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GENERALIZED ARCHIMEDEAN IDEALS

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1. An ideal A in a compact semigroup S is called *generalized Archimedean* if $x, y \in A$ implies $\Gamma(x) \cap AyA \neq \emptyset$, where $\Gamma(x)$ is the closure of the monothetic semigroup generated by x. In this paper, several results are presented which relate generalized Archimedean ideals to $Q^* = \bigcap$ open prime ideals and $M^* = \bigcap$ maximal ideals. It is shown that if S is compact and semi-normal semigroup, then A is a generalized Archimedean ideal of S if and only if $A \subset Q^*$. Moreover, if $S^2 = S$, then M^* is a generalized Archimedean ideal of S if and only if $M^* = Q^*$. These results are the generalizations of the results obtained by M. Satyanarayana on Archimedean semigroups in [6]. Moreover, an open problem posed in [6] is also answered.

Throughout this paper we shall use the following notation and definitions

Q^* denotes the intersection of all open prime ideals of S.

M^* denotes the *Frattini ideal* of S, that is, M^* is the intersection of all maximal ideals of S.

E denotes the set of all idempotents in S. E is known to be a closed subset of S, and it is not empty if S is compact.

If A is a subset of S, then $J_0(A)$ is the union of all ideals of S which are contained in A, that is, if $J_0(A) \neq \emptyset$, then it is the largest ideal contained in A.

An ideal I of S is said to be *topologically semiprime* if $x \notin I$ implies $\Gamma(x) \cap I = \emptyset$.

A semigroup S is said to be *semi-normal* if $eS = Se$ for all $e \in E$.

An element x of a semigroup is said to be a *topological compressed element* of S if $\Gamma(x) \cap K \neq \emptyset$, where K is the kernel of S, that is, the minimal ideal of S.

The set of all topological compressed elements of S is termed the *topological radical* of S and is denoted by $\mathscr{T}(K)$.

The reader is referred to [2] for terminology and definitions not given in this paper.

2. A characterization of generalized Archimedean ideals. The following lemma is to identify all the members of the ideal Q^* of a compact semi-normal semigroup S. In fact, we show that the elements of Q^* are precisely the topological compressed elements of S.

Lemma 2.1. *Let S be a compact semi-normal semigroup. Then $Q^* = \mathcal{T}(K)$.*

Proof. It has been proved in [9] that $\mathcal{T}(K)$ is an ideal of S. As $\mathcal{T}(K)$ is topologically semiprime, by Theorem 2.9 in [9], $\mathcal{T}(K)$ is the intersection of all open prime ideals containing $\mathcal{T}(K)$. Thus, $Q^* \subset \mathcal{T}(K)$. On the other hand, let P_α be an arbitrary open prime ideal of S containing K. Since S is compact, it is well known that P can be expressed in the form $J_0(S - e_\alpha)$, for some $e_\alpha^2 = e_\alpha \notin K$ [5]. Consequently $e_\alpha \notin \mathcal{T}(K)$. Thus, $\mathcal{T}(K) \subset J_0(S - e_\alpha)$. Hence, every open prime ideal containing K also contains $\mathcal{T}(K)$, that is, $\mathcal{T}(K) \subset Q^*$. Thus, $Q^* = \mathcal{T}(K)$.

Note. Schwarz noticed in [7] that Q^* may be empty. But if the semigroup S is compact and semi-normal, then lemma 2.1 says that Q^* is always non-empty.

The following corollaries are easy to observe. They are the topological generalization of the results of M. Satyanarayana and R. Fulp.

