_k(E)$ and any points $t_i \in E_i$, $i = 1, 2, \ldots, n$,

$$|S_E \boldsymbol{f} \, \mathrm{d}\boldsymbol{m} - S_{\pi(E)}(\boldsymbol{m}, \boldsymbol{f}, t_1, \ldots, t_n)| < \frac{1}{k} .$$

For each $k = 1, 2, \ldots$ put

$$\pi_k'(E) = \{(F_k \cap E_{k,i})_{i=1}^{i_k} , \ ((E - F_k) \cap E_{k,i})_{i=1}^{i_k}\} .$$

Then $\pi_k'(E) \geqq \pi_k(E)$ for each $k = 1, 2, \ldots$. Let us choose points $t_{k,i} \in E_{k,i}$, $k = 1, 2, \ldots$, $i = 1, \ldots, i_k$, and define the simple integrable functions $\boldsymbol{f}_k$, $k = 1, 2, \ldots$, as follows

$$\boldsymbol{f}_k(t) = \boldsymbol{f}_{n_k}'(t) \quad \text{if} \quad t \in F_k \cup (T - E)$$

and

$$\boldsymbol{f}_k(t) = \boldsymbol{f}(t_{k,i}) \quad \text{if} \quad t \in (E - F_k) \cap E_{k,i} .$$

Then clearly $\boldsymbol{f}_k(t) \to \boldsymbol{f}(t)$ for each $t \in T - N$. Since $\boldsymbol{f}_{n_k}'$, $k = 1, 2, \ldots$, are $\mathscr{P}$-simple functions, they are of the form $\boldsymbol{f}_{n_k}' = \sum_{j=1}^{j_k} \boldsymbol{x}_{n_k,j} \chi_{A_{n_k,j}}$ with pairwise disjoint $A_{n_k,j} \in \mathscr{P}$,

$j = 1, 2, \dots, j_k$ for each $k = 1, 2, \dots$. For each $k = 1, 2, \dots$ put $A_{n_k,0} = F_k - \bigcup_{j=1}^{j_k} A_{n_k,j}$ if the right hand side is not equal to $\emptyset$. Then for each $k = 1, 2, \dots$ obviously $\pi_k(F_k) = \{F_k \cap E_{k,i} \cap A_{n_k,j},\ i = 1, \dots, i_k,\ j = 0, 1, \dots, j_k\}$ is a partition of F_k. Take any points $t_{k,i,j} \in F_k \cap E_{k,i} \cap A_{n_k,j}$ and put

$$g_k = \sum_{i=1}^{i_k} \sum_{j=0}^{j_k} f(t_{k,i,j}) \cdot \chi_{F_k \cap E_{k,i} \cap A_{n_k,j}}, \quad k = 1, 2, 3, \dots .$$

Then g_k, $k = 1, 2, \dots$, are simple integrable functions,

$$\hat{m}(E) \cdot \|g_k - f'_{n_k}\|_{F_k} < \frac{1}{k},$$

and

$$S_{\pi_k(F_k)}(m, f, t_{k,1,0}, t_{k,2,1}, \dots, t_{k,i_k,j_k}) = \int_{F_k} g_k \, dm .$$

If we now take the partition

$$\pi''_k(E) = \{\pi_k(F_k), ((E - F_k) \cap E_{k,i})_{i=1}^{i_k}\} \geqq \pi_k(E)$$

and the above choosen points $t_{k,i}$ and $t_{k,i,j}$, then

$$\left|S_E f \, dm - \int_E f_k \, dm\right| \leqq \left|S_E f \, dm - S_{\pi_k''(E)}(m, f, t_{k,1,0}, t_{k,2,1}, \dots, t_{k,i_k,j_k}, t_{k,1}, \dots, t_{k,i_k})\right| +$$

$$+ \left|S_{\pi_k''(E)}(m, f, t_{k,1,0}, t_{k,2,1}, \dots, t_{k,i_k,j_k}, t_{k,1}, \dots, t_{k,i_k}) - \int_E f_k \, dm\right| < \frac{1}{k} +$$

$$+ \left|\int_{F_k} g_k \, dm - \int_{F_k} f'_{n_k} \, dm\right| < \frac{1}{k} + \|g_k - f'_{n_k}\|_{F_k} \cdot \hat{m}(E) < \frac{2}{k} \to 0 \quad \text{as} \quad k \to \infty .$$

Let now $A \in E \cap \mathscr{P}$. Then replacing in the considerations above E by A and F_k by $F_k \cap A$ we again obtain that $|S_A f \, dm - \int_A f_k \, dm| \to 0$ as $k \to \infty$. Thus by Theorem 7 in part I the function $f \cdot \chi_E$ is integrable and $\int_A f \, dm = S_A f \, dm$ for each set $A \in E \cap \mathscr{P}$. The theorem is proved.

Remark. If the Banach space X is separable, then according to Theorem 13-1) from part III, which was used in the proofs of Theorems 1 and 6 for each $E \in \sigma(\mathscr{P})$ there is a countably additive measure $\lambda_E \colon \sigma(\mathscr{P}) \cap E \to [0, 1]$ such that $N \in \sigma(\mathscr{P}) \cap E$ and $\lambda_E(N) = 0 \Rightarrow \hat{m}(N) = 0$. The following simple example shows that for non separable X such a measure not always exists. Let Γ be an uncountable set with discrete topology, let $\mathscr{P} = 2^\Gamma$ and define the measure $m \colon \mathscr{P} \to L(l_1(\Gamma), c_0(\Gamma))$ by the equality $(m(E)\, x)(\gamma) = x(\gamma) \cdot \chi_E(\gamma)$, $\gamma \in \Gamma$. Then $|m(E)| = 1$ for each $E \in \mathscr{P}$, $E \neq \emptyset$, particularly $|m(\{\gamma\})| = 1$ for each $\gamma \in \Gamma$. Since Γ is uncountable, the required countably additive measure $\lambda \colon \mathscr{P} \to [0, 1]$ cannot exists.

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