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LOSIK COHOMOLOGY OF THE LIE ALGEBRA OF INFINITESIMAL AUTOMORPHISMS OF A G-STRUCTURE

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INTRODUCTION

In this paper, we shall work exclusively with objects of class C^∞. All manifolds are supposed to be paracompact and Hausdorff, and the terms Lie group and Lie subgroup are used in the sense of [9, chap. I, § 4]. When working with sheaves we keep the terminology and notation of [3] preferring, however, to define a sheaf as a presheaf satisfying certain axioms [3, pp. 5–6].

Let M be a manifold with $\dim M = m$, let $\mathscr{S}$ be the sheaf of all real functions on M, and let $\mathscr{L}$ be a topological Lie algebra sheaf on M, i.e. a topological Lie algebra subsheaf of the topological Lie algebra sheaf $\mathscr{X}$ of all vector fields on M. (The topology on each Lie algebra $\mathscr{L}(U)$, where U is an open subset of M, ist thus the C^∞-topology.) Considering $\mathscr{L}$ as a topological vector space sheaf, let us denote by $\Pi^p\mathscr{L}$ the direct product of p copies of $\mathscr{L}$ and define $C^p(\mathscr{L};\mathscr{S})$ to be the vector space of all alternating multilinear maps $\alpha\colon \Pi^p\mathscr{L}\to\mathscr{S}$ of topological vector space sheafs. As usual, the formula

$$(\mathrm{d}\alpha)(X_0,\ldots,X_p) = \sum_{i=0}^{p}(-1)^i X_i\alpha(X_0,\ldots,\hat{X}_i,\ldots,X_p) + \\ + \sum_{i<j}(-1)^{i+j}\alpha([X_i,X_j], X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_p)\,,$$

where $\alpha\in C^p(\mathscr{L};\mathscr{S})$, $X_0,\ldots,X_p\in\mathscr{L}(U)$, U an open subset of M, defines a differential

$$\mathrm{d}\colon C^p(\mathscr{L};\mathscr{S})\to C^{p+1}(\mathscr{L};\mathscr{S})\,,$$

and the formula

$$(\alpha\wedge\beta)(X_1,\ldots,X_{p+q}) = \\ = \frac{1}{p!\,q!}\sum_\pi \operatorname{sgn}\pi\,.\,\alpha(X_{\pi(1)},\ldots,X_{\pi(p)})\,.\,\beta(X_{\pi(p+1)},\ldots,X_{\pi(p+q)})\,,$$

where $\alpha\in C^p(\mathscr{L};\mathscr{S})$, $\beta\in C^q(\mathscr{L};\mathscr{S})$, $X_1,\ldots,X_{p+q}\in\mathscr{L}(U)$, U an open subset of M, and π runs over all permutations of the set $\{1,2,\ldots,p+q\}$, defines a multiplication

$$\wedge\colon C^p(\mathscr{L};\mathscr{S})\times C^q(\mathscr{L};\mathscr{S})\to C^{p+q}(\mathscr{L};\mathscr{S})\,.$$

If we set

$$C(\mathscr{L}; \mathscr{S}) = \bigoplus_{p=0}^{\infty} C^p(\mathscr{L}; \mathscr{S}),$$

then together with the differential d and the multiplication $\wedge$ the graded vector space $C(\mathscr{L}; \mathscr{S})$ becomes a commutative (in the graded sense) associative differential graded algebra (over $\boldsymbol{R}$) with a unit.

Let $C(\mathscr{L}(M), \mathscr{S}(M))$ be the differential graded algebra of continuous cochains of the topological Lie algebra $\mathscr{L}(M)$ with coefficients in the topological $\mathscr{L}(M)$-module $\mathscr{S}(M)$. There is a canonical homomorphism

$$C(\mathscr{L}; \mathscr{S}) \to C(\mathscr{L}(M), \mathscr{S}(M))$$

of differential graded algebras, and it is easy to see that the image of this homomorphism is contained in the diagonal differential graded subalgebra $C_\Delta(\mathscr{L}(M), \mathscr{S}(M))$ consisting in degree p of all cochains α satisfying the support condition

$$\operatorname{supp} \alpha(X_1, \ldots, X_p) \subset \bigcap_{i=1}^{p} \operatorname{supp} X_i \quad \text{for all} \quad X_1, \ldots, X_p \in \mathscr{L}(M).$$

In the special case when $\mathscr{L} = \mathscr{X}$ one can easily verify that this homomorphism is an isomorphism onto $C_\Delta(\mathscr{L}(M), \mathscr{S}(M))$. In this sense the differential graded algebra $C(\mathscr{L}; \mathscr{S})$ can be considered a natural generalization of the diagonal differential graded algebra $C_\Delta(\mathscr{X}(M), \mathscr{S}(M))$ to the case when there are not sufficiently many globally defined sections of $\mathscr{L}$. Of course, the term "sufficiently many" can be given many various meanings. Probably the most natural one is that the sheaf $\mathscr{L}$ is fine in the sense that the sheaf $\operatorname{Hom}_{\boldsymbol{R}}(\mathscr{L}; \mathscr{L})$ of germs of endomorphisms of the topological vector space sheaf $\mathscr{L}$ is soft. For such a sheaf $\mathscr{L}$ one can easily verify that $C(\mathscr{L}; \mathscr{S}) \approx C_\Delta(\mathscr{L}(M), \mathscr{S}(M))$.

Under a certain hypothesis on $\mathscr{L}$ there is a differential graded subalgebra of $C(\mathscr{L}; \mathscr{S})$ that can be considered a generalization of the diagonal algebra $C_\Delta(\mathscr{X}(M), \mathscr{S}(M))$, too. Let $r \geqq 0$ be an integer. An element $\alpha \in C^p(\mathscr{L}; \mathscr{S})$ will be called a p-cochain of order $\leqq r$ if it satisfies the following condition: For any open subset $U \subset M$ and aby $X_1, \ldots, X_p \in \mathscr{L}(U)$, the value of the function $\alpha(X_1, \ldots, X_p)$ at a point $x \in U$ depends only on the r-jets of the vector fields $X_1, \ldots, X_p$ at the point x. Let $C_{(r)}(\mathscr{L}; \mathscr{S})$ be the graded vector subspace of $C(\mathscr{L}; \mathscr{S})$ generated by all cochains of order $\leqq r$. Clearly, $C_{(r)}(\mathscr{L}; \mathscr{S})$ is a graded subalgebra of $C(\mathscr{L}; \mathscr{S})$, but in general it need not be invariant under the differential of $C(\mathscr{L}; \mathscr{S})$. If, however, the function $d_{(r)}: M \to \boldsymbol{R}$, $d_{(r)}(x) = \dim_{\boldsymbol{R}} J^r\mathscr{L}(x)$, where $J^r\mathscr{L}(x)$ denotes the vector space of r-jets $j^r_x(X)$ of germs $X \in \mathscr{L}(x)$, is locally constant, then one can show (compare with Proposition 1.2) that $C_{(r)}(\mathscr{L}; \mathscr{S})$ is a differential graded subalgebra of $C(\mathscr{L}; \mathscr{S})$. We can also introduce the graded subalgebra $C_{(\infty)}(\mathscr{L}; \mathscr{S})$ of all cochains which are locally of finite order. To this end let us define a differential graded algebra sheaf

$\mathscr{C}(\mathscr{L}; \mathscr{S})$ and its graded algebra subsheaves $\mathscr{C}_{(r)}(\mathscr{L}; \mathscr{S})$, $0 \leqq r < \infty$, putting

$$\mathscr{C}(\mathscr{L}; \mathscr{S})(U) = C(\mathscr{L} \mid U, \mathscr{S} \mid U),$$

$$\mathscr{C}_{(r)}(\mathscr{L}; \mathscr{S})(U) = C_{(r)}(\mathscr{L} \mid U, \mathscr{S} \mid U)$$

for each open subset U of M. Further, let us denote by $\mathscr{C}_{(\infty)}(\mathscr{L}; \mathscr{S})$ the least graded vector space subsheaf of $\mathscr{C}(\mathscr{L}; \mathscr{S})$ containing all the subsheaves $\mathscr{C}_{(r)}(\mathscr{L}; \mathscr{S})$, $0 \leqq r < \infty$, and, finally, let us set

$$C_{(\infty)}(\mathscr{L}; \mathscr{S}) = \mathscr{C}_{(\infty)}(\mathscr{L}; \mathscr{S})(M).$$

Again $C_{(\infty)}(\mathscr{L}; \mathscr{S})$ is a differential graded subalgebra of $C(\mathscr{L}; \mathscr{S})$, if the function $d_{(r)}$ is locally constant for all sufficiently large r. If $\mathscr{L} = \mathscr{X}$, then by virtue of J. Peetre's theorem [11, Theorem 3.3.3] we have $C_{(\infty)}(\mathscr{L}; \mathscr{S}) = C(\mathscr{L}; \mathscr{S}) \approx C_\Delta(\mathscr{L}(M), \mathscr{S}(M)) = C$, where C is the M. V. Losik's differential graded algebra introduced in [10]. Consequently, in general case of a sheaf $\mathscr{L}$ such that the function $d_{(r)}$ is locally constant for all sufficiently large r, the differential graded algebra $C_{(\infty)}(\mathscr{L}; \mathscr{S})$ can also be considered a generalization of the diagonal algebra $C_\Delta(\mathscr{X}(M), \mathscr{S}(M))$ or, equivalently, of Losik's algebra C.

The main aim of this paper is to calculate the cohomology algebra $H_{(r)}(\mathscr{L}; \mathscr{S})$ of the differential graded algebra $C_{(r)}(\mathscr{L}; \mathscr{S})$ for $r = 1$ in the special case when $\mathscr{L}$ is the sheaf of all infinitesimal automorphisms of a G-structure on M. Of course, the class of G-structures under consideration is subject to certain restrictions which ensure especially that the differential graded algebras $C_{(r)}(\mathscr{L}; \mathscr{S})$ are defined, and which will be specified in Section 1. Our main theorem is Theorem 1.5, which generalizes the first part of a result by M. V. Losik [10], who calculated the cohomology algebra $H_{(1)}(\mathscr{X}; \mathscr{S})$ and proved that $H_{(1)}(\mathscr{X}; \mathscr{S}) \approx H_{(\infty)}(\mathscr{X}; \mathscr{S})$. (Let us recall that $C_{(1)}(\mathscr{X}; \mathscr{S})$ and $C_{(\infty)}(\mathscr{X}; \mathscr{S})$ are canonically isomorphic to Losik's algebras B and C, respectively.) The case $1 < r \leqq \infty$ will be studied in a subsequent paper. Let us remark that the main theorem of the present paper was announced in [1].

1. MAIN RESULTS

Throughout this section, let M be a manifold with $\dim M = m$, let G be a Lie subgroup of $GL(m, \boldsymbol{R})$, let $\xi = (P, p, M, G)$ be a G-structure on M, i.e. a reduction in the sense of [9] of the principal $GL(m, \boldsymbol{R})$-bundle $\beta = (B_M, p_M, M, GL(m, \boldsymbol{R}))$ of all frames on M to the subgroup G, and let $\mathscr{L}_\xi$ be the sheaf of all infinitesimal automorphisms of the G-structure ξ. It is clear that $\mathscr{L}_\xi$ can be considered a topological Lie algebra subsheaf of the topological Lie algebra sheaf $\mathscr{L}_\beta = \mathscr{X}$ of all vector fields on M.

1.1. Definition. Let r be a non-negative integer, and let $J^r\mathscr{L}_\xi(x)$ be the vector space of r-jets $j^r_x(X)$ of all germs X from the stalk $\mathscr{L}_\xi(x)$ of the sheaf $\mathscr{L}_\xi$ at the point x.

A G-structure ξ will be called *r-regular* if the function $x \mapsto \dim J^r \mathscr{L}_\xi(x)$ is locally constant on M.

The following proposition can be proved by using special forms and computations of [10].

1.2. Proposition. *If a G-structure ξ is r-regular, then $C_{(r)}(\mathscr{L}_\xi; \mathscr{S})$ is a differential graded subalgebra of $C(\mathscr{L}_\xi; \mathscr{S})$.*

1.3. Definition. A *G-structure ξ will be called 1-transitive* if $\dim J^1 \mathscr{L}_\xi(x) = m + \dim G$ for all $x \in M$.

1.4. Remarks. (a) Clearly any 1-transitive G-structure is 1-regular.

(b) If a G-structure ξ is transitive in the sense of [12], then it is r-regular for any $r \geqq 0$. If, moreover, M is connected, then a transitive G-structure ξ is 1-transitive if and only if $\dim J^1 \mathscr{L}_\xi(x) = m + \dim G$ for at least one point $x \in M$.

(c) It is not difficult to see that all locally flat G-structures are 1-transitive. Moreover, it can be easily proved that almost complex and almost symplectic structures are 1-transitive if and only if they are locally flat, and that riemannian and pseudo-riemannian structures are 1-transitive if and only if they have constant sectional curvature.

For the sake of formulation of our main result we remark that by the complexification of a Lie subgroup G of $GL(m, \mathbf{R})$ we mean the integral subgroup $\hat{G} \subset GL(m, \mathbf{C})$ of the complexification of the Lie algebra of G in $\mathfrak{gl}(m, \mathbf{C})$. By definition, $\hat{G}$ is a complex Lie subgroup of the complex Lie group $GL(m, \mathbf{C})$, and $\dim_{\mathbf{R}} \hat{G} = 2 \dim_{\mathbf{C}} \hat{G} = 2 \dim_{\mathbf{R}} G$.

1.5. Theorem. *Let us suppose that the Lie group G is connected and reductive, and that the G-structure ξ is 1-transitive. Let $\hat{G}$ be the complexification of G, and let $\hat{\xi} = (\hat{P}, \hat{p}, M, \hat{G})$ be an extension of ξ to the group $\hat{G}$. Finally, let K be a maximal compact subgroup of $\hat{G}$, let $\eta = (Q, q, M, K)$ be a reduction of $\hat{\xi}$ to the subgroup K, and let $d = \dim G - \dim K$. Then there exists an isomorphism of graded algebras over $\mathbf{R}$*

$$H_{(1)}(\mathscr{L}_\xi; \mathscr{S}) \approx H^*_\Delta(Q; \mathbf{R}) \otimes_{\mathbf{R}} \Lambda(\mathbf{R}^d),$$

where $H^*_\Delta(Q; \mathbf{R})$ denotes the singular cohomology algebra of Q with real coefficients. If G is semisimple, then $d = 0$.

