Oldřich Kowalski; Lieven Vanhecke

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THE VOLUME OF GEODESIC DISKS IN A RIEMANNIAN MANIFOLD

OLDŘICH KOWALSKI, Praha and LIEVEN VANHECKE, Leuven

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1. INTRODUCTION

Let (M, g) be a connected analytic Riemannian manifold of dimension n. For any unit vector $x \in T_m M$, $m \in M$, we shall denote by $D_m^x(r)$ the *geodesic disk* with center m and radius r (we always suppose r sufficiently small). More specifically, we put

$$D_m^x(r) = \{m' \in M \mid d(m', m) \leqq r\} \cap \exp_m(\{x\}^\perp)$$

where $\exp_m$: $T_m M \to M$ is the exponential map at m. Further, we shall denote by $V_m^x(r)$ the $(n - 1)$-dimensional volume of $D_m^x(r)$.

Let ∇, R, ϱ, τ denote the Riemannian connection, the Riemann curvature tensor, the Ricci tensor and the scalar curvature on (M, g), respectively. In [4] we proved

Theorem 1. *For any point $m \in M$, any unit vector $x \in T_m M$ and any small radius $r > 0$, we have*

$$V_m^x(r) = V_0^{n-1}(r)\{1 + A(x) r^2 + B(x) r^4 + O(r^6)\} \tag{1}$$

where the coefficients $A(x)$, $B(x)$ are given, with respect to an adapted orthonormal basis $\{e_1 = x, e_2, \ldots, e_n\}$, by the following formulas:

$$A(x) = -\frac{1}{6(n+1)}(\tau - 2\varrho_{11})(m),$$

$$B(x) = \frac{1}{360(n+1)(n+3)}\Big(-3\|R\|^2 + 8\|\varrho\|^2 + 5\tau^2 - 18\Delta\tau + $$
$$+ 24\sum R_{1ijk}^2 - 8\sum \varrho_{1i}^2 + 12\varrho_{11}^2 - 16\sum \varrho_{ij}R_{1i1j} +$$
$$+ 20\sum R_{1i1j}^2 - 20\tau\varrho_{11} + 36\sum \nabla_{ii}^2\varrho_{11} + 18\nabla_{11}^2\tau\Big)(m).$$

In the second formula, all the summations range over $i, j, k = 2, 3, \ldots, n$. Finally, $V_0^{n-1}(r)$ denotes the volume of a Euclidean ball of dimension $n - 1$ and radius r.

It is easy to see that in a two-point homogeneous space the volume $V_m^x(r)$ does not depend on m and x, i.e., these spaces are *strongly disk-homogeneous* (see [4]). One of the purposes of this paper is to provide complete formulas for $V_m^x(r)$ when (M, g) is a two-point homogeneous space (Section 3). As far as we know these *complete* formulas are not available in the literature, at least when M is neither the Euclidean space *n*or a space of constant curvature.

In [2] the volume of a geodesic sphere or geodesic ball has been determined, at least in the form of a power series expansion. Complete formulas are given when the ambient space is two-point homogeneous. At the same time, the authors of [2] tried to characterize the two-point homogeneous spaces by means of the complete formulas for the geodesic balls, but up to now, this problem has remained open in the general case. Similar problems are stated in [3] where the volume of tubes about curves is used. In contrast to the problem for geodesic balls, the authors have given positive answers in this case. It is not surprising, as we shall show in Section 2, that similar characterizations can be given by using the volume of geodesic disks.

Finally, in Section 4, we characterize locally symmetric spaces and spaces of constant curvature using volume-preserving geodesic symmetries and harmonicity of the disks.

