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Czechoslovak Mathematical Journal, Vol. 35 (1985), No. 1, 59–65

Persistent URL: http://dml.cz/dmlcz/101997

Czechoslovak Mathematical Journal, 35 (110) 1985, Praha

HOLOMORPHIC EXTENSION OF A FUNCTION WHOSE ODD DERIVATIVES ARE SUMMABLE

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(Received September 15, 1983)

Some applications in the physics of elementary particles lead to interesting problems of analytic functions. In discussing the theory of particle scattering [1] the problem arises whether a function $f \in C^\infty$ for which $\sum_{n=0}^{\infty} f^{(2n+1)}(t)$ converges for every t is holomorphic.

We shall solve the problem in a slightly more general setting. Let R^1 be the real line, I an open interval in R^1. $C^\infty(I)$ is the class of all complex valued functions defined on I having all derivatives on I.

Theorem. *Let $f \in C^\infty(I)$. If*

$$\liminf_{n\to\infty} \left|f^{(2n+1)}(t)\right|^{-1/n} \geqq C \quad \text{for every} \quad t \in I$$

where C is a positive constant, then f can be uniquely extended to an entire function in the complex plane.

Proof. The assumption of the theorem yields $\left|f^{(2n+1)}(s)\right| \leqq M(s)\,(2/C)^n$ for $s \in I$, $n = 0, 1, \ldots$ where $M(s)$ is a positive function. Choose $h > 0$, $h^2 < C/2$ and denote $g(t) = f(s)$, $s = ht$, $J = \{s/h : s \in I\}$. Then

$$\text{(A)} \qquad \sum_{n=0}^{\infty} \left|g^{(2n+1)}(t)\right| \quad \text{converges for every} \quad t \in J\,.$$

Conversely if the theorem is proved for g i.e. g can be extended to an entire function, then the same is valid for the original function f. This enables us to replace the assumption of the theorem by assumption (A). Certainly we can restrict ourselves to the case of real valued functions. The investigation of the class of functions fulfilling (A) requires a series of lemmas.

Lemma 1. *Let I be an interval and F a closed subset of I. Let f_n be continuous functions, $f_n: I \to R^1$. If $\lim_{n\to\infty} f_n(t)$ exists at every point $t \in I$, then there exists an open interval $(a, b) \subset I$, $(a, b) \cap F \neq \emptyset$ and a number M so that*

$$\left|f(t)\right| \leqq M\,, \quad \left|f_n(t)\right| \leqq M \quad \text{for} \quad t \in (a, b) \cap F$$

where $f(t) = \lim_{n\to\infty} f_n(t)$.

Proof. Define $g_n(t) = \max\{|f_k(t)|;\ k = 1, \ldots, n\}$, $g(t) = \sup\{|f_k(t)|;\ k = 1, \ldots\} = \lim_{n\to\infty} g_n(t)$. The functions g_n are continuous and since f_n converge we have $g(t) < \infty$ for $t \in I$. Since g is a function of the first class the restriction of g to F has a point of continuity at F by Baire's direct theorem [2]. We conclude that $g(t)$ and hence $g_n(t), f_n(t), f(t)$ are bounded in a certain neighbourhood of the point of continuity in F.

Lemma 2. *Let $f \in C^\infty(a, b)$ and let $|f^{(2n+1)}(t)| \leqq M$ for $t \in (a, b)$, $n = 0, 1, \ldots$. Then $|f^{(2n)}(t)| \leqq MM_1$ for $t \in (a, b)$, $n = 1, 2, \ldots$ where $M_1 = 4/(b - a) + (b - a)/4$.*

Proof. We have

$$f^{(2n+1)}(t) - f^{(2n+1)}(t_0) = f^{(2n+2)}(t_0)(t - t_0) + \int_{t_0}^{t} (t - s) f^{(2n+3)}(s)\, ds$$

so that $|f^{(2n+2)}(t_0)| \leqq (2M + M(t - t_0)^2/2)/|t - t_0| \leqq 4M/(b - a) + M(b - a)/4$ for every $t_0 \in [a, b]$.

Now we derive a result which is used in [1] and which is very close to Lemma 2.