1.6. Remark. The conclusion of Theorem 1.5 holds also in a more general case when the assumption that G is connected is replaced by the assumption that G is a regular real form in the sense of Definition 2.13 of its complexification $\hat{G}$. This will be clear from the proof of Theorem 1.5.

1.7. Remark. In fact, we shall prove, see Remark 5.4, that under the assumptions of Theorem 1.5 (or of the preceding remark) there is a commutative diagram of graded algebras over $\boldsymbol{R}$

$$\begin{array}{ccc} H_{DR}(M;\boldsymbol{R}) & \xrightarrow{\text{de Rham}} & H^*_\Delta(M;\boldsymbol{R}) \\ \downarrow & & \downarrow{\scriptstyle q^*\otimes 1} \\ H_{(1)}(\mathscr{L}_\xi;\mathscr{S}) & \approx & H^*_\Delta(Q;\boldsymbol{R})\otimes_{\boldsymbol{R}} \Lambda(\boldsymbol{R}^d) \end{array}$$

where the vertical arrow on the left denotes a canonical homomorphism.

2. REAL FORMS OF COMPLEX LIE GROUPS AND PRINCIPAL BUNDLES WITH COMPLEX LIE STRUCTURE GROUPS

2.1. If G is a (real) Lie group we denote by $L(G)$ its Lie algebra, and if $f\colon G \to H$ is a homomorphism of Lie groups we denote by $L(f)\colon L(G) \to L(H)$ the corresponding homomorphism of their Lie algebras. For a complex Lie group $\hat{G}$, we mean by $L(\hat{G})$ the Lie algebra of $\hat{G}$ considered as a real Lie group. In this case, however, $L(\hat{G})$ has a canonical complex Lie algebra structure, and if $\hat{f}\colon \hat{G} \to \hat{H}$ is a homomorphism of complex Lie groups then $L(\hat{f})$ is a homomorphism of complex Lie algebras.

As usual, the Lie algebras of the linear groups $GL(n, \boldsymbol{R})$ and $GL(n, \boldsymbol{C})$ are denoted by $\mathfrak{gl}(n, \boldsymbol{R})$ and $\mathfrak{gl}(n, \boldsymbol{C})$, respectively. Let us remark that $GL(n, \boldsymbol{C})$ is a complex Lie group, and that the complex Lie algebra $\mathfrak{gl}(n, \boldsymbol{C})$ can be canonically identified with $\mathfrak{gl}(n, \boldsymbol{R}) \otimes_{\boldsymbol{R}} \boldsymbol{C}$.

2.2. Definition. Let $\hat{\mathfrak{g}}$ be a complex Lie algebra. An involution of $\hat{\mathfrak{g}}$ is an automorphism σ of $\hat{\mathfrak{g}}$ considered as a real Lie algebra such that $\sigma^2 = \mathrm{id}$ and $\sigma(\mathrm{i}x) = -\mathrm{i}\sigma(x)$ for all $x \in \hat{\mathfrak{g}}$. A real form of $\hat{\mathfrak{g}}$ is a real Lie subalgebra $\mathfrak{g}$ of the real Lie algebra $\hat{\mathfrak{g}}$ such that $\hat{\mathfrak{g}} = \mathfrak{g} \oplus \mathrm{i}\mathfrak{g}$ in the category of real vector spaces.

It is easy to see that there is a one-to-one correspondence between the involutions of $\hat{\mathfrak{g}}$ and the real forms of $\hat{\mathfrak{g}}$. For an involution σ of $\hat{\mathfrak{g}}$ the corresponding real form $\hat{\mathfrak{g}}^\sigma$ of $\hat{\mathfrak{g}}$ is given by $\hat{\mathfrak{g}}^\sigma = \{x \in \hat{\mathfrak{g}}\colon \sigma(x) = x\}$.

2.3. Definition. Let $\hat{G}$ be a connected complex Lie group. An involution of $\hat{G}$ is an automorphism σ of the real Lie group $\hat{G}$ such that $L(\sigma)$ is an involution of the complex Lie algebra $L(\hat{G})$.

2.4. Let $\hat{G}$ and σ be as in Definition 2.3. It is easy to see that $\hat{G}^\sigma = \{g \in \hat{G}\colon \sigma(g) = g\}$ is a closed Lie subgroup of the real Lie group $\hat{G}$ and $L(\hat{G}^\sigma) = L(\hat{G})^{L(\sigma)}$. Moreover, it is clear that two involutions σ, τ of $\hat{G}$ coincide if and only if $L(\hat{G}^\sigma) = L(\hat{G}^\tau)$, which in turn is equivalent to the assertion that the connected components of the unit element in the groups $\hat{G}^\sigma$ and $\hat{G}^\tau$ coincide.

2.5. Definition. Let $\hat{G}$ be a connected complex Lie group. A real form of $\hat{G}$ is a closed real Lie subgroup G of $\hat{G}$ for which there exists an involution σ of $\hat{G}$ such that $G \subset \hat{G}^\sigma$ and $L(G) = L(\hat{G}^\sigma)$.

A real form G of $\hat{G}$ will be called *quasicompact*, if $G = \hat{G}^\sigma$ and if G is the direct product (both in algebraic and topological sense) of a maximal compact subgroup K of $\hat{G}$ and a closed central subgroup R of $\hat{G}$ isomorphic to $\boldsymbol{R}^d$.

2.6. Remarks. (a) Let G be a real form of a connected complex Lie group $\hat{G}$. According to 2.4, there exists exactly one involution σ of $\hat{G}$ satisfying $L(G) = L(\hat{G}^\sigma)$. We shall say that the involution σ is associated with the real form G of $\hat{G}$, and also that the real form G is associated with the involution σ, even if G is not determined by σ uniquely.

(b) Let G be a quasicompact real form of $\hat{G}$, and let K, R and d be as in Definition 2.5. It is well known, see e.g. [8], that the maximal compact subgroups of a connected Lie group are connected and mutually conjugate. Consequently, G is connected. Since clearly $d = \dim_{\boldsymbol{C}} \hat{G} - \dim_{\boldsymbol{R}} K$, it also follows that d depends only on $\hat{G}$.

It follows easily from the definition that a connected complex Lie group $\hat{G}$ having a quasicompact real form is necessarily reductive (in the sense that the Lie algebra $L(\hat{G})$ is reductive) and contains no non-discrete compact complex Lie subgroups. The aim of the next part of this section is to prove the converse assertion. We shall also prove that, under the conditions just mentioned, for any connected real form G of $\hat{G}$ there is a quasicompact real form of $\hat{G}$ which is in a "nice" position with respect to G. Our results may be considered a generalization of E. Cartan's results on semi-simple Lie groups, see e.g. [7, Chap. 3 and 6].

2.7. Lemma. *Let $\hat{G}$ be a connected commutative complex Lie group without non-discrete compact complex Lie subgroups, σ an involution of $\hat{G}$, and G a connected component of the unit element in $\hat{G}^\sigma$. Then there is an involution τ of $\hat{G}$ such that $\sigma \circ \tau = \tau \circ \sigma$, $\hat{G}^\tau$ is the direct product (both in algebraic and topological sense) of a (unique) maximal compact subgroup K of $\hat{G}$ and a closed subgroup $R \subset G$ isomorphic to $\boldsymbol{R}^d$, and $G \cap K$ is a (unique) maximal compact subgroup of G.*

Proof. First let us remark that according to the definition of the involution, the complex Lie algebra $L(\hat{G})$ can be identified canonically with $L(G) \otimes_{\boldsymbol{R}} \boldsymbol{C}$ and that then $L(\sigma)(x \otimes c) = x \otimes \bar{c}$ for $x \in L(G)$ and $c \in \boldsymbol{C}$.

Now let K be a maximal compact subgroup of $\hat{G}$. Since $\hat{G}$ clearly contains exactly one maximal compact subgroup, the subalgebras $L(K)$ and $\hat{L}(K) = L(K) + \mathrm{i}L(K)$ of $L(\hat{G})$ are $L(\sigma)$-invariant, and hence $\hat{L}(K) = (\hat{L}(K) \cap L(G)) \otimes_{\boldsymbol{R}} \boldsymbol{C}$. This implies that for any subalgebra $\mathfrak{r} \subset L(G)$ satisfying $L(G) = \mathfrak{r} \oplus (\hat{L}(K) \cap L(G))$ we have also $L(\hat{G}) = (\mathfrak{r} \otimes_{\boldsymbol{R}} \boldsymbol{C}) \oplus \hat{L}(K)$. Using the σ-invariance of K and the assumption that $\hat{G}$ does not contain non-discrete compact complex subgroups we further find that $L(K) \cap \mathrm{i}L(K) = 0$, and consequently $\hat{L}(K) = L(K) \oplus \mathrm{i}L(K)$. These direct decompositions of $L(\hat{G})$ and $\hat{L}(K)$ imply that $\hat{G}$ is the direct product of the integral subgroup $\hat{R}$ of $\mathfrak{r} \otimes_{\boldsymbol{R}} \boldsymbol{C}$ and the integral subgroup $\hat{K}$ of $\hat{L}(K)$, with $\hat{R}$ being simultaneously the direct product of the integral subgroup R of $\mathfrak{r}$ and the integral subgroup R_i of $\mathrm{i}\mathfrak{r}$, and $\hat{K}$ being the direct product of K and the integral subgroup K_i of $\mathrm{i}L(K)$.

It follows that we can define an automorphism τ of the real Lie group $\hat{G}$ by putting $\tau(r \,.\, r_i \,.\, k \,.\, k_i) = r \,.\, r_i^{-1} \,.\, k \,.\, k_i^{-1}$ for $r \in R$, $r_i \in R_i$, $k \in K$ and $k_i \in K_i$, and one easily shows that τ is an involution of $\hat{G}$ having all the required properties.

2.8. Remark. As is clear from the proof of Lemma 2.7, we have proved at the same time that a connected commutative complex Lie group without non-discrete compact complex Lie subgroups has quasicompact real forms.

2.9. Proposition. *Let $\hat{G}$ be a connected reductive Lie group without any nondiscrete compact complex Lie subgroup, σ an involution of $\hat{G}$, and G the connected component of the unit element in $\hat{G}^\sigma$. Then there is an involution τ of $\hat{G}$ such that $\sigma \circ \tau = \tau \circ \sigma$ and $\hat{G}^\tau$ is the direct product (both in algebraic and topological sense) of a maximal compact subgroup K of $\hat{G}$ and a closed central subgroup R of $\hat{G}$ isomorphic to $\boldsymbol{R}^d$. Moreover, $R \subset G$, and $G \cap K$ is a maximal compact subgroup of G. If $\hat{G}$ is semisimple, then $d = 0$.*

Proof. The proposition is true for $\hat{G}$ commutative by Lemma 2.7 and represents a classical result of E. Cartan for $\hat{G}$ semisimple – see e.g. [7, Chaps. 3 and 6].

In the general case, let $\hat{G}_c \subset \hat{G}$ be the integral subgroup of the centre of the Lie algebra $L(\hat{G})$ and $\hat{G}_s \subset \hat{G}$ the integral subgroup of the derived subalgebra $[L(\hat{G}), L(\hat{G})]$ of $L(\hat{G})$. By the reductivity of $L(\hat{G})$, $L(\hat{G}) = L(\hat{G}_c) \oplus L(\hat{G}_s)$, and $L(\hat{G}_c)$ is a maximal commutative ideal in $L(\hat{G})$. It follows that the subgroup $\hat{G}_c \subset \hat{G}$ is closed, and that the canonical homomorphism $\hat{p}\colon \hat{G}_c \times \hat{G}_s \to \hat{G}$ is a covering map with multiplicity card $(\hat{G}_c \cap \hat{G}_s)$. We remark that $\hat{p}$ is really a homomorphism because $\hat{G}_c$ is a central subgroup of $\hat{G}$. Now $\hat{G}_c \cap \hat{G}_s$ is a closed central subgroup of the semisimple complex Lie group $\hat{G}_s$, and hence it is not only discrete but even finite. Consequently, the multiplicity of $\hat{p}$ is finite, which easily yields that $\hat{G}_s$ is closed in $\hat{G}$, and that every maximal compact subgroup K of $\hat{G}$ is of the form $K = K_c \,.\, K_s$ where $K_c \subset \hat{G}_c$ and $K_s \subset \hat{G}_s$ are maximal compact subgroups. Moreover, as one easily shows, $K_c = K \cap \hat{G}_c$ and $K_s = K \cap \hat{G}_s$.

Similarly, let $G_c \subset G$ be the integral subgroup of the centre of the Lie algebra $L(G)$ and $G_s \subset G$ the integral subgroup of the derived subalgebra $[L(G), L(G)]$ of $L(G)$. Since the reductivity of $L(\hat{G}) = L(G) \otimes_{\boldsymbol{R}} \boldsymbol{C}$ is equivalent to the reductivity of $L(G)$ the same argument as above yields that G_c is a closed subgroup of G, and that the canonical homomorphism $p\colon G_c \times G_s \to G$ is a covering map with multiplicity card $(G_c \cap G_s)$. The inclusion $G_c \cap G_s \subset \hat{G}_c \cap \hat{G}_s$ further implies that this multiplicity is finite, and hence G_s is closed in G and the maximal compact subgroups of G are related to those of G_c and G_s in the same way as the maximal compact subgroups of $\hat{G}$ are related to those of $\hat{G}_c$ and $\hat{G}_s$.