2. CHARACTERIZATION BY MEANS OF VOLUMES OF GEODESIC DISKS

We shall make use of the following

Theorem 2. *Let (M, g) be an n-dimensional Riemannian manifold $(n > 2)$ with the property that for all small r and all x and m the coefficient $A(x)$ in $V_m^x(r)$ is the same as in the corresponding formula for an n-dimensional Einstein space M'. Then (M, g) is an Einstein space with the same scalar curvature as M'.*

Proof. For M', $A'(x)$ reduces to

$$-\frac{1}{6(n+1)}\left(\frac{n-2}{n}\right)\tau'$$

and hence, we have

$$\tau - 2\varrho_{11} = \frac{n-2}{n}\,\tau'\,. \tag{2}$$

Because $e_1 = x$ is arbitrary, we obtain from (2):

$$(n-2)\,\tau = (n-2)\,\tau'$$

and so, $\tau = \tau'$. Finally, (2) implies

$$\varrho_{xx} = \frac{\tau}{n},$$

which proves the required result.

Next we characterize, at least locally, the two-point homogeneous spaces. For this purpose, we shall use characterizations of these spaces by means of quadratic

Riemannian invariants. In [2] and [3] references are given to the works of Berger, Calabi, Donelly, Watanabe and Alekseevskij, where all the details can be found. We always suppose $\dim M > 2$.

We start with the spaces of constant curvature.

Theorem 3. *Let (M, g) be an n-dimensional Riemannian manifold with the property that for all small r and all x and m, M has the same volume function $V_m^x(r)$ as an n-dimensional space M' of constant curvature λ. Then M has constant curvature λ.*

Proof. Theorem 2 implies that M is an Einstein space. In particular, the Ricci curvature ϱ_{ij}, the scalar curvature τ, and $\|\varrho\|^2$ are given by

(3) $$\varrho_{ij} = (n-1)\lambda\delta_{ij}, \quad \tau = n(n-1)\lambda, \quad \|\varrho\|^2 = n(n-1)^2\lambda^2.$$

By equating the coefficients $B(x)$ for both spaces, using (3), we get

(4) $$-3\|R\|^2 + 24\sum_{i,j,k=2}^{n} R_{1ijk}^2 + 20\sum_{i,j=2}^{n} R_{1i1j}^2 = -3\|R'\|^2 + 24\sum_{i,j,k=2}^{n} R'^2_{1ijk} + 20\sum_{i,j=2}^{n} R'^2_{1i1j}.$$

Since $e_1 = x$ is arbitrary, we can use the same integration technique as in [1]. Then, integration of (4) over the unit sphere S^{n-1} gives

(5) $$-3\|R\|^2 + \frac{24(n-1)}{n(n+2)}\left(\|R\|^2 - \frac{2}{n-1}\|\varrho\|^2\right) + \frac{20}{n(n+2)}\left(\frac{3}{2}\|R\|^2 + \|\varrho\|^2\right) = -\frac{3(n^2-6n-2)}{n(n+2)}\|R\|^2 - \frac{28}{n(n+2)}\|\varrho\|^2 = -\frac{3(n^2-6n-2)}{n(n+2)}\|R'\|^2 - \frac{28}{n(n+2)}\|\varrho'\|^2.$$

In virtue of (3), the equality (5) implies

$$\|R\|^2 = \|R'\|^2.$$

Now, M' is a space of constant curvature, and hence

$$\|R'\|^2 = \frac{2}{n-1}\|\varrho'\|^2.$$

Combined with (3), this implies

$$\|R\|^2 = \frac{2}{n-1}\|\varrho\|^2$$

and this is characteristic for (M, g) to be a space of constant curvature. Since $\tau = \tau'$, this curvature equals λ.

Now we characterize Kähler manifolds with constant holomorphic sectional curvature.

Theorem 4. *Let (M, g) be a Kähler manifold of complex dimension n with the property that for all small r and all x and m, M has the same volume function $V_m^x(r)$ as an n-dimensional Kähler manifold M' of constant holomorphic sectional curvature μ. Then M has constant holomorphic sectional curvature μ.*

Proof. Proceeding in the same way as in the proof of Theorem 3 we get

$$\varrho_{ij} = \tfrac{1}{2}(n + 1)\,\mu\delta_{ij}\,, \quad \tau = n(n + 1)\,\mu\,, \quad \|\varrho\|^2 = \tfrac{1}{2}n(n + 1)^2\,\mu^2 \tag{6}$$

and

$$\|R\|^2 = \|R'\|^2\,. \tag{7}$$

The relation

$$\|R'\|^2 = \frac{4}{n + 1}\,\|\varrho'\|^2 \tag{8}$$

holds since M' has constant holomorphic sectional curvature. Hence (6), (7) and (8) imply

$$\|R\|^2 = \frac{4}{n + 1}\,\|\varrho\|^2\,,$$

and this means that M also has constant holomorphic sectional curvature. The rest follows from (6).