Corollary. *Let $f \in C^\infty(a, b)$, let $\sum_{k=0}^{\infty} f^{(2k+1)}(t)$ converge for $t \in (a, b)$. If there exists a constant M such that $|\sum_{k=0}^{n} f^{(2k+1)}(t)| \leqq M$ for $t \in (a, b)$ and every nonnegative integer n, then $\sum_{k=0}^{\infty} f^{(2k)}(t)$ converges for every $t \in (a, b)$.*

Proof. Using the first equality from the proof of Lemma 2 we conclude

$$\left|\sum_{k=s}^{n} f^{(2k+2)}(t_0)\right| \leqq \left|\sum_{k=s}^{n} f^{(2k+1)}(t) - \sum_{k=s}^{n} f^{(2k+1)}(t_0)\right| |t - t_0|^{-1} + M|t - t_0|$$

$$\text{for} \quad t, t_0 \in (a, b).$$

Choose $t_0 \in (a, b)$. For a given $\varepsilon > 0$ we find $t \in (a, b)$ so that $t \neq t_0$, $M|t - t_0| < \varepsilon/2$. Since the odd derivatives are convergent we can find n_0 so that the first term on the right-hand side is smaller than $\varepsilon/2$ for $n, s \geqq n_0$. By Bolzano-Cauchy Theorem the sum of even derivatives is convergent for $t_0 \in (a, b)$. The corollary is proved.

Remark 1. *Let $f \in C^\infty(a, b)$ and $\sum_{n=0}^{\infty} |f^{(2n)}(t)| < \infty$ for $t \in (a, b)$. Then there exists $t_0 \in (a, b)$ such that $\sum_{n=0}^{\infty} |f^{(n)}(t_0)| < \infty$.*

Proof. Applying Lemma 1 to functions $\sum_{k=0}^{n} |f^{(2k)}(t)|$ we conclude that there exists an interval (t_1, t_2), $a \leqq t_1 < t_2 \leqq b$ and a constant M' such that $\sum_{k=0}^{n} |f^{(2k)}(t)| \leqq M'$

for $t \in (t_1, t_2)$, $n \geqq 0$. As in the proof of Corollary we obtain

$$\sum_{k=s}^{n} |f^{(2k+1)}(t_0)| \leqq (\sum_{k=s}^{n} |f^{(2k)}(t)| + \sum_{k=s}^{n} |f^{(2k)}(t_0)|)/|t - t_0| + M'|t - t_0|$$

$$\text{for} \quad t, t_0 \in (t_1, t_2),$$

which yields that $\sum_{k=0}^{\infty} |f^{(2k+1)}(t_0)| < \infty$.

Let a function g fulfil condition (A) on J. Choose an interval $[a_0, b_0]$, $a_0 < b_0$, $[a_0, b_0] \subset J$. Denote

$$(1) \qquad S(t) = \sum_{n=0}^{\infty} g^{(2n+1)}(t), \quad S_n(t) = \sum_{k=0}^{n} g^{(2k+1)}(t).$$

The functions S, S_n fulfil the assumptions of Lemma 1 with $F = [a_0, b_0]$. There exists an interval (a_1, b_1), $a_0 \leqq a_1 < b_1 \leqq b_0$ and a number M such that

$$|S_n(t)| \leqq M, \quad |S(t)| \leqq M \quad \text{for} \quad t \in (a_1, b_1).$$

Thus $|g^{(2n+1)}(t)| \leqq 2M$ for $t \in (a_1, b_1)$, $n = 0, 1, \ldots$. By Lemma 2 we have $|g^{(n)}(t)| \leqq 2MM_1$ for $t \in (a_1, b_1)$, $n = 1, 2, \ldots$.

Denote $g(t, t_0) = \sum_{n=0}^{\infty} g^{(n)}(t_0)(t - t_0)^n/n!$.

We have proved

Lemma 3. *Let a function g fulfil (A) and let an interval $[a_0, b_0] \subset J$ be given. There exists an interval (a_1, b_1), $a_0 \leqq a_1 < b_1 \leqq b_0$ so that $g(t, t_0)$ is defined for all complex t and real t_0, $t_0 \in (a_1, b_1)$. $g(t, t_0)$ is an entire function in t if $t_0 \in (a_1, b_1)$ and $g(t) = g(t, t_0)$ for $t, t_0 \in (a_1, b_1)$.*

Let t_0 be chosen in (a_1, b_1). Consider a maximal open interval I_1 containing (a_1, b_1) so that $g(t) = g(t, t_0)$ for $t \in I_1$.

