Czechoslovak Mathematical Journal

David Kenoyer

Recognizability in the lattice of convex ℓ-subgroups of a lattice-ordered group

Czechoslovak Mathematical Journal, Vol. 34 (1984), No. 3, 411–416

Persistent URL: http://dml.cz/dmlcz/101966

Czechoslovak Mathematical Journal, 34 (109) 1984, Praha

RECOGNIZABILITY IN THE LATTICE OF CONVEX ℓ-SUBGROUPS OF A LATTICE-ORDERED GROUP

DAVID KENOYER, Lawrence

(Received December 20, 1982)

The intent of this note is to show that several types of lattice-ordered groups and convex ℓ-subgroups are impossible to recognize from the lattice structure of the lattice of convex ℓ-subgroups of a lattice-ordered group. In particular we show that one cannot recognize if a lattice-ordered group belongs to any nontrivial proper variety, if it is archimedean, or if it is completely distributive, and also that one cannot tell if a given convex ℓ-subgroup is closed. It is already known that no nontrivial proper variety other than $\mathcal{N}$ (the variety of normal-valued ℓ-groups) is recognizable, but our proof that $\mathcal{N}$ is not recognizable yields the fact for all nontrivial proper varieties at once. Our method is to consider two specific ℓ-groups which we label G and G'. We show that the lattices $\mathscr{C}(G)$ and $\mathscr{C}(G')$ of convex ℓ-subgroups are isomorphic as lattices, and then consider the recognizability questions.

Let $A(\mathbb{R})$ represent the lattice-ordered group of order-preserving permutations of the real numbers with pointwise meet and join as lattice operations, and composition for the group operation. Let $C(\mathbb{R})$ be the ℓ-group of continuous functions $f : \mathbb{R} \to \mathbb{R}$, with pointwise meet, join and addition. We define a continuous function $f : \mathbb{R} \to \mathbb{R}$ to be *finitely piecewise linear* if there is a finite set of real numbers $a_1 < a_2 < \ldots \ldots < a_n$ such that for each interval I of $\mathbb{R}$ which has no a_k in its interior, there are real numbers m and b such that $f(x) = mx + b$ for each $x \in I$. We will say that such a function f *has finitely many* pieces over any interval $J \subseteq \mathbb{R}$, where a *piece of f over* J is defined to be f restricted to a subinterval J' of J that is maximal with respect to the property that there are real numbers m and b with $f(x) = mx + b$ for all x in J'. We also define the 0-support of a function $f : \mathbb{R} \to \mathbb{R}$ to be $\{x \in \in \mathbb{R} \mid f(x) \neq 0\}$ and the 1-support of $f = \{x \in \mathbb{R} \mid f(x) \neq x\}$. We will use $\operatorname{supp}(f)$ to represent both the 0-support of a function in $C(\mathbb{R})$ and the 1-support of a function in $A(\mathbb{R})$, since it will be clear from the context just which is intended.

We let $G = \{f \in C(\mathbb{R}) \mid f$ is finitely piecewise linear and $\operatorname{supp}(f)$ is bounded$\}$ and let $G' = \{f \in A(\mathbb{R}) \mid f$ is finitely piecewise linear and $\operatorname{supp}(f)$ is bounded$\}$. Since G is an ℓ-subgroup of $C(\mathbb{R})$ and G' is an ℓ-subgroup of $A(\mathbb{R})$, we know that G and G' are ℓ-groups. Ball invented the group G' to show that an ℓ-group can have the DCC on regular subgroups and still fail to be normal-valued. In fact, G' has no nontrivial proper normal subgroups at all. A variation of this example due to Conrad can be used to show that a locally flat ℓ-group need not be archimedean (take $G'' = G' \cap \cap A(\emptyset)$; neither G' nor G'' is archimedean).

Now we show that the lattices $\mathscr{C}(G)$ and $\mathscr{C}(G')$ are isomorphic. We begin by identifying all of the prime subgroups of G. For each $r \in \mathbb{R}$, let

$$\begin{cases} G_r &= \{f \in G \mid f(r) = 0\} \\ G_{r+} &= \{f \in G \mid \text{for some } \varepsilon > 0 \text{ and for all } x \text{ in } [r, r + \varepsilon], f(x) = 0\} \\ G_{r-} &= \{f \in G \mid \text{for some } \varepsilon > 0 \text{ and for all } x \text{ in } [r - \varepsilon, r], f(x) = 0\}. \end{cases}$$

It is easily seen that these are prime subgroups, using the finiteness in the cases of G_{r+} and G_{r-}. The next three lemmas together show that these are the only prime subgroups of G.