Next, let $M' = \mathbb{Q}(v)$ denote the quaternionic projective space $\mathbb{Q}P^n(v)$ (with maximum sectional curvature $v > 0$) or its non-compact dual.

Theorem 5. *Let (M, g) be a $4n$-dimensional Riemannian manifold whose holonomy group is a subgroup of* $\mathrm{Sp}(n)\,.\,\mathrm{Sp}(1)$. *Further, suppose that for all small r, all x and m, M has the same volume function $V_m^x(r)$ as $\mathbb{Q}(v)$. Then M is locally isometric to $\mathbb{Q}(v)$.*

Proof. Again, we proceed in the same way. First we obtain

$$\varrho_{ij} = (n + 2)\,v\delta_{ij}\,, \quad \tau = 4n(n + 2)\,v\,, \quad \|\varrho\|^2 = 4n(n + 2)^2\,v^2\,. \tag{9}$$

Then we get

$$\|R\|^2 = \|R'\|^2\,. \tag{10}$$

Since

$$\|R'\|^2 = \frac{(5n + 1)\,\tau'^2}{4n(n + 2)^2} \tag{11}$$

we obtain from (9), (10) and (11):

$$\|R\|^2 = \frac{(5n + 1)\,\tau^2}{4n(n + 2)^2}\,.$$

This is a characteristic condition for M to be locally isometric to a $\mathbb{Q}(\tilde{\nu})$. The required result follows now at once since (9) implies $\tilde{\nu} = \nu$.

We need not characterize the Cayley plane. A well-known theorem of Alekseevskij says that a manifold whose holonomy group is contained in Spin (9) is either locally flat or locally isometric to the Cayley plane or to its noncompact dual.

3. THE COMPLETE FORMULAS FOR TWO-POINT HOMOGENEOUS SPACES

In this section we compute the complete formulas for $V_m^x(r)$ when (M, g) is a two-point homogeneous space. We use the Jacobi vector field technique, similar to the method used in [3] for the computation of the volumes of tubes.

Let $m \in M$ and, as before, let $\{e_1 = x, e_2, \dots, e_n\}$ be an adapted orthonormal basis at m. Further, let γ be a geodesic of (M, g) such that $\gamma(0) = m$ and $\|\gamma'(t)\| = 1$. In what follows we always choose the basis $\{e_1, \dots, e_n\}$ such that

$$\gamma'(0) = e_2 .$$

Next, on the geodesic disk $D_m^x(r)$, we choose the coordinate system $(x_2, \dots, x_n)$ defined by

$$x_i(\exp_m \sum_{i=2}^{n} t_i e_i) = t_i , \quad i = 2, \dots, n ,$$

where $\exp_m \colon T_m M \to M$ denotes the exponential map at m.

Let $S_m^x(t)$, $t \leqq r$, denote the $(n-2)$-dimensional volume of the boundary of the geodesic disk $D_m^x(t)$, i.e. the volume of

$$d_m^x(t) = \{m' \in M \mid d(m', m) = t\} \cap \exp_m(\{x\}^\perp) .$$

Using the Gauss lemma, one sees at once that

$$V_m^x(r) = \int_0^r S_m^x(t) \, \mathrm{d}t . \tag{12}$$

Further, we have

$$S_m^x(t) = t^{n-2} \int_{S^{n-2}(1)} (\sqrt{\det \bar{g}}) (\exp_m tu) \, \mathrm{d}u \tag{13}$$

where $\mathrm{d}u$ denotes the volume element of the $(n-2)$-dimensional unit sphere $S^{n-2}(1)$ in E^{n-2} and $(\bar{g}_{\alpha\beta})$ is the matrix formed by the elements

$$\bar{g}_{\alpha\beta} = g\left(\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_\beta}\right), \quad \alpha, \beta = 2, \dots, n .$$