Lemma 4. *The function $g(t, t_1)$ is defined for $t_1 \in \bar{I}_1$ and $g(t) = g(t, t_1)$ is valid for $t, t_1 \in \bar{I}_1$. ($\bar{I}_1$ is the closure of I_1.)*

Proof. Since $g(t, t_0)$ is an entire function we have

$$g(t, t_0) = \sum_{n=0}^{\infty} g^{(n)}(t_1, t_0)(t - t_1)^n/n! \quad \text{for every} \quad t_1.$$

Choose t_1 from $\bar{I}_1$. The definition of I_1 gives $g^{(n)}(t_1) = g^{(n)}(t_1, t_0)$ so that

$$g(t, t_0) = \sum_{n=0}^{\infty} g^{(n)}(t_1)(t - t_1)^n/n! = g(t, t_1).$$

Since $g(t, t_1)$ does not depend on t_1 if $t_1 \in \bar{I}_1$ we can denote $g(t, t_1)$ as $g(t; I_1)$.

Either $I_1 = J$ or we can repat this construction in $J - \bar{I}_1$ .We conclude that there exists a countable family $\mathscr{F}$ of disjoint intervals I_k and entire functions $g(t; I_k)$ so that the intervals I_k are maximal in the sense that

$$g(t) = g(t; I_k) \quad \text{for} \quad t \in I_k.$$

Remark 2. The family $\mathscr{F}$ can be constructed so that $\{\bigcup I_k : I_k \in \mathscr{F}\}$ is dense in J.

Proof. If $G = J - \overline{\bigcup I_k} \neq \emptyset$ we can repeat the construction in G.

Remark 3. If $I_k \cap I_s = \emptyset$ then $\bar{I}_k \cap \bar{I}_s = \emptyset$.

Assume $\bar{I}_k \cap \bar{I}_s \neq \emptyset$. Choose $t_0 \in \bar{I}_k \cap \bar{I}_s$. Since $g(t) = g(t; I_k)$ for $t \in \bar{I}_k$ we obtain $g^{(n)}(t_0) = g^{(n)}(t_0; I_k)$ for $n = 0, 1, \ldots$. Similarly $g^{(n)}(t_0) = g^{(n)}(t_0; I_s)$ for $n = 0, 1, \ldots$. Since $g(t; I_k)$ and $g(t; I_s)$ are entire functions we have $g(t; I_k) = g(t; I_s) = g(t)$ for $t \in I_k \cup I_s$. This contradicts the fact that I_k, I_s are maximal.

We shall need a result from the theory of entire functions and an auxiliary statement.

Lemma 5. *Let numbers z_k fulfil*

$$\sum_{k=0}^{n} z_k/(2n - 2k + 1)! = 0 \quad \textit{for} \quad n = 1, 2, \ldots, z_0 = 1\,. \tag{2}$$

Then $|z_k| \leqq 2c^{2k+1}$ where $c = 1/2$.

Proof. Assume $|z_k| \leqq 2c^{2k+1}$ for $k = 0, 1, \ldots, n$. By (2) we conclude

$$|z_{n+1}| \leqq 2(c/(2n+3)! + c^3/(2n+1)! + \ldots + c^{2n+1}/3!) =$$
$$= 2c^{2n+4}(c^{-(2n+3)}/(2n+3)! + c^{-(2n+1)}/(2n+1)! + \ldots + c^{-3}/3!)\,.$$

Since $c = 1/2$ we have $(\exp(1/c) - \exp(-1/c))/2 - 1/c = \sum_{k=1}^{\infty} c^{-(2k+1)}/(2k+1)! \leqq$ $\leqq 1/c$ so that $|z_{n+1}| \leqq 2c^{2n+3}$. The lemma is proved. We conclude that

$$\sum_{k=0}^{\infty} |z_k| \leqq 4/3\,. \tag{3}$$

Lemma 6. *Let f be an entire function fulfilling $\sum_{k=0}^{\infty} |f^{(2k)}(t)| < \infty$ for $t \in (0, h)$, $h > 0$. Then $\sum_{k=0}^{\infty} |f^{(k)}(w)| < \infty$ for every complex w and*

$$\sum_{n=0}^{\infty} z_n \sum_{k=n}^{\infty} f^{(2k+1)}(0)/(2k - 2n + 1)! = f'(0)\,,$$

where z_n are the numbers given by (2).