Corollary 2.2. (Satyanarayana [6]) *If S is a compact semi-normal semigroup, then Q^* is an ideal extension of a simple semigroup K by a topological nil semigroup.*

Corollary 2.3. (R. Fulp [1]). *If S is a compact semigroup in which every ideal of S is topologically semiprime, then Q^* is the kernel of K. In particular, if S is a finite semilattice, then $Q^* = \{0\}$.*

Proposition 2.4. *Let S be a compact semi-normal semigroup. Then Q^* is a generalized archimedean ideal of S.*

Proof. Replace Q^* with $\mathcal{T}(K)$ (lemma 2.1). Let $x, y \in Q^* = \mathcal{T}(K)$. By definition of $\mathcal{T}(K)$, $\Gamma(x) \cap K \neq \emptyset$. By the minimality of K, we have $Q^*yQ^* \supset K$, for Q^*yQ^* is an ideal of S. Therefore $\Gamma(x) \cap Q^*yQ^* \neq \emptyset$, so Q^* is generalized Archimedean.

Corollary 2.5. *All ideals contained in Q^* are generalized Archimedean.*

The following theorem asserts that Q^* is the maximal generalized Archimedean ideal if the semigroup S is a compact semi-normal semigroup.

Theorem 2.6. *Let S be a compact semi-normal semigroup. Then an ideal A of S is a generalized Archimedean ideal of S if and only if $A \subset Q^*$.*

Proof. Since Q^* is non-empty, we can choose some element $q \in Q^*A \subset Q^* \cap A$. Let $x \in A$. Since A is a generalized Archimedean ideal, $\Gamma(x) \cap AqA \neq \emptyset$. Since $AqA \subseteq Q^* = \mathcal{T}(K)$, we have $\Gamma(x) \cap \mathcal{T}(K) \neq \emptyset$. As $\mathcal{T}(K)$ is topologically semiprime, this means that $x \in \mathcal{T}(K)$. Thus $A \subseteq \mathcal{T}(K) = Q^*$. The converse statement follows trivially from corollary 2.5.

Corollary 2.7. *Let A be an ideal of a finite semi-normal semigroup. Then $A \subset Q^*$ if and only if A is Archimedean.*

In [6], M. Satyanarayana proved that if Q^* is a non-empty completely semiprime ideal of a semigroup S, then an ideal A of S is Archimedean if and only if $A \subset Q^*$. He then asked whether his result is true in the arbitrary case, that is, whether the assumption of Q^* being completely semiprime can be removed [6, p. 291]. In fact, this is not possible even for finite semigroups. The following semigroup of T. Hall serves as a counter example.

Example 2.8. (Howie [3; p. 252].) Let $S = \{o, e, f, g, x, y\}$ be the semigroup with Cayley table

	o	e	f	g	x	y
o	o	o	o	o	o	o
e	o	e	o	o	x	o
f	o	o	f	g	o	y
g	o	o	g	g	o	y
x	o	o	x	x	o	e
y	o	y	o	o	g	o

Then it is easy to see that $Q^* = \{0\}$, and $A = \{o, e, g, x, y\}$ is an Archimedean ideal of S. Clearly $\{o, e, g, x, y\} \not\subset \{0\}$.

However, if the semigroup S is finite and semi-normal, then the problem of Satyanarayana can be answered affirmatively by Corollary 2.7.

3. An analogue of the Hilbert Nullstellensatz. We now study the Frattini ideal in compact semigroups. It is clear that the Frattini ideal of a semigroup need not be Archimedean in general. The following theorem gives a criterion for the Frattini ideal M^* of a compact semigroup to have the generalized Archimedean property.

Theorem 3.1. *The Frattini ideal M^* of a compact semi-normal semigroup is a generalized Archimedean ideal if and only if $M^* \cap E = K \cap E$.*

In order to prove Theorem 3.1, the following lemma is needed.