We shall need the following lemma, which is easy to prove (see [3]):

Lemma 6. *$\partial/\partial x_2\,(t)$, $t(\partial/\partial x_3)\,(t), \dots, t(\partial/\partial x_n)\,(t)$ are Jacobi vector fields on (M, g)*

along γ and the initial conditions are

$$\left(\frac{\partial}{\partial x_2}\right)(0) = e_2\,, \quad \left(\frac{\partial}{\partial x_2}\right)'(0) = 0\,,$$

$$\left(t\,\frac{\partial}{\partial x_a}\right)(0) = 0\,, \quad \left(t\,\frac{\partial}{\partial x_a}\right)'(0) = e_a\,, \quad a = 3, \ldots, n\,.$$

Hence, to compute the volume, we only need to compute these Jacobi vector fields explicitly and then to carry out the integration. We do this for all the two-point homogeneous spaces. Although the formulas for E^n, $S^n(\lambda)$ and $H^n(-\lambda)$, $\lambda > 0$, can be written down immediately, we prefer to give complete proofs as models for the other cases. Note that the expressions for these Jacobi vector fields are in fact also derived in [3], at least in some cases.

Let $\{E_1, \ldots, E_n\}$ denote the orthonormal frame field along γ obtained by a parallel transport of $\{e_1, \ldots, e_n\}$ along γ.

Theorem 7. *Let (M, g) be the Euclidean space E^n. Then we have*

$$S_m^x(r) = r^{n-2}\, S_0^{n-2}(1) \tag{14}$$

where $S_0^{n-2}(1) = (n-1)\,\pi^{(n-1)/2}/\Gamma((n+1)/2)\,.$

Proof. We express the Jacobi vector fields $\partial/\partial x_2$, $t(\partial/\partial x_a)$, $a = 3, \ldots, n$ in terms of the appropriate parallel frame field $\{E_1, \ldots, E_n\}$. Then we obtain

$$\frac{\partial}{\partial x_2}_{|\gamma(t)} = \gamma'(t) = E_{2|\gamma(t)}\,,$$

$$\frac{\partial}{\partial x_a}_{|\gamma(t)} = E_{a|\gamma(t)}\,, \quad a = 3, \ldots, n\,.$$

Hence

$$\det \bar{g} = 1$$

and the required result follows at once.

Now, we settle the rank one symmetric spaces. For simplicity we treat only the case of positive curvature; the case of negative curvature can be completed just by replacing all the trigonometric functions in the formulas by the corresponding hyperbolic functions.

Theorem 8. *Let (M, g) be a space of constant curvature $\lambda > 0$. Then*

$$S_m^x(r) = \left(\frac{\sin(\sqrt{\lambda})\,r}{\sqrt{\lambda}}\right)^{n-2} S_0^{n-2}(1)\,. \tag{15}$$

Proof. This time, the Jacobi vector fields we need are given by

$$\frac{\partial}{\partial x_2}_{|\gamma(t)} = \gamma'(t) = E_{2|\gamma(t)}\,,$$

$$\frac{\partial}{\partial x_{a|\gamma(t)}} = \left(\frac{\sin(\sqrt{\lambda})\,t}{\sqrt{(\lambda)}\,t}\right) E_{a|\gamma(t)}, \quad a = 3, \ldots, n .$$

Hence, we have

$$\det \bar{g} = \left(\frac{\sin(\sqrt{\lambda})\,t}{\sqrt{(\lambda)}\,t}\right)^{2n-4} . \tag{16}$$

Now the required result easily follows from (13) and (16).