Let σ_c and σ_s denote the restrictions of the involution σ to the subgroups $\hat{G}_c$ and $\hat{G}_s$, respectively. By Lemma 2.7 there exists an involution τ_c of $\hat{G}_c$ such that $\sigma_c \circ \tau_c = \tau_0 \circ \sigma_c$, $\hat{G}_c^{\tau_c}$ is the direct product of a maximal compact subgroup K_c of $\hat{G}_c$ and of a closed subgroup $R \subset G_c$ isomorphic to $\boldsymbol{R}^d$, and $K_c \cap G_c$ is a maximal compact subgroup of G_c. By the first paragraph of this proof there also exists an involution τ_s of the group $\hat{G}_s$ such that $\sigma_s \circ \tau_s = \tau_s \circ \sigma_s$, $K_s = \hat{G}_s^{\tau_s}$ is a maximal compact subgroup of $\hat{G}_s$, and $K_s \cap G_s$ is a maximal compact subgroup of G_s. Let us consider the automorphism $\tau_c \times \tau_s$ of the group $\hat{G}_c \times \hat{G}_s$. The kernel of $\hat{p}$ consists of all pairs (g, g^{-1}) where $g \in \hat{G}_c \cap \hat{G}_s$. We have seen, however, that $\hat{G}_c \cap \hat{G}_s$ is a finite central subgroup of $\hat{G}$, and hence it is contained in $K_c \cap K_s$. This immediately implies that the automorphism $\tau_c \times \tau_s$ restricts to identity on $\ker \hat{p}$, and hence it induces an automorphism $\tau: \hat{G} \to \hat{G}$ satisfying $\hat{p} \circ (\tau_c \times \tau_s) = \tau \circ \hat{p}$. Using the fact that $L(\hat{p})$ is an isomorphism of complex Lie algebras we easily deduce that τ is an involution of the complex Lie group $\hat{G}$. It remains to show that the subgroups R and $K = K_c \,.\, K_s$ and the involution τ have all the desired properties.

The properties of R are obvious, and the assertion that $K \subset \hat{G}$ and $K \cap G \subset G$ are maximal compact subgroups is a consequence of what has been said in the second and third paragraphs of the proof. The equality $\sigma \circ \tau = \tau \circ \sigma$ is an immediate consequence of the definition of τ, as well as the inclusion $R \,.\, K \subset \hat{G}^\tau$. We now show that, in fact, $R \,.\, K = \hat{G}^\tau$. To this aim it suffices to prove that the equality $\tau(g_c \,.\, g_s) = g_c \,.\, g_s$, where $g_c \in \hat{G}_c$ and $g_s \in \hat{G}_s$, implies $\tau(g_c) = g_c$. If, however, $\tau(g_c \,.\, g_s) = g_c \,.\, g_s$, then $g_c^{-1} . \tau(g_c) = g_s . \tau(g_s)^{-1} \in \hat{G}_c \cap G_s = K_c \cap K_s$, and hence $g_c^{-1} . \tau(g_c) = \tau(g_c^{-1} \,.\, \tau(g_c)) = \tau(g_c)^{-1} \,.\, g_c$, which further yields $g_c^2 = \tau(g_c^2)$. Since the kernel of the exponential homomorphism $\exp: L(\hat{G}_c) \to \hat{G}_c$ is clearly contained in $L(K_c)$, this equality easily implies that $g_c = \tau(g_c)$. It remains to show that the decomposition $R \,.\, K = \hat{G}^\tau$ is direct both in algebraic and topological sense. But this is an easy consequence of the equality $R \cap K = R \cap K_c = \{1\}$, the compactness of K and the closedness of R in $\hat{G}$.

2.10. Remark. Similar arguments prove that every connected reductive complex Lie group without non-discrete compact complex Lie subgroups has quasicompact real forms.

2.11. Corollary. *Let $G \subset GL(n, \boldsymbol{R})$ be a reductive integral subgroup, and let $\hat{G} \subset GL(n, \boldsymbol{C})$ be the integral subgroup of the complex Lie subalgebra $\hat{L}(G) = L(G) \otimes_{\boldsymbol{R}} \boldsymbol{C}$ of $\mathfrak{gl}(n, \boldsymbol{C})$. Then the following assertions hold:*

(a) *$\hat{G}$ is a complex Lie group containing together with a matrix g also its complex conjugate matrix $\bar{g}$, the map $\sigma: \hat{G} \to \hat{G}$ defined by $\sigma(g) = \bar{g}$ is an involution of $\hat{G}$, and G is a connected component of the unit element 1 of $\hat{G}^\sigma$.*

(b) *There exists an involution τ of $\hat{G}$ such that $\sigma \circ \tau = \tau \circ \sigma$, and the subgroup $\hat{G}^\tau$ is the direct product (both in algebraic and topological sense) of a maximal compact subgroup K of $\hat{G}$ and a closed central subgroup R of $\hat{G}$ isomorphic to $\boldsymbol{R}^d$. Moreover, $R \subset G$, and $G \cap K$ is a maximal compact subgroup of G. If G is semisimple, then $d = 0$.*

Proof. The assertion (a) is obvious, and (b) follows from Proposition 2.9 since it is an easy consequence of the Jordan canonical form of a matrix that $GL(n, \boldsymbol{C})$ does not contain non-discrete compact complex Lie subgroups.

2.12. Let us keep the notation of Proposition 2.9. The generalization of this proposition to principal bundles given below (see Proposition 2.18) uses the well known fact (see e.g. [8]) that the homogeneous space $G/G \cap K$ is diffeomorphic to $\boldsymbol{R}^s$ for some s. If G_1 is a real form of $\hat{G}$ having G as the connected component of its unit element then the space $G/G \cap K$ imbeds canonically in $G_1/G_1 \cap K$ as a closed-open subspace. It follows that $G_1/G_1 \cap K$ is diffeomorphic to $\boldsymbol{R}^s$ if and only if $G/G \cap K = G_1/G_1 \cap K$, which is, however, equivalent to the assertion that K intersects all the connected components of G_1.

It is therefore natural to ask whether for G_1 as above there is an involution τ of $\hat{G}$ having all the properties listed in Proposition 2.9 and such that K intersects all the connected components of G_1. Such an involution exists, for example, if $\hat{G} = GL(n, \boldsymbol{C})$, $G_1 = GL(n, \boldsymbol{R})$ and $G = GL^+(n, \boldsymbol{R})$, as shown in Remark 2.14 below. Unfortunately, we do not know any simple conditions ensuring the existence of such an involution τ in general. This fact and the effort to cover also this important special case motivate the following definition.

2.13. Definition. Let $\hat{G}$ be a connected complex Lie group, G a real form of $\hat{G}$ associated with an involution σ of $\hat{G}$, and $G_0 \subset G$ the connected component of the unit element of G. We shall say that G is a *regular real form of the group* $\hat{G}$ if there exists an involution τ of $\hat{G}$ such that $\sigma \circ \tau = \tau \circ \sigma$, $\hat{G}^\tau$ is a quasicompact real form of $\hat{G}$, the maximal compact subgroup K of $\hat{G}^\tau$ intersects all the connected components of G, and $G_0 \cap K$ is a maximal compact subgroup of G_0.

2.14. Remark. Let $\hat{G}$ be a connected reductive complex Lie group without non-discrete compact complex Lie subgroups. Proposition 2.9 says that every connected real form of $\hat{G}$ is regular.

An example of a non-connected regular real form is the subgroup $GL(n, \boldsymbol{R})$ of $GL(n, \boldsymbol{C})$. In fact, let σ be the involution of $GL(n, \boldsymbol{C})$ defined by $\sigma(g) = \bar{g}$, where $\bar{g}$ denotes the complex conjugate matrix to g, and let τ be the involution of $GL(n, \boldsymbol{C})$ defined by $\tau(g) = (g^*)^{-1}$, where g^* denotes the hermitian conjugate matrix to g. Clearly $GL(n, \boldsymbol{C})^\sigma = GL(n, \boldsymbol{R})$, $GL(n, \boldsymbol{C})^\tau = U(n)$ and $\sigma \circ \tau = \tau \circ \sigma$. Moreover, $U(n)$ is a maximal compact subgroup of $GL(n, \boldsymbol{C})$ having a non-empty intersection with each of the both connected components of $GL(n, \boldsymbol{R})$, and $U(n) \cap GL^+(n, \boldsymbol{R}) = O^+(n)$ is a maximal compact subgroup of the connected component $GL^+(n, \boldsymbol{R})$ of the unit element of $GL(n, \boldsymbol{R})$. Consequently, $GL(n, \boldsymbol{R})$ is a regular real form of $GL(n, \boldsymbol{C})$ as we have claimed.

In the remaining part of this section we generalize the above notions and results to principal bundles having a connected complex Lie group as a structure group.

2.15. Definition. Let $\hat{G}$ be a connected complex Lie group. An *involution of a principal $\hat{G}$-bundle* $\hat{\xi} = (\hat{P}, \hat{p}, M, \hat{G})$ is a pair (Σ, σ) consisting of an involution σ of $\hat{G}$ and a σ-automorphism Σ of the principal $\hat{G}$-bundle $\hat{\xi}$ such that $\Sigma^2 = \mathrm{id}$, $\hat{P}^\Sigma = \{x \in \hat{P} : \Sigma(x) = x\}$ is a closed submanifold of $\hat{P}$ and $\hat{\xi}^\Sigma = (\hat{P}^\Sigma, \hat{p}^\Sigma, M, \hat{G}^\sigma)$, where $\hat{p}^\Sigma$ is the restriction of $\hat{p}$ on $\hat{P}^\Sigma$, is a principal $\hat{G}^\sigma$-bundle.

2.16. Definition. Let $\hat{G}$ be a connected complex Lie group and $\hat{\xi} = (\hat{P}, \hat{p}, M, \hat{G})$ a principal $\hat{G}$-bundle. A *real form of* $\hat{\xi}$ is a reduction $\xi = (P, p, M, G)$ of $\hat{\xi}$ to a subgroup $G \subset \hat{G}$ for which there is an involution (Σ, σ) of $\hat{\xi}$ such that G is a real form of $\hat{G}$ associated with σ and $P \subset \hat{P}^{\Sigma}$.

A real form ξ of $\hat{\xi}$ is said to be *quasicompact* or *regular* if G is a quasicompact or regular form of $\hat{G}$, respectively.

2.17. Remark. It is clear that for a given reduction ξ of $\hat{\xi}$ there exists at most one involution (Σ, σ) of $\hat{\xi}$ with the properties listed in Definition 2.16. This allows us to call (Σ, σ) the involution associated to the real form ξ of $\hat{\xi}$.

2.18. Proposition. *Let $\hat{G}$ be a connected complex Lie group, $\xi = (P, p, M, G)$ a regular real form of a principal $\hat{G}$-bundle $\hat{\xi} = (\hat{P}, \hat{p}, M, \hat{G})$, and (Σ, σ) the involution of $\hat{\xi}$ associated with ξ. Let G_0 be the connected component of the unit element of G and let τ be any involution of $\hat{G}$ such that $\sigma \circ \tau = \tau \circ \sigma$, $\hat{G}^{\tau}$ is a quasicompact real form of $\hat{G}$, the maximal compact subgroup K of $\hat{G}^{\tau}$ intersects all the components of G, and $G_0 \cap K$ is a maximal compact subgroup of G_0.*

Under these assumptions there exists an involution (T, τ) of the principal $\hat{G}$-bundle $\hat{\xi}$ such that $T \circ \Sigma = \Sigma \circ T$ and $\hat{\xi}^{T}$ is the extension to the principal $\hat{G}^{\tau}$-bundle of a reduction of ξ to the subgroup $G \cap K$.

Proof. It follows from our assumptions and from [8] (see 2.12) that $G/G \cap K$ is diffeomorphic to $\boldsymbol{R}^s$ for some s. Consequently, applying Proposition 5.6 and Theorem 5.7 of [9, Chapter I] we get a reduction $\xi_1 = (P_1, p_1, M, G \cap K)$ of ξ to the subgroup $G \cap K$. Supposing $P_1 \subset P$ and $p_1 = p/P_1$, which is clearly admissible, one easily verifies that the required τ-automorphism T of $\hat{\xi}$ may be defined by the formula $T(x \,.\, g) = x \,.\, \tau(g)$, where $x \in P_1$ and $g \in G$.

2.19. Remark. Using Remark 2.10, we can show in a similar way that for any principal $\hat{G}$-bundle, where $\hat{G}$ is a connected reductive complex Lie group, there exist quasicompact real forms.

3. TRANSGRESSION THEOREM AND HOMOLOGY OF REAL FORMS OF COMMUTATIVE DIFFERENTIAL GRADED ALGEBRAS OVER $\boldsymbol{C}$

Let k be a commutative field of characteristic 0.

3.1. Starting from this section, by a *G-k*-algebra (*DG-k*-algebra) A we mean a graded algebra (differential graded algebra) over k the graduation $\{A^n\}$ of which is non-negative, i.e. $A^n = 0$ for all $n < 0$.

By a filtration of a *G-k*-algebra (*DG-k*-algebra) A we mean a decreasing filtration $FA = \{F^pA\}$ of the graded (differential graded) algebra A satisfying $F^0A = A$, $F^{n+1}A^n = 0$ for all n.

By an FG-k-algebra (FDG-k-algebra) we mean a G-k-algebra (DG-k-algebra) equipped with a fixed filtration.

All algebras under consideration are supposed to be associative, and commutativity of a graded algebra means commutativity in the graded sense.

3.2. A spectral sequence $\{E_r, d_r, \iota_r\}_{r \geqq r_0}$ over k is called *multiplicative* if for any $r \geqq r_0$,

(a) E_r is a bigraded k-algebra,

(b) the differential d_r has bidegree $(r, 1 - r)$ and is a derivation in the graded sense with respect to the total degree in E_r,

(c) $\iota_r\colon E_{r+1} \to H(E_r, d_r)$ is an isomorphism of bigraded k-algebras, and

(d) $E_r^{p,q} = 0$ if either $p < 0$ or $q < 0$.

Let us remark that by virtue of (d) every multiplicative spectral sequence converges.

Homomorphisms of multiplicative spectral sequences are defined in the obvious way.