Lemma 3.2. *Let M^* be a non-empty Frattini ideal of a compact semi-normal semigroup S. Then $M^* \cap E = K \cap E$ if and only if $M^* \subset Q^*$.*

Proof. a) Let $t \in M^*$. Since S is compact, we have $f^2 = f \in \Gamma(t) \subset tS^1 \subset M^*$. This implies $f \in K$, as we assume that $M^* \cap E = K \cap E$. Thus, $\Gamma(t) \cap K \neq \emptyset$. By lemma 2.1, we have $t \in Q^*$. Thus $M^* \subset Q^*$.

b) It is trivial to see that $M^* \cap E \supset K \cap E$. Suppose if possible that $K \cap E \subsetneq M^* \cap E$. Then there exists an idempotent $e^2 = e \in M^* - K$. Hence, $K \subset J_0(S - e)$, which is an open prime ideal of S. Thus, $Q^* \subset J_0(S - e)$. By assumption, we have $e \in M^* \subset Q^* \subset J_0(S - e)$, a contradiction. Thus $K \cap E = M^* \cap E$.

The proof of Theorem 3.1 now follows immediately from lemma 3.2 and Theorem 2.6.

We have shown that under certain conditions, the Frattini ideal M^* of a semigroup S is a generalized Archimedean ideal. It is natural to ask when M^* is the maximal Archimedean ideal of S, that is, when $M^* = Q^*$?

For globally idempotent semigroups, we have the following theorem.

Theorem 3.3. *Let S be a compact semi-normal semigroup with $S^2 = S$. Then M^* is a generalized Archimedean ideal if and only if $M^* = Q^*$.*

Proof. Clearly, $M^* \subset Q^*$ by theoıem 2.6, for M^* is generalized Archimedean. Since maximal ideals are prime in globally idempotent semigroup [7, Theorem 1] and maximal ideals of a compact semigroup are open [4], $Q^* \subset M^*$. Hence $Q^* = M^*$. The converse statement is evident from Proposition 2.4.

For non-globally idempotent semigroups, we have the following theorem.

Theorem 3.4. *Let S be a compact semi-normal semigroup. Then $M^* = Q^*$ if and only if M^* is a generalized Archimedean topologically semiprime ideal of S.*

Proof. Suppose $M^* = Q^*$. As Q^* is a generalized Archimedean ideal and is topologically semiprime, so is M^*. Conversely, Suppose M^* is a generalized Archimedean ideal. Then by virtue of theorem 2.6, $M^* \subset Q^*$. If $M^* \subsetneqq Q^*$, then there is an element $y \in Q^* - M^*$. Since M^* is topologically semiprime, $\Gamma(y) \cap M^* = \emptyset$. Hence $f^2 = f \notin M^*$, where $f^2 = f \in \Gamma(y)$. This implies $f \in K$ for $M^* \supset K$. However, $f \in \Gamma(y) \subset Q^* = \mathcal{T}(K)$ implies that $f \in K$. which is a contradiction. Hence $M^* = Q^*$.

The above two theorems extend a result of M. Satyanarayana in [6, Theorem 11(i) (ii)] for finite semigroups.

The Hilbert Nullstellensatz says that if R is a finitely generated algebra over a field K then the intersection of the maximal ideals is the nilradical of R [10]. In view of lemma 2.1, Theorem 3.3 and Theorem 3.4 we obtain the following analogue of the Hilbert Nullstellensatz for compact semigroups.

Theorem 3.5. (Hilbert Nullstellensatz) (i) *Let S be a compact semi-normal globally idempotent semigroup with zero. If the Frattini ideal of S is a generalized Archimedean ideal, then the intersection of the maximal ideals of S is the topological nilradical of S.*

(ii) *Let S be a compact semi-normal semigroup with zero. If the Frattini ideal of S is a generalized Archimedean topologically semiprime ideal, then the intersection of the maximal ideals of S is the topological nilradical of S.*

In [6], Satyanarayana noticed that the maximal Archimedean ideal is related to the semisimplicity of the semigroup S. But since the ideal maximal with respect to the property of not containing a semisimple element x need not be an open prime ideal, the concept of semisimplicity cannot be extended analogously to "general" topological semigroups.

Finally, we remark that the equivalent statements of Theorem 6 in [6] also hold for finite semi-normal semigroups, and can be extended analogously to compact seminormal semigroups by imposing the concepts of generalized Archimedean and topologically semiprime ideals on the Frattini ideal M^* of S.

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