Geodesic disks in other rank one symmetric spaces are more complicated. This is due to the fact that the isotropy group of a vector $x \in T_m M$ no longer acts transitively on the unit sphere of $\{x\}^{\perp} \subset T_m M$.

Theorem 9. *Let (M, g, J) be a Kähler manifold of complex dimension n with almost complex structure J and of constant holomorphic sectional curvature $\mu > 0$. Then*

$$S_m^x(r) = \left(\frac{2}{\sqrt{\mu}} \sin \frac{\sqrt{\mu}}{2} r\right)^{2n-2} \left(\cos \frac{\sqrt{\mu}}{2} r\right) \int_{S^{2n-2}(1)} \left\{1 + a^2(u)\, \mathrm{tg}^2 \frac{\sqrt{\mu}}{2} r\right\}^{1/2} \mathrm{d}u \tag{17}$$

where $a^2(u) = \cos^2 \theta(u)$ and $\theta(u)$ is the angle of the unit vector u with Jx. More explicitly,

$$S_m^x(r) = \left(\frac{2}{\sqrt{\mu}} \sin \frac{\sqrt{\mu}}{2} r\right)^{2n-2} \left(\cos \frac{\sqrt{\mu}}{2} r\right) S_0^{2n-3}(1) \int_{-1}^{+1} (1 + c^2 t^2)^{1/2} (1 - t^2)^{n-2}\, \mathrm{d}t , \tag{18}$$

where $c^2 = \mathrm{tg}^2 \frac{1}{2}(\sqrt{\mu})\, r$.

Proof. There is a parallel vector field Z along γ with $\gamma'(0) = u$ such that

$$\|Z\| = 1 , \quad g(Z, J\gamma') = 0 , \quad g(Z, \gamma') = 0$$

and

$$x = e_1 = aJ\gamma'(0) + b\, Z(0) ,$$

where $a^2 + b^2 = 1$. With respect to the parallel frame field $\{\gamma', J\gamma', Z, E_4, \ldots, E_{2n}\}$ we have

$$\frac{\partial}{\partial x_{2|\gamma(t)}} = \gamma'(t) ,$$

$$\frac{\partial}{\partial x_{3|\gamma(t)}} = \left\{-\left(\frac{b}{(\sqrt{\mu})\,t} \sin(\sqrt{\mu})\,t\right) J\gamma' + \left(\frac{2a}{(\sqrt{\mu})\,t} \sin \frac{\sqrt{\mu}}{2} t\right) Z\right\}_{|\gamma(t)} ,$$

$$\frac{\partial}{\partial x_{a|\gamma(t)}} = \left(\frac{2}{(\sqrt{\mu})\,t} \sin \frac{\sqrt{\mu}}{2} t\right) E_{a|\gamma(t)} , \quad a = 4, \ldots, 2n .$$

Hence we obtain

$$\det \bar{g} = \left(\frac{4a^2}{\mu t^2} \sin^2 \frac{\sqrt{\mu}}{2} t + \frac{b^2}{\mu t^2} \sin^2 (\sqrt{\mu})\, t\right) \left(\frac{2}{(\sqrt{\mu})\,t} \sin \frac{\sqrt{\mu}}{2} t\right)^{4n-6} =$$

$$= \left(a^2 + b^2 \cos^2 \frac{\sqrt{\mu}}{2} t\right)\left(\frac{2}{(\sqrt{\mu})\, t} \sin \frac{\sqrt{\mu}}{2} t\right)^{4n-4} =$$

$$= \left(\cos^2 \frac{\sqrt{\mu}}{2} t + a^2 \sin^2 \frac{\sqrt{\mu}}{2} t\right)\left(\frac{2}{(\sqrt{\mu})\, t} \sin \frac{\sqrt{\mu}}{2} t\right)^{4n-4},$$

where

$$a = a(u) = g(Je_2, e_1) = -g(\gamma'(0), Je_1).$$

This together with (13) yields (17).