Proof. By Remark 1 there exists $t_0 \in (0, h)$ such that $\sum_{k=0}^{\infty} |f^{(k)}(t_0)| < \infty$. Since f is entire we have

$$\sum_{k=0}^{\infty} |f^{(k)}(w)| \leqq \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} |f^{(n)}(t_0)| \cdot |w - t_0|^{n-k}/(n-k)! =$$
$$\sum_{n=0}^{\infty} |f^{(n)}(t_0)| \sum_{k=0}^{n} |w - t_0|^{n-k}/(n-k)! \leqq e^{|w-t_0|} \sum_{n=0}^{\infty} |f^{(n)}(t_0)| < \infty$$

for every complex w.

We have

$$\sum_{n=0}^{\infty} z_n \sum_{k=n}^{\infty} f^{(2k+1)}(0)/(2k - 2n + 1)! = \sum_{n=0}^{N} z_n \sum_{k=n}^{N} f^{(2k+1)}(0)/(2k - 2n + 1)! +$$

$$+ \sum_{n=0}^{\infty} z_n \sum_{k=\max(N+1,n)}^{\infty} f^{(2k+1)}(0)/(2k - 2n + 1)! .$$

By the first statement of Lemma ($w \approx 0$) and by (3) the first and the last terms converge. Denote the last term by Q_N. Since $|z_k| \leqq 4^{-k}$ (see Lemma 5) we conclude

$$|Q_N| \leqq \sum_{n=0}^{\infty} 4^{-n} \sum_{k=\max(N+1,n)}^{\infty} |f^{(2k+1)}(0)| \to 0 \quad \text{for} \quad N \to \infty .$$

The definition of the numbers z_n (Lemma 5) yields

$$\sum_{n=0}^{N} z_n \sum_{k=n}^{N} f^{(2k+1)}(0)/(2k - 2n + 1)! = f'(0) \quad \text{for} \quad N \geqq 0 .$$

The lemma is proved.

Lemma 7. *Let g be an entire function fulfilling* (A) *on* $(0, h)$, $0 < h \leqq 1$. *If*

(4) $$|g^{(2n+1)}(0)| \leqq 1 , \quad |g^{(2n+1)}(h)| \leqq 1 \quad \textit{for} \quad n = 0, 1, \ldots .$$

then

(5) $$|g^{(2n+1)}(t)| \leqq M_0 \quad \textit{for} \quad t \in [0, h] , \quad n = 0, 1, \ldots ,$$

where $M_0 = 4e(1 + e)/3$.

Proof. Denote

(6) $$f(t) = g'(s) , \quad th = s .$$

The function f is entire again and since $h \leqq 1$,

(7) $$|f^{(2n)}(0)| \leqq 1 , \quad |f^{(2n)}(1)| \leqq 1 \quad \text{for} \quad n = 0, 1, \ldots .$$

Define

$$f_1(t) = \sum_{k=0}^{\infty} f^{(2k+1)}(0)\, t^{2k+1}/(2k + 1)! ,$$

$$f_2(t) = \sum_{k=0}^{\infty} f^{(2k)}(0)\, t^{2k}/(2k)! .$$

Notice that the series are absolutely convergent since f is entire. By (7) we have

$$|f_2^{(2n)}(1)| = \left|\sum_{k=n}^{\infty} f^{(2k)}(0)/(2k - 2n)!\right| \leqq e .$$

Again due to (7) we conclude

(8) $$|f_1^{(2n)}(1)| \leqq 1 + e .$$

The numbers $f^{(2k+1)}(0)$ fulfil

$$\sum_{k=n}^{\infty} f^{(2k+1)}(0)/(2k - 2n + 1)! = f_1^{(2n)}(1) .$$

Since g is entire and fulfils the condition (A) the function f fulfils the assumptions of Lemma 6 so that

$$f'(0) = \sum_{n=0}^{\infty} z_n \sum_{k=n}^{\infty} f^{(2k+1)}(0)/(2k - 2n + 1)! = \sum_{n=0}^{\infty} z_n f_1^{(2n)}(1)$$

and due to (3),

$$|f^{(1)}(0)| \leqq 4(1 + e)/3 .$$

Since $f^{(2n)}$ fulfils the conditions (7) again we obtain

$$|f^{(2n+1)}(0)| \leqq 4(1 + e)/3 \quad \text{for} \quad n = 0, 1, \ldots .$$

Taking into consideration the first part of the inequalities (7),

$$|f^{(n)}(0)| \leqq 4(1 + e)/3 \quad \text{for} \quad n = 0, 1, \ldots$$

so that

$$|f(t)| = \left|\sum_{n=0}^{\infty} f^{(n)}(0)\, t^n/n!\right| \leqq 4(1 + e)/3 \sum_{n=0}^{\infty} t^n/n! \leqq 4e(1 + e)/3 \quad \text{for} \quad t \in [0, 1] .$$

The identity (6) yields $|g'(t)| \leqq M_0$ for $t \in [0, h]$ if g is entire. Since the derivatives $g^{(2n)}(t)$ fulfil the same condition (4) the lemma is proved.