3.3. A diagram

$$\{E_r, d_r, \iota_r\}_{r \geqq 0} \xRightarrow[(k)]{\iota_\infty} H \text{ rel. } FH$$

will mean that $\{E_r, d_r, \iota_r\}_{r \geqq r_0}$ is a multiplicative spectral sequence over k (with the limit E_∞), H is a G-k-algebra equipped with a filtration FH, and ι_∞ is an isomorphism of the bigraded k-algebra E_∞ with the bigraded k-algebra $\mathscr{G}rFH = \bigoplus\limits_{p,q} \mathscr{G}r^{p,q}FH$ associated with the filtration FH; here $\mathscr{G}r^{p,q}FH = F^pH^{p+q}/F^{p+1}H^{p+q}$.

Now let us suppose we are given a diagram

$$\begin{array}{ccc} \{E_r, d_r, \iota_r\}_{r \geqq r_0} & \xRightarrow[(k)]{\iota_\infty} & H \text{ rel. } FH \\ \Big\downarrow{\scriptstyle \varepsilon} & & \Big\downarrow{\scriptstyle h} \\ \{E'_r, d'_r, \iota'_r\}_{r \geqq r_0} & \xRightarrow[(k)]{\iota'_\infty} & H' \text{ rel. } FH' \end{array} \tag{3.1}$$

with the rows having the meaning described above, where ε is a homomorphism of multiplicative spectral sequences, and h is a homomorphism of G-k-algebras compatible with the filtrations FH and FH'. We shall say that this diagram commutes if $(\mathscr{G}rh) \circ \iota_\infty = \iota'_\infty \circ \varepsilon_\infty$, where $\varepsilon_\infty\colon E_\infty \to E'_\infty$ and $\mathscr{G}rh;\ \mathscr{G}rFH \to \mathscr{G}rFH'$ are homomorphisms induced by ε, h, respectively. We shall also describe such a situation by saying that the homomorphisms ε and h are compatible with respect to ι_∞ and ι'_∞.

We recall the well-known fact that commutativity of the diagram (3.1) implies that h is an isomorphism of G-k-algebras if $\varepsilon_r\colon E_r \approx E'_r$ for some r.

3.4. It is well known that with each FDG-k-algebra A a diagram

$$\{E_r, d_r, \iota_r\}_{r \geqq 1} \xRightarrow[(k)]{\iota_\infty} H(A) \text{ rel. } FH(A)$$

is associated, where $H(A)$ is the homology G-k-algebra of the DG-k-algebra A. This can be shown in several equivalent ways. Using the definition of H. Cartan and S. Eilenberg [4, Chap. XV] we have

$$E_r^{p,q} = Z_r^{p,q}/B_r^{p,q}$$

where

$$Z_r^{p,q} = \operatorname{Im}\{H^{p+q}(F^pA/F^{p+r}A) \to H^{p+q}(F^pA/F^{p+1}A)\}\,,$$
$$B_r^{p,q} = \operatorname{Im}\{H^{p+q-1}(F^{p-r+1}A/F^pA) \to H^{p+q}(F^pA/F^{p+1}A)\}\,,$$

and

$$F^p H(A) = \operatorname{Im}\{H(F^pA) \to H(A)\}\,.$$

If the FDG-k-algebra A is commutative then each algebra E_r is commutative in the graded sense with respect to the total degree. If A has the unit element then each E_r has the unit element. Analogous assertions hold of course also for the algebras $H(A)$ and $\mathcal{G}rFH(A)$.

We remark that the DG-k-subalgebra $E_1^{*,0} = \bigoplus_p E_1^{p,0}$ of E_1 can be canonically identified with the DG-k-subalgebra $B = \bigoplus_p B^p$ of A, where $B^p = F^pA^p \cap \cap\, d^{-1}(F^{p+1}A^{p+1})$. Clearly, $\ker d_1^{p,0}$ is identified with $Z^p(B) = B^p \cap \ker d$ via this identification.

3.5. The transgression homomorphism (briefly transgression) in the first quadrant spectral sequence $\{E_r, d_r, \iota_r\}_{r \geqq 1}$ is the homomorphism

$$\bigoplus_q d_{q+1}^{0,q}\colon \bigoplus_q E_{q+1}^{0,q} \to \bigoplus_q E_{q+1}^{q+1,0}$$

of total degree 1. Let $v_q\colon E_{q+1}^{0,q} \to E_2^{0,q}$ be the canonical monomorphisms and $\pi_{q+1}\colon \ker d_1^{q+1,0} \to E_{q+1}^{q+1}$ the canonical epimorphisms. An element $x \in E_2^{0,q}$ is called *transgressive* if it belongs to $\operatorname{Im} v_q$. By a transgression of a transgressive element $x \in E_2^{0,q}$ we shall mean any element $y \in \ker d_1^{q+1,0}$ such that $\pi_{q+1}(y) = d_{q+1}^{0,q}(v_q^{-1}(x))$, or its image in $E_2^{q+1,0}$.

If the spectral sequence under consideration is associated with a FDG-k-algebra A, then for any integer q we have the following commutative diagram

$$\begin{array}{ccccccccc}
F^1A^q \cap d^{-1}(F^{q+1}A^{q+1}) & \subset & A^q \cap d^{-1}(F^{q+1}A^{q+1}) & \xrightarrow{\pi_q'} & E_{q+1}^{0,q} & \xrightarrow{v_q} & E_2^{0,q} & & \\
\big\downarrow d & & \big\downarrow d & & \big\downarrow d_{q+1}^{0,q} & & & & \\
 & & Z^{q+1}(B) & & & & & & \\
 & & \| & & & & & & \\
0 \longrightarrow B_{q+1}^{q+1,0} & \longrightarrow & \ker d_1^{q+1,0} & \longrightarrow & E_{q+1}^{q+1,0} & \longrightarrow & 0 & & \\
\downarrow & & & & & & & & \\
0 & & & & & & & &
\end{array}$$

in which the left column and the lower row are exact, and π_q' is a canonical epimorphism. This diagram implies that $y \in \ker d_1^{q+1,0}$ is a transgression of $x \in E_2^{0,q}$ if and only if there is an element $z \in A^q \cap d^{-1}(F^{q+1}A^{q+1})$ such that $y = d(z)$ and $v_q \circ \pi_q'(z) = x$.

The following proposition formalizes Borel's transgression theorem [2, Chap. VI, § 24].

3.6. Proposition. *Let A be a commutative FDG-k-algebra with the unit element, and let us suppose that the spectral sequence $\{E_r, d_r, \iota_r\}_{r\geqq 1}$ associated with A satisfies the following conditions:*

(a) *The G-k-algebra $E_2^{0,*} = \bigoplus_q E_2^{0,q}$ is a free G-k-algebra $L(P) = \Lambda(P_0) \otimes_k S(P_e)$ over $P = P_0 \oplus P_e$ where $P_0 \subset E_2^{0,*}$ ($P_e \subset E_2^{0,*}$) is a graded k-vector subspace spanned by transgressive elements of odd (even) degree.*

(b) *The canonical homomorphism $E_2^{p,0} \otimes_k E_2^{0,q} \to E_2^{p,q}$ is an isomorphism for all p, q.*

Let $B \otimes_k L(P)$ be the tensor product (in the graded sense) of the G-k-algebra $B = E_1^{*,0}$ (see 3.5) *and the G-k-algebra $L(P)$. Let us choose a homogeneous basis $\{a_i\}_{i\in I}$ of the graded k-vector space P and set $q_i =$* deg a_i. *For each $i \in I$ let us further choose a transfression $b_i \in B^{q_i+1}$ of a_i and a representative $c_i \in A^{q_i} \cap d^{-1}(F^{q_i+1}A^{q_i+1})$ of a_i* (see 3.5) *such that $dc_i = b_i$. Then the following assertions hold:*

(α) *The k-linear map $d\colon B \otimes_k P \to B \otimes_k L(P)$ defined by*

$$d(b \otimes a_i) = db \otimes a_i + (-1)^q\, b \,.\, b_i \otimes 1 \quad (b \in B^q,\ i \in I)$$

extends in a unique way to the differential d of the G-k-algebra $B \otimes_k L(P)$ so that $B \otimes_k L(P)$ together with this d is a DG-k-algebra C.

The differential d is explicitly given by the formula

$$(3.2) \qquad d(b \otimes a_{i_1} \wedge \ldots \wedge a_{i_r} \otimes a_{j_1} \ldots a_{j_s}) =$$
$$= db \otimes a_{i_1} \wedge \ldots \wedge a_{i_r} \oplus a_{j_1} \ldots a_{j_s} +$$
$$+ (-1)^q \,.\, \sum_{k=1}^{r} (-1)^{k-1}\, b \,.\, b_{i_k} \oplus a_{i_1} \wedge \ldots \wedge \hat{a}_{i_k} \wedge \ldots \wedge a_{i_r} \otimes$$
$$\otimes a_{j_1} \ldots a_{j_s} + (-1)^q \,.\, \sum_{k=1}^{s} b \,.\, b_{j_k} \otimes a_{i_1} \wedge \ldots \wedge a_{i_r} \otimes a_{j_1} \ldots \hat{a}_{j_k} \ldots a_{j_s}$$

where $b \in B^q$ and $i_1, \ldots, i_r, j_1, \ldots, j_s$ are any elements of I such that $a_{i_1}, \ldots, a_{i_r} \in P_0$ and $a_{j_1}, \ldots, a_{j_s} \in P_e$.

(β) *The k-linear map $\varphi\colon B \otimes_k P \to A$ defined by $\varphi(b \otimes a_i) = bc_i$ ($b \in B$, $i \in I$) extends in a unique way to a homomorphism $\varphi\colon C \to A$ of DG-k-algebras and the induced homomorphism $\varphi_*\colon H(C) \to H(A)$ of homology G-k-algebras is an isomorphism.*

The homomorphism φ is explicitly given by the formula

$$\varphi(b \otimes a_{i_1} \wedge \ldots \wedge a_{i_r} \otimes a_{j_1} \ldots a_{j_s}) = bc_{i_1} \ldots c_{i_r} c_{i_r} \ldots c_{j_s}$$

where b and $i_1, \ldots, i_r, j_1, \ldots, j_s$ have the same meaning as in (3.2).

Proof. The assertion (α) is trivial as well as the part of (β) concerning the existence and the explicit expression of $\varphi: C \to A$. Thus it suffices to prove that φ_* is an isomorphism.

Let us define a filtration FC of the DG-k-algebra C by the formula $F^pC = \bigoplus_{r \geqq p} B^r \otimes_k \otimes_k L(P)$. It is easy to see that φ is compatible with the filtrations FC and FA, so that φ induces a homomorphism $\varepsilon = \{\varepsilon_r\}_{r \geqq 1}$ of the spectral sequence $\{\bar{E}_r, \bar{d}_r, \bar{\iota}_r\}_{r \geqq 1}$ associated with the filtration FC of the DG-k-algebra C into the spectral sequence $\{E_r, d_r, \iota_r\}_{r \geqq 1}$. It is also clear that ε is compatible with φ_*. An easy calculation shows that we can identify $\bar{E}_2$ with $E_2^{*,0} \otimes_k L(P) = E_2^{*,0} \otimes_k E_2^{0,*}$. Under this identification the homomorphism $\varepsilon_2: \bar{E}_2 \to E_2$ coincides with the canonical isomorphism $E_2^{*,0} \otimes_k \otimes_k E_2^{0,*} \to E_2$. Therefore ε_2, and consequently φ_* is an isomorphism.

3.7. Corollary. *Let A and A' be commutative FDG-k-algebras with the unit element, and let us suppose that the spectral sequences $\{E_r, d_r, \iota_r\}_{r \geqq 1}$ and $\{E'_r, d'_r, \iota'_r\}_{r \geqq 1}$ associated with A and A', respectively, satisfy the conditions* (a) *and* (b) *of Proposition* 3.6 *with P and P', respectively. Further, let us suppose that there exists a homomorphism $\varphi: E_1^{*,0} \to E_1'^{*,0}$ of DG-k-algebras inducing an isomorphism $\varphi_*: E_2^{*,0} \approx \approx E_2'^{*,0}$ of G-k-algebras, and an isomorphism $\psi: P \to P'$ of graded k-vector spaces, which commute with the transgression in the following sense: for each $x \in P$ there exists its transgression $y \in E_1^{*,0}$ such that $\varphi(y)$ is a transgression of $\psi(x)$. Then the G-k-algebras $H(A)$ and $H(A')$ are isomorphic.*

Proof. Let us keep the notation from the proof of Proposition 3.6. We may suppose that the b_i's have been chosen in such a way that $b'_i = \varphi(b_i)$ is a transgression of $a'_i = \psi(a_i)$ for all $i \in I$. Using the elements a'_i and b'_i $(i \in I)$, let us define a DG-k-algebra C' analogously as we have defined C. Clearly φ and ψ induce a homomorphism $\varphi \otimes L(\psi): C \to C'$ of DG-k-algebras. This homomorphism is compatible with the filtrations FC and FC', and one easily verifies that it induces an isomorphism of the spectral sequences associated with FC and FC', respectively. Thus $H(C) \approx \approx H(C')$, which by virtue of Proposition 3.6 completes the proof.

3.8. Remark. As can be easily seen from the proofs of 3.6 and 3.7, we have in fact proved somewhat more, namely, that under the assumptions of Corollary 3.7 there exists an isomorphism $H(A) \approx H(A')$ of G-k-algebras compatible with the filtrations $FH(A)$ and $FH(A')$, and such that the diagram

$$\begin{tikzcd} H(B) \approx E_2^{*,0} \arrow[r, "\varphi_*"] \arrow[d] & E_2'^{*,0} \approx H(B') \arrow[d] \\ H(A) \arrow[r, "\approx"] & H(A') \end{tikzcd}$$

where the vertical homomorphisms are induced by the inclusions, commutes.

An analogous remark holds for the following special case of Corollary 3.7.

3.9. Corollary. *Let A and A' be commutative FDG-k-algebras with unit elements, and let us suppose that the multiplicative spectral sequences $\{E_r, d_r, \iota_r\}_{r\geqq 1}$ and $\{E'_r, d'_r, \iota'_r\}_{r\geqq 1}$ associated with A and A', respectively, satisfy the conditions* (a) *and* (b) *of Propositions* 3.6 *with P and P', respectively. Further, let us suppose that there is a homomorphism $\varepsilon: \{E_r, d_r, \iota_r\}_{r\geqq 1} \to \{E'_r, d'_r, \iota'_r\}_{r\geqq 1}$ of spectral sequences over k, not necessarily compatible with their multiplicative structures, which maps P isomorphicaly onto P', and the restriction $\varepsilon_1^{*,0}: E_1^{*,0} \to E_1'^{*,0}$ of which is an isomorphism of DG-k-algebras. Then the G-k-algebras $H(A)$ and $H(A')$ are isomorphic.* (See also the preceding remark.)