To compute (18) we take a system of polar coordinates on $S^{2n-2}(1)$ in $\{e_1\}^{\perp}$. In this space we take a basis $\{U_1, \ldots, U_{2n-1}\}$ such that $U_{2n-1} = Je_1$. We have

$$(19) \qquad \begin{aligned} u_1 &= \cos\theta_1 \cos\theta_2 \ldots \cos\theta_{2n-3} \cos\theta_{2n-2}, \\ u_2 &= \cos\theta_1 \cos\theta_2 \ldots \cos\theta_{2n-3} \sin\theta_{2n-2}, \\ u_3 &= \cos\theta_1 \cos\theta_2 \ldots \sin\theta_{2n-3}, \\ &\ldots \\ u_{2n-1} &= \sin\theta_1 \end{aligned}$$

where

$$0 \leqq \theta_{2n-2} \leqq 2\pi, \quad -\frac{\pi}{2} \leqq \theta_a \leqq \frac{\pi}{2} \quad \text{for} \quad a = 1, \ldots, 2n-3.$$

Then we have

$$(20) \qquad a = -\sin\theta_1$$

and

$$(21) \qquad *1 = (\cos\theta_1)^{2n-3} (\cos\theta_2)^{2n-4} \ldots \cos\theta_{2n-3}\, d\theta_1 \ldots d\theta_{2n-2}.$$

By virtue of (20) and (21) the integral in (17) becomes

$$\int_{S^{2n-2}(1)} (1 + c^2 a^2)^{1/2} *1 = S_0^{2n-3}(1) \int_{-\pi/2}^{+\pi/2} (1 + c^2 \sin^2\theta_1)^{1/2} (\cos\theta_1)^{2n-3}\, d\theta_1$$

where $c^2 = \operatorname{tg}^2 \frac{1}{2}(\sqrt{\mu})\, r$. Further we have

$$(22) \quad \int_{-\pi/2}^{+\pi/2} (1 + c^2 \sin^2\theta_1)^{1/2} (\cos\theta_1)^{2n-3}\, d\theta_1 = \int_{-1}^{+1} (1 + c^2 t^2)^{1/2} (1 - t^2)^{n-2}\, dt$$

which gives the required expression (18).

Note that the integral in (22) can be computed by using standard methods. We omit the clumsy explicit expression.

Theorem 10. *Let (M, g) be the quaternionic projective space $QP^n(v)$ (with maximum sectional curvature $v > 0$). Then*

$$(23) \quad S_m^x(r) = \left(\frac{2}{\sqrt{v}} \sin \frac{\sqrt{v}}{2} r\right)^{4n-2} \cos^3 \frac{\sqrt{v}}{2} r \int_{S^{4n-2}(1)} \left\{1 + a^2(u) \operatorname{tg}^2 \frac{\sqrt{\mu}}{2} r\right\}^{1/2} du$$

where $a^2(u) = \cos^2\theta(u)$ *and* $\theta(u)$ *is the angle of the unit vector u with the subspace spanned by Ix, Jx and Kx. I, J, K are the local almost complex structures.*

Proof. Along γ, with $\gamma'(0) = u$, there exists a parallel frame field $\{\gamma', I\gamma', J\gamma', K\gamma', Z, E_6, \dots, E_{4n}\}$ such that

$$\|Z\| = 1\,, \quad g(Z(0), \gamma'(0)) = 0\,, \quad g(Z(0), I\gamma'(0)) = 0\,,$$
$$g(Z(0), J\gamma'(0)) = 0\,, \quad g(Z(0), K\gamma'(0)) = 0$$

and

$$e_1 = aI\gamma'(0) + bZ(0)\,, \quad a^2 + b^2 = 1\,,$$

where I, J, K denote appropriately chosen almost complex structures. With respect to this parallel frame field we have:

$$\frac{\partial}{\partial x_{2|\gamma(t)}} = \gamma'(t)\,,$$

$$\frac{\partial}{\partial x_{3|\gamma(t)}} = \left\{-\left(\frac{b}{(\sqrt{v})\,t}\sin(\sqrt{v})\,t\right)I\gamma' + \left(\frac{2a}{(\sqrt{v})\,t}\sin\frac{\sqrt{v}}{2}\,t\right)Z\right\}_{|\gamma(t)},$$