Now we are able to prove

Lemma 8. *The family $\mathscr{F}$ (which is defined after Lemma 4) is a one-element set.*

Proof. Assume that $\mathscr{F}$ contains intervals $I_1 = (u_1, v_1)$, $I_2 = (u_2, v_2)$, $v_1 \leqq u_2$. By Remark 3, $v_1 < u_2$. Denote

$$F = [v_1, u_2] - \{\bigcup I_k : I_k \in \mathscr{F}\} .$$

The set F is certainly closed, nonempty since $v_1, u_2 \in F$, and has no isolated points due to Remark 3.

Applying Lemma 1 to this F and to the functions $S_n(t)$ we conclude that there exists an interval (x, y), $x < y < x + 1$, $v_1 \leqq x < y \leqq u_2$, $(x, y) \cap F \neq \emptyset$ and a number M so that

$$|g^{(2n+1)}(t)| \leqq M \quad \text{for} \quad t \in (x, y) \cap F , \quad n = 0, 1, \ldots, .$$

Since F has no isolated points there exist infinitely many points of F in (x, y) so that we can additionally assume $x, y \in F$. Let $I_k = (u_k, v_k)$ be from $\mathscr{F}$ such that $I_k \subset (x, y)$. Since the end-points of I_k belong to $(x, y) \cap F$ we have

$$|g^{(2n+1)}(u_k)| \leqq M , \quad |g^{(2n+1)}(v_k)| \leqq M , \quad n = 0, 1, \ldots .$$

By the definition of I_k we have $g(t) = g(t; I_k)$ on I_k where $g(t; I_k)$ is an entire function. By Lemma 7,

$$|g^{(2n+1)}(t)| = |g^{(2n+1)}(t; I_k)| \leqq MM_0 \quad \text{for} \quad t \in I_k .$$

Since this bound is independent of such I_k we obtain

$$|g^{(2n+1)}(t)| \leqq MM_0 \quad \text{for} \quad t \in (x, y) , \quad n = 0, 1, \ldots .$$

By Lemma 2, $|g^{(2n)}(t)| \leqq M M_0 M_1$ for $t \in (x, y)$, $n = 1, 2, \ldots$. By Remark 2 and due to the fact that $(x, y) \cap F \neq \emptyset$ there exists $I_n \in \mathscr{F}$ such that $\bar{I}_n \subset (x, y)$. The previous estimates for $g^{(2n+1)}$ and $g^{(2n)}$ yield that the interval I_n is not maximal. This contradiction proves the lemma.

The proof of the theorem is now very simple. By Lemma 7 the family $\mathscr{F}$ contains only one element I_1. By Remark 2 we have $I_1 = J$. The theorem is proved.

Conclusion 1. *Let $f \in C^\infty(R^1)$. If*

$$\liminf_{n \to \infty} |f^{(2n+1)}(t)|^{-1/n} \geqq g(t) > 0 \quad \textit{for every} \quad t \in R^1$$

where g is a continuous function then f can be uniquely extended to an entire function.

Let $I_n = (-n, n)$. Since $\bar{I}_n$ is compact, there exists $C_n > 0$ such that the assumption of the theorem is fulfilled on I_n where C is replaced by C_n. By the theorem f is holomorphic on every I_n and the extensions of f coincide.

Conclusion 2. *If $M(t) = \sup\{|f^{(2n+1)}(t)|;\ n = 0, 1, \ldots\} < \infty$ for every $t \in I$ (we do not assume that $M(t)$ is bounded) then the statement of the theorem is valid.*

Since $|f^{(2n+1)}(s)| \leqq M(s)$ for $s \in I$ we can transform f to g by $f(s) = g(t)$, $s = ht$, $0 < h < 1$ similarly as at the beginning of the proof of the theorem so that condition (A) is valid.

Conclusion 3. *If $\sum_{n=0}^{\infty} f^{(2n+1)}(t)$ converges for every $t \in I$ then the statement of the theorem is valid.*

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