The remaining part of this section is devoted to *FDG-$\boldsymbol{C}$*-algebras and their real forms. Let $\hat{\mathscr{A}}$ denote the category of all *G-$\boldsymbol{C}$*-algebras or *FG-$\boldsymbol{C}$*-algebras or *DG-$\boldsymbol{C}$*-algebras or *FDG-$\boldsymbol{C}$*-algebras or bigraded $\boldsymbol{C}$-algebras, and let $\mathscr{A}$ denote the corresponding category of $\boldsymbol{R}$-algebras.

3.10. Definition. Let $\hat{A}$ be an object of $\hat{\mathscr{A}}$. An *involution of $\hat{A}$* is an automorphism of $\hat{A}$ considered as an object of $\mathscr{A}$ such that $\sigma^2 = \mathrm{id}$ and $\sigma(\mathrm{i}a) = -\mathrm{i}\sigma(a)$ for all $a \in \hat{A}$. A real form of $\hat{A}$ is a subobject A of $\hat{A}$ considered as an object of $\mathscr{A}$ such that $\hat{A} = A \oplus \mathrm{i}A$ in the corresponding category of $\boldsymbol{R}$-vector spaces.

Clearly there is a one-to-one correspondence between involutions of $\hat{A}$ and real forms of $\hat{A}$. For an involution σ of $\hat{A}$ the corresponding real form of $\hat{A}$ is $\hat{A}^\sigma = \{x \in \hat{A} : \sigma(x) = x\}$.

3.11. Let σ be an involution of $\hat{A} \in \hat{\mathscr{A}}$. Clearly we have a canonical isomorphism $\hat{A}^\sigma \otimes_{\boldsymbol{R}} \boldsymbol{C} \approx \hat{A}$ in $\hat{\mathscr{A}}$ compatible with the canonical involution of $\hat{A}^\sigma \otimes_{\boldsymbol{R}} \boldsymbol{C}$ and the involution σ of $\hat{A}$.

If $\hat{A}$ is a *DG-$\boldsymbol{C}$*-algebra or an *FDG-$\boldsymbol{C}$*-algebra, this isomorphism induces an isomorphism $H(\hat{A}^\sigma) \otimes_{\boldsymbol{R}} \boldsymbol{C} \approx H(\hat{A})$ of *G-$\boldsymbol{C}$*-algebras or *FG-$\boldsymbol{C}$*-algebras, respectively. On the other hand, σ clearly induces an involution of $H(\hat{A})$, which we also denote by σ, and there is a canonical isomorphism $H(\hat{A})^\sigma \otimes_{\boldsymbol{R}} \boldsymbol{C} \approx H(\hat{A})$. It is easy to see that both these isomorphisms can be identified by means of the canonical isomorphism $H(\hat{A}^\sigma) \approx H(\hat{A})^\sigma$ induced by the inclusion $\hat{A}^\sigma \subset \hat{A}$.

Similarly, if $\hat{A}$ is an *FG-$\boldsymbol{C}$*-algebra, there is an induced involution σ of $\mathscr{G}rF\hat{A}$, the canonical isomorphism $\hat{A}^\sigma \otimes_{\boldsymbol{R}} \boldsymbol{C} \approx \hat{A}$ induces a canonical isomorphism $(\mathscr{G}rF\hat{A}^\sigma) \otimes_{\boldsymbol{R}} \otimes_{\boldsymbol{R}} \boldsymbol{C} \approx \mathscr{G}rF\hat{A}$ of bigraded $\boldsymbol{C}$-algebras, and $\mathscr{G}rF\hat{A}^\sigma \approx (\mathscr{G}rF\hat{A})^\sigma$ canonically.

In what follows we shall regard all the canonical isomorphisms of this section as identifications.

3.12. Lemma. *Let $\hat{A}$ be a G-$\boldsymbol{C}$-algebra (FG-$\boldsymbol{C}$-algebra, DG-$\boldsymbol{C}$-algebra, FDG-$\boldsymbol{C}$-algebra, bigraded $\boldsymbol{C}$-algebra), and let σ and τ be two commuting involutions in $\hat{A}$. Let us define $\boldsymbol{C}$-linear maps $\varphi, \psi: \hat{A} \to \hat{A}$ by*

$$\varphi = \frac{1}{2}(\mathrm{id} + \sigma \circ \tau) + \frac{\mathrm{i}}{2}(\mathrm{id} - \sigma \circ \tau), \tag{3.3}$$

$$\psi = \frac{1}{2}(\mathrm{id} + \sigma \circ \tau) - \frac{\mathrm{i}}{2}(\mathrm{id} - \sigma \circ \tau).$$

Then: (a) φ *and* ψ *are automorphisms of the G-**C**-vector space (FG-**C**-vector space, DG-**C**-vector space, FDG-**C**-vector space, bigraded **C**-vector space)* $\hat{A}$ *and* $\varphi \circ \psi = \psi \circ \varphi = \mathrm{id}$.

(b) $\varphi(\hat{A}^\sigma) = \hat{A}^\tau$, $\varphi(\hat{A}^\tau) = \hat{A}^\sigma$, $\psi(\hat{A}^\sigma) = \hat{A}^\tau$, $\psi(\hat{A}^\tau) = \hat{A}^\sigma$.

Proof is trivial. We only remark that the multiplicative structure of $\hat{A}$ plays no role here.

3.13. Lemma. *Let $\hat{A}$ be a G-**C**-algebra, and let σ and τ be two commuting involutions of $\hat{A}$. If $\hat{A}^\sigma$ is a free commutative G-**R**-algebra $L_{\boldsymbol{R}}(P)$ with unit over a τ-invariant G-**R**-submodule $P \subset \hat{A}^\sigma$, then $\hat{A}^\tau$ is a free commutative G-**R**-algebra with unit over the σ-invariant G-**R**-submodule $\varphi(P) = \psi(P) \subset \hat{A}^\tau$.*

Proof. Clearly $\hat{A} = \hat{A}^\sigma \otimes_{\boldsymbol{R}} \boldsymbol{C}$ is a free commutative G-$\boldsymbol{C}$-algebra over P. However, since $P = P^\tau \oplus P^{-\tau}$, where $P^\tau = P \cap \hat{A}^\tau$ and $P^{-\tau} = P \cap \mathrm{i}\hat{A}^\tau$, and $\varphi(a) = a$ for $a \in \hat{A}^\sigma \cap \hat{A}^\tau$ and $\varphi(b) = \mathrm{i}b$ for $b \in \hat{A}^\sigma \cap \mathrm{i}A^\tau$, it is also a free commutative G-$\boldsymbol{C}$-algebra $L_{\boldsymbol{C}}(\varphi(P))$ with unit over $\varphi(P) = \varphi(P^\tau) \oplus \varphi(P^{-\tau}) = P^\tau \oplus \mathrm{i}P^{-\tau}$. Using the obvious fact that the canonical isomorphism $L_{\boldsymbol{C}}(\varphi(P)) \approx \hat{A}$ induced by the inclusion $\varphi(P) \subset \hat{A}^\tau \subset \hat{A}$ is compatible with the canonical involution of $L_{\boldsymbol{C}}(\varphi(P)) = L_{\boldsymbol{R}}(\varphi(P)) \otimes_{\boldsymbol{R}} \boldsymbol{C}$ and the involution τ of $\hat{A}$, we immediately conclude that $\hat{A}^\tau$ is a free commutative G-$\boldsymbol{R}$-algebra with unit over $\varphi(P)$.

3.14. Remark. Let $\mu_\sigma\colon L_{\boldsymbol{R}}(P) \to \hat{A}^\sigma$ and $\mu_\tau\colon L_{\boldsymbol{R}}(\varphi(P)) \to A$ extend the inclusions $P \subset \hat{A}^\sigma$ and $\varphi(P) \subset \hat{A}^\tau$, respectively, and let us define an $\boldsymbol{R}$-linear map $\Phi\colon L_{\boldsymbol{R}}(P) \to L_{\boldsymbol{R}}(\varphi(P))$ by putting

$$\Phi(a_1 \dots a_k b_1 \dots b_l) = (-1)^{[l/2]}\, \varphi(a_1) \dots \varphi(a_k) \,.\, \varphi(b_1) \dots \varphi(b_l) = (-1)^{[l/2]}\, a_1 \dots a_k \,.\, (\mathrm{i}b_1) \dots (\mathrm{i}b_l)$$

for $a_1, \dots, a_k \in P^\tau$ and $b_1, \dots, b_l \in P^{-\tau}$, where $[l/2]$ denotes the integer part of $l/2$. It is easy to verify that Φ is a well defined isomorphism of G-$\boldsymbol{R}$-modules and that the diagram

$$\begin{array}{ccc} L_{\boldsymbol{R}}(P) & \xrightarrow{\mu_\sigma} & \hat{A}^\sigma \\ \approx\big\downarrow \Phi & & \approx\big\downarrow \varphi \\ L_{\boldsymbol{R}}(\varphi(P)) & \xrightarrow{\mu_\tau} & \hat{A}^\tau \end{array}$$

commutes. This gives an alternative proof of the preceding lemma.

3.15. Lemma. *Let $\hat{A}$ be a complex bigraded algebra, and let σ be an involution of $\hat{A}$. Then the canonical homomorphism $(\hat{A}^\sigma)^{p,0} \otimes_{\boldsymbol{R}} (\hat{A}^\sigma)^{0,q} \to (\hat{A}^\sigma)^{p,q}$ is an isomorphism if and only if the canonical homomorphism $\hat{A}^{p,0} \otimes_{\boldsymbol{C}} \hat{A}^{0,q} \to \hat{A}^{p,q}$ is an isomorphism.*

3.16. Definition. An involution of a multiplicative spectral sequence $\hat{E} =$

$= \{\hat{E}_r, \hat{d}_r, \iota_r\}_{r \geqq k}$ over the field $\boldsymbol{C}$ is an automorphism σ of the multiplicative spectral sequence $\hat{E}$ over $\boldsymbol{R}$ such that $\sigma^2 = \mathrm{id}$ and $\sigma(ia) = -i\delta(a)$.

Clearly, for every $r \geqq k$ we then have a differential $\hat{d}_r^\sigma \colon \hat{E}_r^\sigma \to \hat{E}_r^\sigma$ induced by the differential $\hat{d}_r$, an isomorphism $\iota_r^\sigma \colon \hat{E}_{r+1}^\sigma \approx H(\hat{E}_r^\sigma, \hat{d}_r^\sigma)$ of bigraded $\boldsymbol{R}$-algebras induced by ι_r, and $\hat{E}^\sigma = \{\hat{E}_r^\sigma, \hat{d}_r^\sigma, \iota_r^\sigma\}_{r \geqq k}$ is a multiplicative spectral sequence over $\boldsymbol{R}$.

We remark that σ induces an involution of $\hat{E}_\infty$ (also denoted by σ), and that $\hat{E}_\infty^\sigma$ can be canonically identified with $(\hat{E}_\infty)^\sigma$.

The following lemma is an immediate consequence of Lemma 3.12.

3.17. Lemma. *Let σ and τ be two commuting involutions of a multiplicative spectral sequence $\hat{E} = \{\hat{E}_r, \hat{d}_r, \iota_r\}_{r \geqq k}$ over $\boldsymbol{C}$, and let us define $\boldsymbol{C}$-linear maps $\varphi, \psi \colon \hat{E} \to \hat{E}$ by the formulae* (3.3). *Then*:

(a) *φ and ψ are automorphisms of the additive spectral sequence $\hat{E}$ over $\boldsymbol{C}$* (i.e. *the multiplicative structure of $\hat{E}$ is not taken into account*), *and $\varphi \circ \psi = \psi \circ \varphi = \mathrm{id}$.*

(b) $\varphi(\hat{E}^\sigma) = \hat{E}^\tau$, $\varphi(\hat{E}^\tau) = \hat{E}^\sigma$, $\psi(\hat{E}^\sigma) = \hat{E}^\tau$, *and* $\psi(\hat{E}^\tau) = \hat{E}^\sigma$.

(Again the multiplicative structure of E plays no role here.)

3.18. Let $\hat{A}$ be an *FDG*-$\boldsymbol{C}$-algebra and let σ be an involution of $\hat{A}$. Then σ induces an involution, which we denote again by σ, of the multiplicative spectral sequence $\hat{E} = \{\hat{E}_r, \hat{d}_r, \iota_r\}_{r \geqq 1}$ over $\boldsymbol{C}$ associated with $\hat{A}$. It is easy to see that we can canonically identify the multiplicative spectral sequence $E_\sigma = \{E_{\sigma r}, d_{\sigma r}, \iota_{\sigma r}\}_{r \geqq 1}$ over $\boldsymbol{R}$ associated with the *FDG*-$\boldsymbol{R}$-algebra $\hat{A}^\sigma$ and the multiplicative spectral sequence $\hat{E}^\sigma = \{\hat{E}_r^\sigma, \hat{d}_r^\sigma, \iota_r^\sigma\}_{r \geqq 1}$ over $\boldsymbol{R}$.

If σ and τ are two commuting involutions of $\hat{A}$, the induced involutions σ and τ of $\hat{E}$ also commute, and the automorphisms φ and ψ of $\hat{E}$ defined in Lemma 3.17 are induced, respectively, by the automorphisms φ and ψ of $\hat{A}$ defined in Lemma 3.12.

The following proposition is the main result of this section.

3.19. Proposition. *Let $\hat{A}$ be a commutative FDG-$\boldsymbol{C}$-algebra with unit, and let σ and τ be two commuting involutions of $\hat{A}$. Let $\hat{E} = \{\hat{E}_r, \hat{d}_r, \iota_r\}_{r \geqq 1}$ be the multiplicative spectral sequence over $\boldsymbol{C}$ associated with $\hat{A}$, and let the induced involutions σ and τ of $\hat{E}$ coincide on $\hat{E}_1^{*,0} = \bigoplus_p \hat{E}_1^{p,0}$. Finally, let φ be the automorphism of $\hat{E}$ defined in Lemma* 3.17.