1. Lemma. *Every proper convex ℓ-subgroup of G is contained in some G_r. Hence each prime subgroup lies in a unique G_r.*

Proof. Let $C \in \mathscr{C}(G)$ and suppose that for all $r \in \mathbb{R}$, $C \nsubseteq G_r$. Let $0 < g \in G$, and let $K = \overline{\text{supp}\,(g)}$. K is compact, and so $g(K)$ is bounded. Let $M = \sup\{g(x) \mid x \in \mathbb{R}\}$. For each $r \in K$, $C \nsubseteq G_r$, so we can find $0 < f_r \in C \setminus G_r$. Since $f_r(r) > 0$, we have $\frac{1}{2} f_r(r) > 0$. Let $U_r = \{x \in \mathbb{R} \mid f_r(x) > \frac{1}{2} f_r(r)\}$. Since each f_r is continuous, each U_r is an open neighbourhood of r, and so $\{U_r \mid r \in K\}$ is an open cover of K. Let $\{U_{r_i} \mid 1 \leqq i \leqq n\}$ be a finite subcover, and let $m = \min\{\frac{1}{2} f_{r_i}(r_i) \mid 1 \leqq i \leqq n\}$. We have $m > 0$, so there must be a positive integer N with $mN > M$. Let $f = \bigvee_{i=1}^{n} Nf_{r_i}$. Since $0 < g \leqq f \in C$, we have $g \in C$, so $C = G$, and every proper convex ℓ-subgroup lies in some G_r. Now, since G_r and G_s are incomparable if $r \neq s$, and since for any prime subgroup the set of convex ℓ-subgroups containing it is totally ordered, each prime subgroup lies in a unique G_r. □

2. Lemma. *If P is a prime subgroup of G with $P \subsetneqq G_r$, then either $P \subseteq G_{r+}$ or $P \subseteq G_{r-}$.*

Proof. Suppose P is a prime subgroup of G with $P \nsubseteq G_{r+}$ and $P \nsubseteq G_{r-}$, but $P \subseteq G_r$. Let $0 < h_1 \in P \setminus G_{r+}$ and $0 < h_2 \in P \setminus G_{r-}$. Let $0 < g \in G_r$. As in the proof of Lemma 1, let $M = \sup\{g(x) \mid x \in \mathbb{R}\}$. Since $0 < h_1 \in P \subseteq G_r$ and $h_1 \notin G_{r+}$, there must exist $\varepsilon_1 > 0$ and $m_1 > 0$ with $h_1(x) = m_1(x - r)$ for all x in $[r, r + \varepsilon_1]$. Similarly, there exist $\varepsilon_2 > 0$ and $m_2 < 0$ with $h_2(x) = m_2(x - r)$ for all x in $[r - \varepsilon_2, r]$. Since g consists of finitely many pieces over $[r, r + \varepsilon_1]$, we let $m_+ = \max\{\text{slopes of pieces of } g \text{ over } [r, r + \varepsilon_1]\}$. Similarly, we set $m_- = \min\{\text{slopes of pieces of } g \text{ over } [r - \varepsilon_2, r]\}$. We let N be a positive integer such that $m_1 N \geqq m_+$ and $m_2 N \leqq m_-$. Then for all x in $[r - \varepsilon_2, r + \varepsilon_1]$ we have $0 \leqq g(x) \leqq [N(h_1 \vee h_2)](x)$, with $N(h_1 \vee h_2) \in P$. Now let $K_0 = \overline{\text{supp}\,(g)} \setminus (r - \varepsilon_2, r + \varepsilon_1)$. K_0 is compact, and $r \notin K_0$. Since P is prime, we know from Lemma 1 that if $s \in K_0$, then $P \nsubseteq G_s$. As in the proof of Lemma 1, we find $f \in P$ with $f > 0$ and $f(x) \geqq g(x)$ for all $x \in K_0$. Then $0 < g \leqq f \vee N(h_1 \vee h_2)$, with $f \vee N(h_1 \vee h_2) \in P$, so $g \in P$, and $P = G_r$. □