$$\frac{\partial}{\partial x_{4|\gamma(t)}} = \left(\frac{1}{(\sqrt{v})\,t}\sin(\sqrt{v})\,t\right)J\gamma'_{|\gamma(t)}\,,$$

$$\frac{\partial}{\partial x_{5|\gamma(t)}} = \left(\frac{1}{(\sqrt{v})\,t}\sin(\sqrt{v})\,t\right)K\gamma'_{|\gamma(t)}\,,$$

$$\frac{\partial}{\partial x_{a|\gamma(t)}} = \left(\frac{2}{(\sqrt{v})\,t}\sin\frac{\sqrt{v}}{2}\,t\right)E_{a|\gamma(t)}\,, \quad a = 6, \dots, 4n\,.$$

Hence

$$\det\bar{g} = \left(\frac{2}{(\sqrt{v})\,t}\sin\frac{\sqrt{v}}{2}\,t\right)^{8n-4}\cos^6\frac{\sqrt{v}}{2}\,t\left(1 + a^2(u)\,\mathrm{tg}^2\frac{\sqrt{v}}{2}\,t\right)$$

and the required formula follows at once.

Using the appropriate almost complex structures $I_0, \dots, I_6$ in the Cayley plane, one obtains, proceeding in a similar way:

Theorem 11. *Let* (M, g) *be the Cayley plane (with maximum sectional curvature* $\zeta > 0$*). Then*

$$(24)\qquad S_m^x(r) = \left(\frac{2}{\sqrt{\zeta}}\sin\frac{\sqrt{\zeta}}{2}\,r\right)^{14}\cos^7\frac{\sqrt{\zeta}}{2}\,r\int_{S^{14}(1)}\left\{1 + a^2(u)\,\mathrm{tg}^2\frac{\sqrt{\zeta}}{2}\,r\right\}^{1/2}du$$

where $a^2(u) = \cos^2\theta(u)$ *and* $\theta(u)$ *is the angle of the unit vector with the subspace spanned by the vectors* $I_0x, \dots, I_6x$.

We proved in [4] that the volume $V_m^x(r)$ of a geodesic disk is given by

$$V_m^x(r) = \int_0^r t^{n-2} \int_{S^{n-2}(1)} \tilde{\theta}(\exp_m tu)\, du\, dt$$

where $\tilde{\theta}$ is the "normal volume function" of the disk D_m^x, i.e.

$$\tilde{\theta}(\exp_m tu) = (\sqrt{\det \tilde{g}})(\exp_m tu)\,.$$

From the formulas given in [4] one easily derives the following power series expansion for $\tilde{\theta}(\exp_m tu)$:

$$\tilde{\theta}(\exp_m tu) = 1 - \tfrac{1}{6}(\varrho_{uu} - R_{1u1u})(m)\, t^2 - \tfrac{1}{12}(\nabla_u \varrho_{uu} - \nabla_u R_{1u1u})(m)\, t^3 + \\ + \tfrac{1}{24}\{-\tfrac{3}{5}(\nabla^2_{uu}\varrho_{uu} - \nabla^2_{uu}R_{1u1u}) + \tfrac{8}{15}\sum_{i=2}^{n} R^2_{1uiu} - \\ - \tfrac{2}{15}\sum_{i,j=2}^{n} R^2_{uiuj} + \tfrac{1}{3}(\varrho_{uu} - R_{1u1u})^2\}(m)\, t^4 + 0(t^5)\,. \tag{25}$$

In what follows we characterize locally symmetric spaces and spaces of constant curvature using weaker geometric properties of the geodesic disks.