Under these assumptions, if the multiplicative spectral sequence $E = \hat{E}^\tau = E_\tau$ over $\boldsymbol{R}$ associated with the FDG-$\boldsymbol{R}$-algebra $\hat{A}^\tau$ satisfies the conditions of Proposition 3.6 *with $P = P_\tau$ invariant under σ, then the multiplicative spectral sequence $E = \hat{E}^\sigma = E_\sigma$ over $\boldsymbol{R}$ associated with the FDG-$\boldsymbol{R}$-algebra $\hat{A}^\sigma$ satisfies these conditions with $P = P_\sigma$, where $P_\sigma = \varphi(P_\tau)$ is invariant under τ, and the G-$\boldsymbol{R}$-algebras $H(\hat{A}^\tau)$ and $H(\hat{A}^\sigma)$ are isomorphic.*

Proof. Applying Lemma 3.13 to the *G*-$\boldsymbol{C}$-algebra $\hat{E}_2^{0,*} = \bigoplus_q \hat{E}_2^{0,q}$ we immediately see that $E_{\sigma,2}^{0,*}$ is a free *G*-$\boldsymbol{R}$-algebra over the *G*-$\boldsymbol{R}$-vector subspace $P_\sigma = \varphi(P_\tau)$. By virtue of Lemma 3.17, $\varphi: E_\tau \to E_\sigma$ is an isomorphism of additive spectral sequences (i.e., the multiplicative structures of E_τ and E_σ are not considered here), which implies that P_τ is spanned by transgressive elements if and only if $P_\sigma = \varphi(P_\tau)$ is. It follows that E_σ satisfies the condition (a) of Proposition 3.6. The condition (b) for E_σ follows from Lemma 3.15 when applied twice to the complex bigraded algebra $\hat{E}_2$. Since φ is the identity on $\hat{E}_1^{*,0}$, the rest of the proposition follows from Corolary 3.9.

3.20. Remark. Applying in the preceding proof Remark 3.8 instead of Corollary 3.9, we get that under that assumptions of Proposition 3.19 there exists a commutative diagram of G-$\boldsymbol{R}$-algebras

$$\begin{tikzcd} & H(B) \arrow[dl] \arrow[dr] & \\ H(\hat{A}^\tau) \arrow[rr, "\approx"] & & H(\hat{A}^\sigma) \end{tikzcd}$$

where $B = E_{\sigma,1}^{*,0} = E_{\tau,1}^{*,0}$ is a DG-$\boldsymbol{R}$-subalgebra of both DG-$\boldsymbol{R}$-algebras $\hat{A}^\sigma$ and $\hat{A}^\tau$ (see 3.4).

4. INVARIANT DE RHAM COHOMOLOGY OF REAL FORMS OF PRINCIPAL BUNDLES WITH COMPLEX LIE STRUCTURE GROUPS

Let Λ be a finite dimensional commutative associative algebra over $\boldsymbol{R}$ with unit, and let $\hat{\Lambda}$ be an algebra of the same type over $\boldsymbol{C}$.

4.1. Given a manifold M, we denote by $A(M; \Lambda)$ the de Rham DG-$\boldsymbol{R}$-algebra of differential forms on M with values in Λ and by $H_{DR}(M; \Lambda)$ its cohomology G-$\boldsymbol{R}$-algebra. For a map $f: M \to M'$, the both induced homomorphisms $A(M'; \Lambda) \to \to A(M; \Lambda)$ and $H_{DR}(M'; \Lambda) \to H_{DR}(M; \Lambda)$ are denoted by f^*.

4.2. Now we shall recall some basic facts about the invariant de Rham cohomology. More details can be found e.g. in [6].

Let G be a Lie group and M a right G-manifold, i.e. a manifold on which G operates from the right. For $g \in G$, let $R_g: M \to M$ be the right translation by g defined by $R_g(x) = x \,.\, g$ for $x \in M$. A form $\omega \in A(M; \Lambda)$ is called *invariant* if $R_g^*(\omega) = \omega$ for all $g \in G$. Clearly, the set of all invariant forms is a DG-$\boldsymbol{R}$-subalgebra $A_I(M; \Lambda)$ of $A(M; \Lambda)$. The cohomology G-$\boldsymbol{R}$-algebra of $A_I(M; \Lambda)$ is denoted by $H_{IDR}(M; \Lambda)$ and called the *invariant de Rham cohomology algebra of M with coefficients in Λ.*

It is easy to see that $A_I(-; \Lambda)$ and $H_{IDR}(-; \Lambda)$ may be considered as contravariant functors on the appropriate category. If $\gamma: G \to G'$ is a homomorphism of Lie groups and $f: M \to M'$ is a γ-equivariant map of a G-manifold M into a G'-manifold M', then f induces homomorphisms $A_I(M'; \Lambda) \to A_I(M; \Lambda)$ and $H_{IDR}(M'; \Lambda) \to \to H_{IDR}(M; \Lambda)$. Both these homomorphisms will be denoted again by f^*. The same argument as in the case of the ordinary de Rham cohomology shows that γ-equivariantly homotopic maps $f_0, f_1: M \to M'$ induce the same homomorphism of invariant de Rham cohomology, i.e. $f_0^* = f_1^*: H_{IDR}(M'; \Lambda) \to H_{IDR}(M; \Lambda)$.

We shall need the following result of C. Chevalley and S. Eilenberg [5] concerning the canonical homomorphism

$$H_{IDR}(M; A) \to H_{DR}(M; A) \tag{4.1}$$

(see also [6, vol. II, p. 163]).

4.3. Proposition. *Let M be a G-manifold. If the Lie group G is compact and connected, then the canonical homomorphism* (4.1) *is an isomorphism.*

4.4. A Lie group G will always be considered as a right G-manifold. Each left translation $L_g: G \to G$ of G by an element $g \in G$ is an equivariant diffeomorphism of the right G-manifold G, and therefore it induces an automorphism L_g^* of the G-$\boldsymbol{R}$-algebra $H_{IDR}(G; A)$. If g belongs to the connected component G_0 of the unit element $e \in G$, then L_g is equivariantly homotopic to $L_e = \mathrm{id}$, and therefore L_g^* is the identity automorphism of $H_{IDR}(G; A)$. It follows that the group $\pi_0(G) = G/G_0$ of connected components of the group G operates (in a canonical way) from the right on the G-$\boldsymbol{R}$-algebra $H_{IDR}(G; A)$.

4.5. Lemma. *Let M be a manifold, and let a Lie group G operate on* $M \times G$ *by the canonical right action. Then the canonical homomorphisms*

$$A(M; \boldsymbol{R}) \otimes_{\boldsymbol{R}} A_I(G; A) \to A_I(M \times G; A),$$

$$H_{DR}(M; R) \otimes_{\boldsymbol{R}} H_{IDR}(G; A) \to H_{IDR}(M \times G; A)$$

are isomorphisms.

In particular, if M is connected and $H^p_{DR}(M; \boldsymbol{R}) = 0$ *for* $p > 0$, *then the canonical projection* $M \times G \to G$ *induces an isomorphism* $H_{IDR}(G; A) \approx H_{IDR}(M \times G; A)$.

Proof is easy.

4.6. Lemma. *Let* $\xi = (P, p, M, G)$ *be a principal G-bundle and let* $\mathscr{H}_{IDR}(p; A)$ *be the sheaf on M generated by the presheaf* $U \mapsto H_{IDR}(p^{-1}(U); A)$. *If the group* $\pi_0(G)$ *operates trivially on* $H_{IDR}(G; A)$, *the sheaf* $\mathscr{H}_{IDR}(p; A)$ *is canonically isomorphic to the constant sheaf on M with the stalk* $H_{IDR}(G; A)$.

Proof follows immediately from the special part of the preceding lemma.

4.7. Now let M be a complex manifold. In this case the tangent bundle of M (considered as a real manifold) has a canonical structure of a holomorphic complex vector bundle so that it makes sense to speak about holomorphic vector fields on open subsets of M and about multiplication of vectors and vector fields by complex numbers. More explicitly, a vector field X on an open subset of M will be called *holomorphic* if it is a holomorphic section of the tangent bundle of M or, equivalently, if $\mathscr{L}_X J = 0$, where $\mathscr{L}_X$ denotes the Lie derivative with respect to X, and J is the associated integrable almost complex structure. We remark that this conception is a little bit nonstandard because our holomorphic fields are real vector fields. Let us notice that a vector field X on an open subset $U \subset M$ is holomorphic if and only if $[X, iY] = i[X, Y]$ for all vector fields Y on U.

A differential form ω on M with values in $\hat{\Lambda}$ will be called holomorphic if it is $\boldsymbol{C}$-linear, and for any holomorphic vector fields $X_1, \ldots, X_p$ on an open subset $U \subset M$, the map $\omega(X_1, \ldots, X_p)\colon U \to \hat{\Lambda}$ is holomorphic. Using the property of holomorphic vector fields mentioned above one can easily verify that holomorphic forms constitute a DG-$\boldsymbol{C}$-subalgebra $A_H(M; \hat{\Lambda})$ of the DG-$\boldsymbol{C}$-algebra $A(M; \hat{\Lambda})$.

4.8. Now let us consider a principal $\hat{G}$-bundle $\hat{\xi} = (\hat{P}, \hat{p}, M, \hat{G})$ with $\hat{G}$ a complex Lie group. A differential form ω on $\hat{P}$ with values in $\hat{\Lambda}$ will be called *vertically holomorphic* if the restriction of ω to any fibre of $\hat{\xi}$ is a holomorphic form (this makes sense since each fibre of $\hat{\xi}$ has a canonical structure of complex manifold). It is clear from the properties of invariant and holomorphic forms that all invariant vertically holomorphic differential forms on $\hat{P}$ with values in $\hat{\Lambda}$ constitute a DG-$\boldsymbol{C}$-subalgebra of the DG-$\boldsymbol{C}$-algebra $A_I(\hat{P}; \hat{\Lambda})$. We shall denote this DG-$\boldsymbol{C}$-algebra by $A_{IVH}(P; \hat{\Lambda})$. It will play an important role in all the rest of this section.

4.9. Lemma. *Let $\hat{\xi} = (\hat{P}, \hat{p}, M, \hat{G})$ be a principal $\hat{G}$-bundle with $\hat{G}$ a complex Lie group, let (Σ, σ) be an involution of $\hat{\xi}$, and let $\hat{\Lambda} = \Lambda \otimes_{\boldsymbol{R}} \boldsymbol{C}$. Then the formula*

$$s(\omega) = \overline{\Sigma^*(\omega)}\,, \quad \omega \in A_{IVH}(\hat{P}; \hat{\Lambda})\,, \tag{4.2}$$

where the bar denotes the canonical involution in $\hat{\Lambda}$, defines an involution s of the DG-$\boldsymbol{C}$-algebra $A_{IVH}(\hat{P}; \hat{\Lambda})$.

Proof. We have to prove that $s(\omega)$ is invariant, that its restriction $s(\omega)/\hat{p}^{-1}(x)$ to a fibre $\hat{p}^{-1}(x)$ is holomorphic for any $x \in M$, and that $d(s(\omega)) = s(d(\omega))$, where d is the exterior differential. The first property follows easily from the relation $\Sigma \circ R_g = R_{\sigma(g)} \circ \Sigma$ holding for all $g \in \hat{G}$ (see Definition 2.15). Since $s(\omega)$ is already known to be invariant, it has the second property if and only if $s(\omega)/\hat{p}^{-1}(x)$ is $\boldsymbol{C}$-multilinear for any $x \in M$. But this is obvious in the special case $M = \{x\}$, $\hat{P} = \hat{G}$ and $\Sigma = \sigma$, and the general case reduces immediately to this special one. Finally, the third property is obvious.

4.10 Lemma. *Let $\hat{\xi} = (\hat{P}, \hat{p}, M, \hat{G})$ be a principal $\hat{G}$-bundle with $\hat{G}$ a complex Lie group, let $\xi = (P, p, M, G)$ be a real form of $\hat{\xi}$ associated with an involution (Σ, σ) of $\hat{\xi}$, let $\hat{\Lambda} = \Lambda \otimes_{\boldsymbol{R}} \boldsymbol{C}$, and let us identify Λ with the subalgebra $\Lambda \otimes 1 \subset \hat{\Lambda}$. If s is the involution of the DG-$\boldsymbol{C}$-algebra $A_{IVH}(\hat{P}; \hat{\Lambda})$ defined by the formula (4.2), then for a form $\omega \in A_{IVH}(\hat{P}; \hat{\Lambda})$ we have $s(\omega) = \omega$ if and only if the restriction ω/P of ω to P is a form with values in Λ, and the correspondence $\omega \mapsto \omega/P$ defines an isomorphism $A_{IVH}(\hat{P}; \hat{\Lambda})^s \approx A_I(P; \Lambda)$ of DG-$\boldsymbol{R}$-algebras.*

Proof. Let $\omega \in A^k_{IVH}(\hat{P}; \hat{\Lambda})$. If $s(\omega) = \omega$, then for any point $y \in \hat{P}$ and any vectors $V_1, \ldots, V_k \in T_yP \subset T_y\hat{P}$ we have

$$\omega(V_1, \ldots, V_k) = s(\omega)\,(V_1, \ldots, V_k) = \overline{\omega(d\Sigma(V_1), \ldots, d\Sigma(V_k))} = \omega(V_1, \ldots, V_k)$$

since $\Sigma = \text{id}$ on P, and therefore $d\Sigma = \text{id}$ on TP. This shows that ω/P takes values in Λ.