3. **Lemma.** *If P is a prime subgroup of G with $P \subseteq G_{r+}$, then $P = G_{r+}$. If P is a prime subgroup of G with $P \subseteq G_{r-}$ then $P = G_{r-}$. Thus $\{C \in \mathscr{C}(G) \mid$ for some $r \in \mathbb{R}$, $C = G_r$ or G_{r+} or $G_{r-}\}$ is a complete list of the prime subgroups of G.*

Proof. Let $P \in \mathscr{C}(G)$, with $P \subsetneqq G_{r+}$. Let $0 < g \in G_{r+} \setminus P$. Then there exists $\varepsilon > 0$ with $g(x) = 0$ for all x in $[r, r + \varepsilon]$. Define f by:

$$f(x) = \begin{cases} x - r, & \text{if } x \in [r, r + \frac{1}{2}\varepsilon] \\ r + \varepsilon - x, & \text{if } x \in [r + \frac{1}{2}\varepsilon, r + \varepsilon] \\ 0, & \text{if } x \notin [r, r + \varepsilon]. \end{cases}$$

Now, $f \notin G_{r+}$, so $f \notin P$, and $g \notin P$, but $f \wedge g = 0$, so P is not prime. A similar example shows that if $P \in \mathscr{C}(G)$ and $P \subsetneqq G_{r-}$, then P is not prime. □

In G', we let $G'_r = \{f \in G' \mid f(r) = r\}$, $G'_{r+} = f \in G' \mid$ for some $\varepsilon > 0$ and for all $x \in [r, r + \varepsilon], f(x) = x\}$, and $G'_{r-} = \{f \in G' \mid$ for some $\varepsilon > 0$ and for all $x \in [r - \varepsilon, r]$, $f(x) = x\}$. Then, as part of Ball's example, it is known that $\{C' \in \mathscr{C}(G') \mid$ for some $r \in \mathbb{R}$, $C' = G'_r$ or G'_{r+} or $G'_{r-}\}$ is a complete list of the prime subgroups of G'. (The three lemmas we have just proved can be modified slightly to obtain this result.)

Now, recall (Bigard, et al, [1], 2.5.5) that a convex ℓ-subgroup is the intersection of the prime subgroups containing it. We use this to set up a lattice isomorphism between $\mathscr{C}(G)$ and $\mathscr{C}(G')$ via a lattice that describes the collections of prime subgroups above elements of $\mathscr{C}(G)$ and $\mathscr{C}(G')$. This isomorphism will send G_r to G'_r, G_{r+} to G'_{r+}, and G_{r-} to G'_{r-}, as one would expect.

Let $\mathscr{S} = \{(S, S_+, S_-) \mid S_+ \cup S_- \subseteq S \subseteq \mathbb{R}$, S is closed in $\mathbb{R}$, and for each sequence $\{s_n\}_{n=1}^{\infty}$ in S with $\lim_{n\to\infty} s_n = s$, we have that if $\{s_n\}_{n=1}^{\infty}$ is strictly increasing, then $s \in S_-$, while if $\{s_n\}_{n=1}^{\infty}$ is strictly decreasing, then $s \in S_+\}$. Order $\mathscr{S}$ by setting $(S, S_+, S_-) \leqq (T, T_+, T_-)$ if and only if $S \supseteq T$ and $S_+ \supseteq T_+$ and $S_- \supseteq T_-$. This is a lattice order on $\mathscr{S}$, with $(S, S_+, S_-) \vee (T, T_+, T_-) = (S \cap T, S_+ \cap T_+, S_- \cap T_-)$ and $(S, S_+, S_-) \wedge (T, T_+, T_-) = (S \cup T, S_+ \cup T_+, S_- \cup T_-)$.