We start with locally symmetric spaces. We have

Theorem 12. *A Riemannian manifold (M, g) is a locally symmetric space if and only if, for each point $m \in M$, the local geodesic symmetry at m preserves the $(n-1)$-dimensional volume element of each geodesic disk centered at m.*

Proof. It is easy to see that D_m^x has the volume-preserving property at m if and only if

$$\tilde{\theta}(\exp_m(-tu)) = \tilde{\theta}(\exp_m tu) \tag{26}$$

for small t and all unit vectors $u \in \{x\}^\perp$. (See for example [6].) Then, from (25) and (26), we have

$$\nabla_u \varrho_{uu} - \nabla_u R_{xuxu} = 0 \tag{27}$$

for all $u \in \{x\}^\perp$. Since this must hold for any x, it follows that

$$\nabla_u \varrho_{uu} = 0\,. \tag{28}$$

So, (27) and (28) imply

$$\nabla_u R_{xuxu} = 0\,.$$

Using Lemma 5.1 of [6], we get $\nabla R = 0$.

Conversely, suppose the manifold is a symmetric space. Along a geodesic γ the Jacobi equation can be solved explicitly and hence $\det \tilde{g}$ can be written down at once. One easily sees that (26) is satisfied.

Next we characterize the spaces of constant curvature. Following [5], the geodesic disk $D_m^x(r)$ is said to be *harmonic at the center m* if and only if the normal volume function of the disk depends only on the distance from the center m and not on the direction u. We have

Theorem 13. *(M, g) is a space of constant curvature if and only if each geodesic disk $D_m^x(r)$ is harmonic at the center m.*

Proof. First, let (M, g) be a space of constant curvature. Then the result follows from (16).

Conversely, suppose $\tilde{\theta}(\exp_m tu)$ is independent of u. Then, Theorem 12 implies that M is locally symmetric. Further, we have from (25):

$$\varrho_{uu} - R_{1u1u} \quad \text{is independent of } u\,. \tag{29}$$

Now, let us first suppose $\dim M = n > 3$. Then we have

$$\varrho_{uu} - R_{1u1u} = \varrho_{e_\alpha e_\alpha} - R_{1e_\alpha 1e_\alpha}\,, \quad \alpha = 2, \ldots, n$$

where $\{e_\alpha\}$ is an orthonormal basis of $\{e_1 = x\}^\perp$. Summation with respect to α gives

$$(n-1)(\varrho_{uu} - R_{1u1u}) = \tau - 2\varrho_{11}\,. \tag{30}$$

Next, we interchange the role of $x = e_1$ and u. This gives

$$(n-1)(\varrho_{11} - R_{1u1u}) = \tau - 2\varrho_{uu}\,. \tag{31}$$

From (30) and (31) we get

$$\varrho_{uu} - \varrho_{11} = \frac{2}{n-1}(\varrho_{uu} - \varrho_{11})$$

or

$$(n-3)(\varrho_{uu} - \varrho_{11}) = 0\,.$$

This implies that M is an Einstein space. Then, (30) gives

$$R_{1u1u} = \frac{\tau}{n(n-1)} \tag{32}$$

and this means that M is a space of constant curvature since (32) holds for all x and $u \in \{x\}^\perp$.

Finally, suppose $\dim M = 3$. The coefficient of t^4 in (25) must be independent of the direction u. Taking into account the local symmetry and the condition given by the coefficient of t^2, we get that

$$4\sum_{i=2}^{3} R_{1uiu}^2 - \sum_{i,j=2}^{3} R_{uiuj}^2 \quad \text{is independent of} \quad u \perp e_1\,. \tag{33}$$

Putting $u = e_2$ and $u = e_3$, we hence obtain $R_{1232}^2 = R_{1323}^2$. If we rotate the basis $\{e_2, e_3\}$ in the plane $\{x\}^\perp$, we finally see that $R_{xyzy} = 0$ holds for any orthonormal triple $x, y, z \in T_mM$. The required result follows from Cartan's lemma.

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Authors' addresses: O. Kowalski, 186 00 Praha 8, Sokolovská 83, Czechoslovakia (Matematicko-fyzikální fakulta UK), L. Vanhecke, Department of Mathematics, Katholieke Universiteit Leuven, Celestijnenlaan 200 B, B-3030 Leuven, Belgium.