Conversely, let us suppose that the restriction ω/P takes values in Λ. Let $\hat{y} \in \hat{P}$, and let $\hat{V}_1, \dots, \hat{V}_k \in T_{\hat{y}}\hat{P}$ be any vectors. We can choose a point $y \in P$, an element $g \in \hat{G}$, and vectors $V_1, \dots, V_k \in T_y P$ such that $\hat{y} = R_g(y)$ and $dR_g(V_j) = \hat{V}_j$ for $j = 1, 2, \dots, k$. It is easy to see that $T_y\hat{P} = T_y P \oplus T_y^v P$, where $T_y^v P$ denotes the tangent space at y to the fibre of ξ through the point y. By virtue of this decomposition we can write $V_j = V_j' + \mathrm{i}V_j''$ with $V_j' \in T_y P$ and $V_j'' \in T_y^v P$ for $j = 1, \dots, k$. Now we have

$$\begin{aligned}
s(\omega)(\hat{V}_1, \dots, \hat{V}_k) &= s(\omega)(dR_g(V_1), \dots, dR_g(V_k)) = s(\omega)(V_1, \dots, V_k) = \\
&= \overline{\omega(d\Sigma(V_1), \dots, d\Sigma(V_k))} = \overline{\omega(V_1' - \mathrm{i}V_1'', \dots, V_k' - \mathrm{i}V_k'')} = \\
&= \omega(V_1' + \mathrm{i}V_1'', \dots, V_k' + \mathrm{i}V_k'') = \omega(V_1, \dots, V_k) = \\
&= \omega(dR_g(V_1), \dots, dR_g(V_k)) = \omega(\hat{V}_1, \dots, \hat{V}_k),
\end{aligned}$$

which proves that $s(\omega) = \omega$. We remark that the fifth equality above holds because ω/P takes values in Λ, and that we have used the fact that the tangent space $T_y^v\hat{P}$ has the canonical structure of a complex vector space, and $d\Sigma\colon T_y^v\hat{P} \to T_y^v\hat{P}$ satisfies $d\Sigma(\mathrm{i}\hat{V}) = -\mathrm{i}d\Sigma(\hat{V})$ for all $\hat{V} \in T_y^v\hat{P}$.

If $s(\omega) = \omega$, the above considerations show that $\omega(\hat{V}_1, \dots, \hat{V}_k) = \omega(V_1, \dots, V_k)$. This implies that $\omega = 0$ if $\omega/P = 0$, and that any form $\omega' \in A_I(P; \Lambda)$ can be uniquely extended to a form $\omega \in A_{IVH}(\hat{P}; \hat{\Lambda})^s$. This proves the last assertion of the lemma and completes the proof.

Now we are ready to prove the main results of this section.

4.11. Proposition. *Let $\hat{\xi} = (\hat{P}, \hat{p}, M, \hat{G})$ be a principal $\hat{G}$-bundle with $\hat{G}$ a reductive connected complex Lie group without non-discrete compact complex Lie subgroups (or equivalently, without complex tori), and let $\xi_i = (P_i, p_i, M, G_i)$, $i = 1, 2$, be any two regular forms of $\hat{\xi}$. Then the G-**R**-algebras $H_{IDR}(P_1; \Lambda)$ and $H_{IDR}(P_2; \Lambda)$ are isomorphic.*

Proof. Since clearly $A_I(P_i; \Lambda) \approx A_I(P_i; \boldsymbol{R}) \otimes_{\boldsymbol{R}} \Lambda$ and therefore $H_{IDR}(P_i; \Lambda) \approx H_{IDR}(P_i; \boldsymbol{R}) \otimes_{\boldsymbol{R}} \Lambda$, if suffices to consider the case $\Lambda = \boldsymbol{R}$. Moreover, obviously we may suppose that M is connected.

(a) Let $i = 1$ or 2 be fixed. Let $\mathscr{A}(M; \boldsymbol{R})$ be the de Rham sheaf on M, i.e. $\mathscr{A}(M; \boldsymbol{R})(U) = A(U; \boldsymbol{R})$ for $U \subset M$, U open, and let us define DG-$\boldsymbol{R}$-algebra sheaves $\mathscr{A}_I(p_i; \boldsymbol{R})$ and $\mathscr{A}_i$ on M by the formulae

$$\mathscr{A}_I(p_i; \boldsymbol{R})(U) = A_I(p_i^{-1}(U); \boldsymbol{R}) \quad \text{for} \quad U \subset M, \quad U \text{ open},$$

$$\mathscr{A}_i = \mathscr{A}(M; \boldsymbol{R}) \otimes_{\boldsymbol{R}} \mathscr{A}_I(p_i; \boldsymbol{R}).$$

There is a canonical monomorphism

$$\mathscr{A}_I(p_i; \boldsymbol{R}) \to \mathscr{A}_i \tag{4.3}$$

of DG-$\boldsymbol{R}$-algebra sheaves, and therefore also a canonical monomorphism

$$A_I(P_i; \boldsymbol{R}) \to A_i \tag{4.4}$$

of DG-$\boldsymbol{R}$-algebras, where we have put $A_i = \mathscr{A}_i(M)$. We shall prove that (4.4) induces an isomorphism $H_{IDR}(P_i; \boldsymbol{R}) \approx H(A_i)$ of cohomology G-$\boldsymbol{R}$-algebras.

Each of the sheaves in (4.3) has a canonical structure of a module over the sheaf $\mathscr{A}^0(M; \boldsymbol{R})$. Since the sheaf $\mathscr{A}^0(M; \boldsymbol{R})$ is fine, the same is true for both sheaves in (4.3). Further, the Künneth theorem yields that (4.3) induces an isomorphism $\mathscr{H}(\mathscr{A}_i(p_i; \boldsymbol{R})) \approx \mathscr{H}(\mathscr{A}_i)$ of homology sheaves. Consequently, we may apply [3, chap. IV, Theorem 2.2], which immediately implies that (4.4) induces an isomorphism of cohomology G-$\boldsymbol{R}$-algebras.

(b) Let (Σ_i, σ_i) be the involution of $\hat{\xi}$ associated with the real form ξ_i, and let us suppose that the involutions (Σ_1, σ_1) and (Σ_2, σ_2) commute. Let us now consider a DG-$\boldsymbol{R}$-algebra sheaf $\mathscr{A}_{IVH}(\hat{p}; \boldsymbol{C})$ on M defined by

$$\mathscr{A}_{IVH}(\hat{p}; \boldsymbol{C})(U) = A_{IVH}((\hat{p})^{-1}(U); \boldsymbol{C}) \quad \text{for} \quad U \subset M\,, \quad U \text{ open}\,,$$

and a FDG-$\boldsymbol{C}$-algebra sheaf $\hat{\mathscr{A}}$ on M defined by the formulae

$$\hat{\mathscr{A}} = \mathscr{A}(M; \boldsymbol{R}) \otimes_{\boldsymbol{R}} \mathscr{A}_{IVH}(\hat{p}; \boldsymbol{C})\,,$$

$$F^p\hat{\mathscr{A}} = \bigoplus_{r \geqq p} \mathscr{A}^r(M; \boldsymbol{R}) \otimes_{\boldsymbol{R}} \mathscr{A}_{IVH}(\hat{p}; \boldsymbol{C})\,.$$

By Lemma 4.9, the involution (Σ_i, σ_i) induces in a canonical way an involution of the DG-$\boldsymbol{C}$-algebra sheaf $\mathscr{A}_{IVH}(\hat{p}; \boldsymbol{C})$, and therefore also an involution of the FDG-$\boldsymbol{C}$-algebra sheaf $\hat{\mathscr{A}}$. We denote both these involutions by s_i. Furthermore the involution s_i of $\hat{\mathscr{A}}$ induces an involution of the FDG-$\boldsymbol{C}$-algebra $\hat{A} = \hat{\mathscr{A}}(M)$, which will be denoted by the same symbol s_i. It is clear that in each case we have $s_1 \circ s_2 = s_2 \circ s_1$. Let us further consider $\mathscr{A}_i$ as an FDG-$\boldsymbol{R}$-algebra sheaf with a filtration given by the formula

$$F^p\mathscr{A}_i = \bigoplus_{r \geqq p} \mathscr{A}^r(M; \boldsymbol{R}) \otimes_{\boldsymbol{R}} \mathscr{A}_I(p_i; \boldsymbol{R})\,,$$

and A_i as a FDG-$\boldsymbol{R}$-algebra with the filtration

$$F^pA_i = (F^p\mathscr{A}_i)(M)\,.$$

Lemma 4.10 implies that the FDG-$\boldsymbol{R}$-algebra sheaf $\hat{\mathscr{A}}^{s_i}$ can be identified with the FDG-$\boldsymbol{R}$-algebra sheaf $\mathscr{A}_i$, and therefore the FDG-$\boldsymbol{R}$-algebra $\hat{A}^{s_i}$ can be identified with the FDG-$\boldsymbol{R}$-algebra A_i. It means that A_1 and A_2 are two commuting real forms of the FDG-$\boldsymbol{C}$-algebra $\hat{A}$. Clearly, $\hat{A}$ is commutative (in graded sense) and has a unit. Consequently, if we knew that $\hat{A}$ and $\sigma = s_1$, $\tau = s_2$ satisfy the conditions of Proposition 3.19, we could conclude by applying this proposition that the cohomology G-$\boldsymbol{R}$-algebras $H(A_1)$ and $H(A_2)$ are isomorphic.

(c) Let $\xi = (P, p, M, G)$ be a quasicompact real form of $\hat{\xi}$, and for ξ let us define a DG-$\boldsymbol{R}$-algebra sheaf $\mathscr{A}_I(p; \boldsymbol{R})$, a FDG-$\boldsymbol{R}$-algebra sheaf $\mathscr{A}$, and a FDG-$\boldsymbol{R}$-algebra A

in the same way as we have defined for ξ_i the sheaves $\mathscr{A}_I(p_i; \boldsymbol{R})$, $\mathscr{A}_i$ and the algebra A_i. We shall show that the multiplicative spectral sequence $E = \{E_r, d_r, \iota_r\}_{r \geqq 1}$ over $\boldsymbol{R}$ associated with the FDG-$\boldsymbol{R}$-algebra A satisfies the conditions of Proposition 3.6 with $P = P_0$ being invariant under any automorphism of E induced by a γ-automorphism of ξ, where γ is an automorphism of G.

An easy calculation using the exactness of the functor $\mathscr{S} \mapsto \mathscr{S}(M)$ on the subcategory of fine sheaves of abelian groups and Lemma 4.6 shows that there is an isomorphism

$$E_2 \approx H_{DR}(M; \boldsymbol{R}) \otimes_{\boldsymbol{R}} H_{IDR}(G; \boldsymbol{R}) \tag{4.5}$$

of bigraded $\boldsymbol{R}$-algebras. Consequently,

$$E_2^{0,*} \approx H_{IDR}(G; \boldsymbol{R}) \tag{4.6}$$

as G-$\boldsymbol{R}$-algebras.

It follows immediately from (4.5) that the spectral sequence E satisfies the condition (b) of Proposition 3.6. Let γ be an automorphism of G, and g a γ-automorphism of ξ. It is easy to see that via (4.6) the automorphism g^* of $E_2^{0,*}$ induced by g becomes the automorphism γ^* of $H_{IDR}(G; \boldsymbol{R})$. Consequently, it remains to show that $H_{IDR}(G; \boldsymbol{R})$ is the exterior algebra over a vector subspace P invariant under all γ^*, γ being an automorphism of G.

Let $L(G)$ be the Lie algebra of G, and let G_s and G_c be the integral subgroups of G corresponding to the derived subalgebra $[L(G), L(G)]$ and the centre of $L(G)$, respectively. Since G_c is a central subgroup of G, the multiplication in G defines a canonical homomorphism $G_s \times G_c \to G$. By assumption, G is a direct product (both in algebraic and topological sense) of a maximal compact subgroup and a subgroup isomorphic to $\boldsymbol{R}^d$ for some d. Using this direct product decomposition, and arguing similarly as in the proof of Proposition 2.9, it is easy to show that G_c is closed, that $G_s \times G_c \to G$ is a covering map with finite multiplicity, and that G_s is compact. It follows that there are canonical isomorphisms

$$A_I(G; \boldsymbol{R}) \approx A_I(G_s \times G_c; \boldsymbol{R}) \approx A_I(G_s; \boldsymbol{R}) \otimes_{\boldsymbol{R}} A_I(G_c; \boldsymbol{R})$$

of DG-$\boldsymbol{R}$-algebras, which together with Künneth theorem and Proposition 4.3 yields a canonical isomorphism

$$H_{IDR}(G; \boldsymbol{R}) \approx H_{DR}(G_s; \boldsymbol{R}) \otimes_{\boldsymbol{R}} H_{IDR}(G_c; \boldsymbol{R}) \tag{4.7}$$

of G-$\boldsymbol{R}$-algebras. By [6, vol. II, Chap. IV, Theorem IV], we have

$$H_{DR}(G_s; \boldsymbol{R}) \approx \Lambda(P_s)\,,$$

where P_s is the subspace of primitive elements in $H_{DR}(G; \boldsymbol{R})$. Further, since G_c is commutative, $H_{IDR}(G_c; \boldsymbol{R}) \approx A_I(G_c; \boldsymbol{R}) \approx \Lambda(A_I^1(G_c; \boldsymbol{R}))$, and therefore

$$H_{IDR}(G_c; \boldsymbol{R}) \approx \Lambda(P_c)$$

with $P_c = H^1_{IDR}(G_c; \boldsymbol{R})$. All these facts together yield that $H_{IDR}(G; \boldsymbol{R})$ is the exterior algebra over the subspace P corresponding via the isomorphism (4.7) to $P_s \otimes 1 + 1 \otimes P_c$. It remains to show that $\gamma^*(P) = P$ for any automorphism γ of G, but this follows immediately from the corresponding property of P_s and P_c since clearly $\gamma(G_s) = G_s$ and $\gamma(G_c) = G_c$.

(d) Let us now return to the situation considered in part (b) and suppose that one of the real forms ξ_1 and ξ_2, say ξ_i, is quasicompact. We shall prove that under this additional assumption the G-$\boldsymbol{R}$-algebras $H(A_1)$ and $H(A_2)$ are isomorphic.