4. **Proposition.** *There is a lattice isomorphism between $\mathscr{C}(G)$ and $\mathscr{S}$.*

Proof. Let $C \in \mathscr{C}(G)$, and set $S_C = \{r \in \mathbb{R} \mid C \subseteq G_r\}$, $S_{C+} = \{r \in \mathbb{R} \mid C \subseteq G_{r+}\}$, and $S_{C-} = \{r \in \mathbb{R} \mid C \subseteq G_{r-}\}$. We want $(S_C, S_{C+}, S_{C-}) \in \mathscr{S}$. Obviously, $S_C \subseteq \mathbb{R}$, and since $G_{r+} \subseteq G_r$ and $G_{r-} \subseteq G_r$ for each $r \in \mathbb{R}$, $S_{C+} \cup S_{C-} \subseteq S_C$. Let $\{s_n\}_{n=1}^{\infty}$ be a sequence in S_C, with $\lim_{n\to\infty} s_n = s \in \mathbb{R}$. Let $f \in C$. Then $f(s_n) = 0$ for all positive integers n, and by continuity, $f(s) = 0$, so $f \in G_s$, and $C \subseteq G_s$. Then $s \in S_C$, and S_C is closed. If, in addition, $\{s_n\}_{n=1}^{\infty}$ is strictly increasing, then because $f \in G$, there must be $\varepsilon > 0$ such that $f(x) = mx + b$ on $[s - \varepsilon, s]$. Since $\lim_{n\to\infty} s_n = s$, there must be a positive integer N such that if $n \geqq N$, then $|s_n - s| < \varepsilon$. Therefore, if $n_1 > n_2 \geqq N$

$s_{n_1} \in [s - \varepsilon, s]$ and $s_{n_2} \in [s - \varepsilon, s]$ with $s_{n_1} \neq s_{n_2}$. Hence $f(s_{n_1}) = f(s_{n_2}) = 0$, so $f(x) \equiv 0$ on $[s - \varepsilon, s]$, and $s \in S_{C-}$. Similarly if $\{s_n\}_{n=1}^{\infty}$ is strictly decreasing. then $s \in S_{C+}$, and so $(S_C, S_{C+}, S_{C-}) \in \mathscr{S}$ for each $C \in \mathscr{C}(G)$. We let $\theta : \mathscr{C}(G) \to \mathscr{S}$ by setting $\theta(C) = (S_C, S_{C+}, C_{C-})$. If $C, D \in \mathscr{C}(G)$ with $C \neq D$, then the sets of primes exceeding C is not equal to the set of primes containing D, and so $\theta(C) \neq \theta(D)$. Thus θ is one-to-one. Let $(S, S_+, S_-) \in \mathscr{S}$, and set $C = (\bigcap_{s \in S} G_s) \cap (\bigcap_{s \in S_+} G_{s+}) \cap \cap (\bigcap_{s \in S_-} G_{s-})$.

Obviously $C \in \mathscr{C}(G)$, and $\theta(C) \leqq (S, S_+, S_-)$. If $s \notin S$, then since S is closed, there is $\varepsilon > 0$ such that if $x \in (s - 2\varepsilon, s + 2\varepsilon)$, then $x \notin S$. Define $f \in G$ by

$$f(x) = \begin{cases} x - s + \varepsilon, & \text{if } s - \varepsilon \leqq x \leqq s \\ -x + s + \varepsilon, & \text{if } s \leqq x \leqq s + \varepsilon \\ 0, & \text{if } x \notin [s - \varepsilon, s + \varepsilon]. \end{cases}$$

If $x \in S$, then $x \notin [s - \varepsilon, s + \varepsilon]$, so $f(x) = 0$. If $x \in S_+ \cup S_-$, then $x \in S$ and $f(x) = 0$, and in addition, for each y with $|y - x| < \varepsilon$, we have $y \notin [s - \varepsilon, s + \varepsilon]$, since $x \notin [s - 2\varepsilon, s + 2\varepsilon]$. Thus $f(y) = 0$, so $f \in G_r$ for all $r \in S$, $f \in G_{r+}$ for all $r \in S_+$, and $f \in G_{r-}$ for all $r \in S_-$; hence $f \in C$. But $f(s) = \varepsilon > 0$, so $f \notin G_s$, and $C \nsubseteq G_s$. Thus $S_C = S$. Suppose $s \notin S_+$. By definition there is $\varepsilon > 0$ such that if $x \in (s, s + 2\varepsilon)$, then $x \notin S$. Define $g \in G$ by

$$g(x) = \begin{cases} 2x - 2s, & \text{if } x \in [s, s + \tfrac{1}{2}\varepsilon] \\ -2x + 2s + 2\varepsilon, & \text{if } x \in [s + \tfrac{1}{2}\varepsilon, s + \varepsilon] \\ 0, & \text{if } x \notin [s, s + \varepsilon]. \end{cases}$$

If $x \in S$, then $x \notin (s, s + \varepsilon)$, so $g(x) = 0$. If $x \in S_-$, then $x \notin (s, s + 2\varepsilon)$, so that for each y in $[x - \varepsilon, x]$, $y \notin (s, s + \varepsilon)$ and $g(y) = 0$. If $x \in S_+$, then $x \neq s$ and $x \notin (s, s + \varepsilon)$ so if $x \geqq s + \varepsilon$, then for all y in $[x, x + \varepsilon]$, $y \geqq s + \varepsilon$ and $g(y) = 0$, while if $x < s$, then set $\delta = \frac{1}{2}(s - x)$. We have $\delta > 0$, and for each y in $[x, x + \delta]$, $y < s$, so $g(y) = 0$. Thus $g \in C$, but $g \notin G_{s+}$ since there is no $\alpha > 0$ with $g(x) = 0$ for all x in $[s, s + \alpha]$. Hence $S_+ = S_{C+}$. A similar argument shows that $S_- = S_{C-}$, so that $\theta(C) = (S, S_+, S_-)$, and θ is onto. If $C, D \in \mathscr{C}(G)$, then since G_r is a prime subgroup of G for each $r \in \mathbb{R}$, we have that if $C \wedge D \subseteq G_r$, then $C \subseteq G_r$ or $D \subseteq G_r$. Thus $r \in S_{C \wedge D} \Leftrightarrow C \wedge D \subseteq G_r \Leftrightarrow C \subseteq G_r$ or $D \subseteq G_r \Leftrightarrow r \in S_C \cup S_D$, so $S_{C \wedge D} = = S_C \cup S_D$. Similarly, $S_{C \wedge D+} = S_{C+} \cup S_{D+}$ and $S_{C \wedge D-} = S_C \cup S_{D-}$, so that $\theta(C \wedge D) = \theta(C) \wedge \theta(D)$. Also, $r \in S_{C \vee D} \Leftrightarrow C \vee D \subseteq G_r \Leftrightarrow C \subseteq G_r$ and $D \subseteq \subseteq G_r \Leftrightarrow r \in S_C \cap S_D$, so that $S_{C \vee D} = S_C \cap S_D$. Similarly, $S_{C \vee D+} = S_{C+} \cap S_{D+}$ and $S_{C \vee D-} = S_{C-} \cap S_{D-}$, so that $\theta(C \vee D) = \theta(C) \vee \theta(D)$, and θ is a lattice isomorphism of $\mathscr{C}(G)$ onto $\mathscr{S}$. □

Essentially the same proof shows that if $\theta' : \mathscr{C}(G') \to \mathscr{S}$. By the rule $\theta'(C') = = (S_{C'}, S_{C'+}, S_{C'-})$, where $S_{C'} = \{r \in \mathbb{R} | C' \subseteq G'_r\}$, $S_{C'+} = \{r \in \mathbb{R} | C' \subseteq G'_{r+}\}$, and $S_{C'+} = \{r \in \mathbb{R} | C' \subseteq G'_{r-}\}$, then θ' is a lattice isomorphism of $\mathscr{C}(G')$ onto $\mathscr{S}$. Thus, the function $(\theta')^{-1} \theta$ is a lattice isomorphism of $\mathscr{C}(G)$ onto $\mathscr{C}(G')$, and it is easily checked that this map sends G_r to G'_r, G_{r+} to G'_{r+}, and G_{r-} to G'_{r-} for each $r \in \mathbb{R}$.

Now, we consider the questions of recognizability. First, G is a member of the variety of abelian lattice-ordered groups, which is a subvariety of all nontrivial varieties, while G' lies outside the variety of normal-valued ℓ-groups, since each G'_r is a regular subgroup of G' but is not normal (however, $G'_{r+} \lhd G'_r$ and $G'_{r-} \lhd G'_r$, for each $r \in \mathbb{R}$). But $\mathcal{N}$ contains all proper varieties, so G' cannot lie in any proper variety, while G is in all nontrivial varieties, and it is impossible to tell if a lattice-ordered group belongs to any nontrivial proper variety by considering only its lattice of convex ℓ-subgroups.