Let j denote the element of the set $\{1, 2\}$ different from i. Clearly, σ_j restricts to the automorphism σ_j/G_i of G_i, and Σ_j restricts to the (σ_j/G_i)-automorphism Σ_j/P_i of ξ_i. Let E_i be the spectral sequence associated with the FDG-$\boldsymbol{R}$-algebra A_i, and let $\hat{E}$ be the spectral sequence associated with the FDG-$\boldsymbol{C}$-algebra $\hat{A}$. The isomorphism $A_i \approx \hat{A}^{s_i}$ of FDG-$\boldsymbol{R}$-algebras induces in isomorphism $E_i \approx \hat{E}_{s_i} = \hat{E}^{s_i}$ (see 3.18 for the notation) of multiplicative spectral sequences over $\boldsymbol{R}$, and it is easy to see that this isomorphism takes the automorphism $(\Sigma_j/P_i)^*$ of E_i induced by Σ_j/P_i into the restriction of the involution s_j of $\hat{E}$ to $\hat{E}_{s_i}$. This, together with (c), implies that the spectral sequence $E = \hat{E}_{s_i}$ satisfies the conditions of Proposition 3.6 with P invariant under s_j. Since clearly s_1 and s_2 coincide on $\hat{E}_1^{*,0} \approx H_{DR}(M; \boldsymbol{R}) \otimes_{\boldsymbol{R}} \boldsymbol{C}$, where they are both induced by the conjugation in $\boldsymbol{C}$, we see that the FDG-$\boldsymbol{C}$-algebra $\hat{A}$ and its involutions $\sigma = s_1$ and $\tau = s_2$ satisfy the conditions of Proposition 3.19, and therefore by (b) the G-$\boldsymbol{R}$-algebras $H(A_1)$ and $H(A_2)$ are isomorphic.

(e) It follows immediately from (a) and (d) that the assertion of the proposition holds if the real forms ξ_1 and ξ_2 commute and one of them is quasicompact.

(f) Now we shall prove that the assertion of the proposition is true if both the real forms ξ_1 and ξ_2 are quasicompact.

Since the real form ξ_i is quasicompact, there is a direct product decomposition $G_i = K_i \times R_i$, where K_i is a maximal compact subgroup of $\hat{G}$ and R_i is a closed central subgroup of G_i isomorphic to $\boldsymbol{R}^d$ with d depending, by Remark 2.6 (b), on G only. The homogeneous space G_i/K_i is diffeomorphic to $\boldsymbol{R}^d$, and therefore there exists a reduction $\eta_i = (Q_i, q_i, M, K_i)$ of ξ_i to the subgroup K_i. The right G_i-manifold P_i is clearly canonically equivariantly diffeomorphic to the right G_i-manifold $Q_i \times R_i$, and therefore we have canonical isomorphisms

$$\begin{aligned}(4.8)\qquad A_I(P_i; \boldsymbol{R}) &\approx A_I(Q_i \times R_i; \boldsymbol{R}) \approx A_I(Q_i; \boldsymbol{R}) \otimes_{\boldsymbol{R}} A_I(R_i; \boldsymbol{R}) \approx \\ &\approx A_I(Q_i; \boldsymbol{R}) \otimes_{\boldsymbol{R}} A_I(\boldsymbol{R}^d; \boldsymbol{R})\end{aligned}$$

of DG-$\boldsymbol{R}$-algebras. Combining (4.8) with the Künneth theorem and Proposition 4.3, we get a canonical isomorphism

$$H_{IDR}(P_i; \boldsymbol{R}) \approx H_{DR}(Q_i; \boldsymbol{R}) \otimes_{\boldsymbol{R}} H_{IDR}(\boldsymbol{R}^d; \boldsymbol{R})$$

of G-$\boldsymbol{R}$-algebras. Consequently, it suffices to prove that the manifolds Q_1 and Q_2 are diffeomorphic.

By [8] any two maximal compact subgroups of $\hat{G}$ are conjugate, and therefore there exists an element $g_0 \in \hat{G}$ such that $K_2 = g_0^{-1} . K . g_0$. Since $(x . g) . g_0 = = (x . g_0) . g_0^{-1} g g_0$ for all $x \in Q_1$ and $g \in K_1$, the formulae $Q_2' = Q_1 . g_0$, $q_2' = = \hat{p}/Q_2'$ define a principal K_2-bundle $\eta_2' = (Q_2', q_2', M, K_2)$ with Q_2' diffeomorphic to Q_1. Further, η_2' and η_2 may be considered as reductions of $\hat{\xi}$ to the maximal compact subgroup K_2 of $\hat{G}$, and therefore they are isomorphic. These two facts imply that the manifolds Q_1 and Q_2 are diffeomorphic.

(g) In general case, the assertion of the proposition follows easily from (e), (f), and Proposition 2.18.

4.12. Remark. If we use in the preceding proof Remark 3.20 instead of Proposition 3.19, we find that under the assumptions of Proposition 4.11 there exists a commutative diagram of G-$\boldsymbol{R}$-algebras

$$\begin{array}{ccc} & H_{DR}(M; \boldsymbol{R}) & \\ \swarrow{\scriptstyle p_1^*} & & \searrow{\scriptstyle p_2^*} \\ H_{IDR}(P_1; \boldsymbol{R}) & \approx & H_{IDR}(P_2; \boldsymbol{R}) \end{array}$$

A similar remark applies to the following two corollaries of Proposition 4.11.

4.13. Corollary. *Let $\hat{\xi} = (\hat{P}, \hat{p}, M, \hat{G})$ be a principal $\hat{G}$-bundle, where $\hat{G}$ is a reductive connected complex Lie group without non-discrete compact complex Lie subgroup (or equivalently, without complex tori), let $\xi = (P, p, M, G)$ be a regular real form of $\hat{\xi}$, and let $\eta = (Q, q, M, K)$ be a reduction of $\hat{\xi}$ to a maximal compact subgroup K of $\hat{G}$. Then there exists an isomorphism of G-$\boldsymbol{R}$-algebras*

$$H_{IDR}(P; \boldsymbol{R}) \approx H_{DR}(Q; \boldsymbol{R}) \otimes_{\boldsymbol{R}} \Lambda(\boldsymbol{R}^d), \tag{4.9}$$

where $d = \dim_{\boldsymbol{C}} \hat{G} - \dim_{\boldsymbol{R}} K$, and the elements of $\boldsymbol{R}^d$ are supposed to have degree 1. If G (or equivalently $\hat{G}$) is semisimple, then $d = 0$.

Proof. By [8] any two maximal compact subgroups of $\hat{G}$ are conjugate. This and Remark 2.10 easily yield that there exists a closed central subgroup R of $\hat{G}$ such that $K \cap R = \{e\}$, and $G' = K . R = K \times R$ is a quasicompact real form of $\hat{G}$. Putting $\xi' = (P', p', M, G')$, where $P' = Q . R$ and $p' = \hat{p}/\hat{P}'$, we evidently get a quasicompact real form of $\hat{\xi}$. By Proposition 4.11, the G-$\boldsymbol{R}$-algebras $H_{IDR}(P; \boldsymbol{R})$ and $H_{IDR}(P'; \boldsymbol{R})$ are isomorphic, and the same argument as in part (f) of the proof of Proposition 4.11 yields an isomorphism

$$H_{IDR}(P'; \boldsymbol{R}) \approx H_{DR}(Q; \boldsymbol{R}) \otimes_{\boldsymbol{R}} H_{IDR}(\boldsymbol{R}^d; \boldsymbol{R}) \tag{4.10}$$

of G-$\boldsymbol{R}$-algebras. Combining (4.10) with the obvious isomorphisms $H_{IDR}(\boldsymbol{R}^d; \boldsymbol{R}) \approx \approx A_I(\boldsymbol{R}^d; \boldsymbol{R}) \approx \Lambda((\boldsymbol{R}^d)^*) \approx \Lambda(\boldsymbol{R}^d)$ we get an isomorphism (4.9).

4.14. Corollary. *Let $G \subset GL(n, \boldsymbol{R})$ be a reductive Lie subgroup, let $\hat{G} \subset GL(n, \boldsymbol{C})$ be its complexification, and let $K \subset \hat{G}$ be a maximal compact subgroup. Let $\xi = (P, p, M, G)$ be a principal G-bundle, let $\hat{\xi} = (\hat{P}, \hat{p}, M, \hat{G})$ be an extension* *of ξ to the group $\hat{G}$, and let $\eta = (Q, q, M, K)$ be a reduction of $\hat{\xi}$ to the group K. If G is a regular real form of $\hat{G}$, then there exists an isomorphism (4.9) of G-$\boldsymbol{R}$-algebras, where $d = \dim_{\boldsymbol{R}} G - \dim_{\boldsymbol{R}} K$. If G is semisimple, then $d = 0$.*

4.15. Remark. By the well known de Rham theorem, we may replace $H_{DR}(Q; \boldsymbol{R})$ in the both preceding corollaries by the singular cohomology algebra $H^*_\Delta(Q; \boldsymbol{R})$. The assertion then remains valid even if the reduction η is not smooth.

5. PROOF OF THE MAIN THEOREM

5.1. Keeping the notation from the beginning of Section 1, for any vector field X on an open subset U of M let us denote by $X^{(1)}$ its natural lift [9, pp. 229–230] to $p_M^{-1}(U)$. It is well known that

(a) the map $X \mapsto X^{(1)}$ is $\boldsymbol{R}$-linear,

(b) $[X, Y]^{(1)} = [X^{(1)}, Y^{(1)}]$,

(c) $X^{(1)}$ is invariant, and

(d) $dp_M(X^{(1)}) = X$.

Moreover, it is easy to check that the value of $X^{(1)}$ at a point $y \in p_M^{-1}(U)$ depends only on the 1-jet $j_x^1(X)$ of X at the point $x = p_M(y)$. It follows that the formula

$$\chi_y(j_x^1(X)) = X^{(1)}(y)\,, \quad x = p_M(y)\,, \quad X \in \mathscr{X}(x)\,,$$

defines an injective $\boldsymbol{R}$-linear map

$$\chi_y\colon J^1\,\mathscr{X}(x) \to T_y(B_M)\,, \tag{5.1}$$

where $T_y(B_M)$ is the tangent space of B_M at the point y.

It is well known that $X \in \mathscr{X}(U)$ is an infinitesimal automorphism of the G-structure ξ if and only if $X^{(1)}$ is tangent to P at all points $y \in p^{-1}(U)$. Consequently, (5.1) restricts to an injective $\boldsymbol{R}$-linear map

$$\chi_{y,\xi}\colon J^1\,\mathscr{L}_\xi(x) \to T_y(P)$$

for any point $y \in P$, and $x = p(y)$. This further implies that in the general case $\dim J^1\,\mathscr{L}_\xi(x) \leqq m + \dim G$, and that ξ is 1-transitive iff $\chi_{y,\xi}$ is bijective for all $y \in P$.

Finally, let us remark that the definition of the G-$\boldsymbol{R}$-algebra $C_{(1)}(\mathscr{L}_\xi; \mathscr{S})$ implies that its arbitrary element α of degree k defines in a canonical way alternating k-linear forms

$$\alpha_x\colon J^1\,\mathscr{L}_\xi(x) \times \ldots \times J^1\,\mathscr{L}_\xi(x) \to \boldsymbol{R} \quad (x \in M)\,.$$

5.2. Lemma. *Let us suppose that the G-structure ξ is 1-transitive. Then for any $\alpha \in C^k_{(1)}(\mathscr{L}_\xi; \mathscr{S})$ the formula*

$$\alpha^{(1)}(V_1, \ldots, V_k) = \alpha_x(\chi_{y,\xi}^{-1}(V_1), \ldots, \chi_{y,\xi}^{-1}(V_k))\,,$$

where $y \in P$, $x = p(y)$ and $V_1, \ldots, V_k \in T_y(P)$, defines an invariant smooth form $\alpha^{(1)}$

on P, and the formula $\mu(\alpha) = \alpha^{(1)}$ *defines an isomorphism*

$$\mu\colon C_{(1)}(\mathscr{L}_\xi; \mathscr{S}) \approx A_I(P; \boldsymbol{R})$$

of DG-$\boldsymbol{R}$-algebras

Proof. Let $\alpha \in C^k_{(1)}(\mathscr{L}_\xi; \mathscr{S})$. It is easy to see that for any point $x_0 \in M$ there is an open neighbourhood U of x_0 and vector fields $X_1, \ldots, X_{m+\dim G} \in \mathscr{L}_\xi(U)$ such that the 1-jets $j^1_x(X_1), \ldots, j^1_x(X_{m+\dim G})$ form a basis of the $\boldsymbol{R}$-vector space $J^1 \mathscr{L}_\xi(x)$ for any $x \in U$. Using the bijectivity of $\chi_{y,\xi}$ for all $y \in P$, we get that $X_1^{(1)}(y), \ldots, X_{m+\dim G}(y)$ is a basis of $T_y(P)$ for all $y \in p^{-1}(U)$. Since clearly $\alpha^{(1)}(X^{(1)}_{i_1}, \ldots, X^{(1)}_{i_k})(y) = \alpha(X_{i_1}, \ldots, X_{i_k})(p(y))$ for any indices $i_1, \ldots, i_k \in \{1, 2, \ldots, m + \dim G\}$ and all $y \in p^{-1}(U)$, we see that $\alpha^{(1)}$ is smooth and invariant.

It is clear that μ is a homomorphism of G-$\boldsymbol{R}$-algebras. To prove that it is bijective and commutes with the differentials, it suffices to notice that the formula

$$\nu(\alpha)(X_1, \ldots, X_k)(p(y)) = \alpha(X_1^{(1)}, \ldots, X_k^{(1)})(y),$$

where $\alpha \in A^k_I(P; \boldsymbol{R})$, $X_1, \ldots, X_k \in \mathscr{L}_\xi(U)$, U is an open subset of M, and $y \in p^{-1}(U)$, defines a homomorphism

$$\nu\colon A_I(P; \boldsymbol{R}) \to C_{(1)}(\mathscr{L}_\xi; \mathscr{S})$$

of DG-$\boldsymbol{R}$-algebras which is inverse to μ.

5.3. Proof of Theorem 1.5. In view of the preceding lemma, it is an immediate consequence of Corollary 4.14 and Remark 4.15.

5.4. Remark. Similarly we can obtain the commutative diagram of Remark 1.7. To this end it suffices to apply also Remark 4.12.

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