Next, G is archimedean, while G' is not, and so it is impossible to recognize from the lattice of convex ℓ-subgroups if a given ℓ-group is archimedean.

Martinez [6] introduced the idea of a torsion class, which generalizes the notion of a variety. The class of archimedean ℓ-groups is not a torsion class, but if we take only those archimedean ℓ-groups with the additional property that every ℓ-homomorphic image is archimedean, then we get the torsion class of hyperarchimedean ℓ-groups. Several characterizations of hyperarchimedean ℓ-groups are known (for a history, see Conrad [3]), including the fact that an ℓ-group is hyperarchimedean if and only if each prime subgroup is a minimal prime subgroup, and so each prime subgroup is a maximal. Since a convex ℓ-subgroup P is prime if and only if the set of convex ℓ-subgroups containing P is totally ordered, we can recognize primes, and certainly maximals are recognizable, so it is possible to recognize if an ℓ-group is hyperarchimedean. Hence some torsion classes are recognizable, while varieties are not. Conrad [4] has considered several other important torsion classes, and determined which are recognizable by looking at torsion radicals. He has shown that the torsion class of divisible abelian ℓ-groups is not recognizable, while the following torsion classes are recognizable:

$\mathcal{A}$ = all hyperarchimedean ℓ-groups
$\mathcal{F}$ = all ℓ-groups such that each bounded disjoint subset is finite
$\mathcal{F}_v$ = all finite-valued ℓ-groups
$\mathcal{D}$ = all ℓ-groups with DCC on the set of regular subgroups
$\mathcal{O}$ = all cardinal sums of o-groups
$\mathcal{R}$ = all cardinal sums of archimedean o-groups
$\mathcal{B}$ = all ℓ-groups such that each prime exceeds a unique minimal prime.

Recall that in an ℓ-group A, a convex ℓ-subgroup C is closed if for each family $\{x_i \mid i \in I\}$ with $e \leqq x_i \in C$ such that $\bigvee_{i \in I} x_i = x$ exists in A, then $x \in C$. In Bigard, et al. [1] (11.1.10), it is shown that in an archimedean ℓ-group, closed convex ℓ-subgroups are polars, so that in G, no G_r, G_{r+}, or G_{r-} can be closed, since they are not polars (for $G_{r+}^{\perp} = G_{r-}^{\perp} = \{0\}$). On the other hand, for each $r \in \mathbb{R}$, G'_r is closed in G'. This is a result of McCleary [7], which generalized the work of Lloyd [5]. Hence it is impossible to recognize closed convex ℓ-subgroups in the lattice of convex ℓ-subgroups of a lattice-ordered group.

Finally, Byrd and Lloyd [2] define the distributive radical of an ℓ-group, and show that it is the intersection of all closed prime subgroups. They also show that an ℓ-group is completely distributive if and only if it has a trivial distributive radical. In our examples, no prime subgroups of G are closed, so that G is its own radical, while all of the G'_r are closed in G', so that the distributive radical of G' is $\{1\}$. Hence it is impossible to recognize the distributive radical, and one cannot tell if an ℓ-group is completely distributive, from the lattice of convex ℓ-subgroups.

Bibliography

[1] *Bigard, A., Keimel, K.*, and *Wolfenstein, S.*: Groupes et Anneaux Réticulés, Springer-Verlag, Berlin (1977).

[2] *Byrd, R. D.*, and *Lloyd, J. T.*: Closed subgroups and complete distributivity in lattice-ordered groups, Math. Zeit., *101* (1967), 123–130.

[3] *Conrad, P.*: Epi-archimedean groups, Czech. Math. J., *24* (1974), 192–218.

[4] *Conrad, P.*: Torsion radicals of lattice-ordered groups, Inst. Naz. di Alta Mate., Symposium Mathematica, *XXI* (1977), 479–513.

[5] *Lloyd, J. T.*: Complete distributivity in certain infinite permutation groups, Mich. Math. J., *14* (1967) 393–400.

[6] *Martinez, J.*: Torsion theory for lattice-ordered groups, Czech. Math. J. *25* (1975), 284–299.

[7] *McCleary, S. H.*: Closed subgroups of lattice-ordered permutation groups, Trans. Amer. Math. Soc., *173* (1972), 303–314.

Author's address: Department of Mathematics, The University of Kansas, Lawrence, Kansas 66045, U.S.A.