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A FORD-FULKERSON TYPE THEOREM CONCERNING VECTOR-VALUED FLOWS IN INFINITE NETWORKS

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1. INTRODUCTION

In this note, we are concerned with the existence of maximal vector-valued flows for a wide class of networks. In this generality, a measure theoretic setting turns out to be quite natural. Our main result immediately includes the well-known Ford-Fulkerson theorem on maximal flows and minimal cuts in the classical situation of finite networks. But even in this special case, our approach substantially differs from the various known proofs involving a suitable combinatorical argument, a labeling process, or the Farkas lemma; see for instance [1] or [7]. Here, the main ingredient will be an appropriate version of the Hahn-Banach theorem, and correspondingly certain sublinear operators will play an essential role. Our techniques are related to those of Fuchssteiner [2] and König-Neumann [3] in somewhat different situations. It should be noted, however, that here we do not need any disintegration type argument. The present approach can also be used to handle rather general supply and demand problems from mathematical economics and to prove the existence of measures with given marginals in various situations. These aspects will be studied in another paper. The author is indebted to Professor Vlastimil Pták for a stimulating remark.

2. FLOWS AND CUTS IN INFINITE NETWORKS

Throughout this note, let S be a non-empty set endowed with some algebra Σ of subsets, and consider a pair of disjoint sets $P, Q \in \Sigma$. Further, let X denote an ordered vector space which is assumed to be Dedekind complete in the sense that each upper bounded subset has a supremum. Finally, let us fix a pair of biadditive set functions $\sigma, \tau : \Sigma \times \Sigma \to X$ satisfying $\sigma(A, B) \leqq \tau(A, B)$ at least for all disjoint sets $A, B \in \Sigma$. In this situation, the tupel $N := (S, \Sigma, P, Q, X, \sigma, \tau)$ is said to be a generalized *network*.

Of course, S will be interpreted as the set of nodes of the given network, P and Q will be viewed as the sets of sinks and sources, respectively, and $\sigma(A, B)$, $\tau(A, B) \in X$

will stand for the lower and upper capacities concerning all arcs starting in $A \in \Sigma$ and leading to $B \in \Sigma$. This is a reasonable generalization of the classical situation, where S is some finite set, Σ is the whole power set $\mathscr{P}(S)$, X is the real line $\mathbb{R}$, and σ, τ are obtained by summation over the respective individual arc capacities; let us refer to [1] or [7] for the classical background material. The following definitions correspond to the classical notions as well.

Definitions. A biadditive set function $v : \Sigma \times \Sigma \to X$ is said to be a *flow* in N, if the following conditions are satisfied:

(1) $\sigma(A, B) \leqq v(A, B) \leqq \tau(A, B)$ for all disjoint $A, B \in \Sigma$;

(2) $v(A, S) = v(S, A)$ for all $A \in \Sigma$ with $A \subset R := \overline{P \cup Q}$,

where the bar $^-$ denotes the complement with respect to S. The *value* of a flow v is defined to be $f(v) := v(Q, S) - v(S, Q)$, and v is termed *maximal* if $f(\tilde{v}) \leqq f(v)$ holds for every flow $\tilde{v}$ in N. Finally, any $A \in \Sigma$ satisfying $Q \subset A \subset Q \cup R$ is called a *cut* in N; the corresponding *cut capacity* is given by $g(A) := \tau(A, \bar{A}) - \sigma(\bar{A}, A)$.

Remarks. It is intuitively clear and easily verified that every flow v in N satisfies $f(v) = v(S, P) - v(P, S)$. Moreover, for every flow v and every cut A in N we have $f(v) \leqq g(A)$ because of $f(v) = v(A, S) - v(S, A) = v(A, \bar{A}) - v(\bar{A}, A) \leqq g(A)$. Hence, if there is at least one flow in N, then

$$c := \inf \{g(A) : A \in \Sigma \text{ cut in } N\} \in X$$

exists according to the Dedekind completeness of X and satisfies $f(v) \leqq c$ for every flow v in N. Note, however, that flows do not exist in general.

Theorem. *There exists at least one flow in the network N if and only if the following condition*

(3) $\tau(A, \bar{A}) \geqq \sigma(\bar{A}, A)$ *for all $A \in \Sigma$ with $A \subset R$ or with $\bar{A} \subset R$ is fulfilled. Moreover, in this case we have:*

(4) $\max \{f(v) : v$ *flow in in* $N\} = \inf \{g(A) : A$ *cut in* $N\}$.

(5) *Under the stronger assumption $\sigma \leqq \tau$ on $\Sigma \times \Sigma$, there exists a maximal flow v in N satisfying $\sigma \leqq v \leqq \tau$ on $\Sigma \times \Sigma$.*

(6) *If $\sigma, \tau \geqq 0$ on $\Sigma \times \Sigma$, then there exists a maximal flow v in N satisfying $\sigma \leqq$ $\leqq v \leqq \sigma + \tau$ on $\Sigma \times \Sigma$.*

The proof of this generalization of the Ford-Fulkerson theorem will be postponed to the last section. The main point will be the construction of a flow v in N satisfying $f(v) \geqq c$ as well as certain additional estimates with respect to σ and τ, which are strong enough to ensure (5) and (6). If X is endowed with a vector space topology such that the positive cone is normal [5; p. 61], and if σ and τ are both assumed to be bi-σ-additive, then the flows v from (5) and (6) are automatically bi-σ-additive as well. These assertions are of particular interest in the context of measures on the product space $S \times S$. To demonstrate this fact, we assume $X = \mathbb{R}$, for simplicity, and suppose that the mappings $\sigma, \tau : \Sigma \times \Sigma \to \mathbb{R}$ are canonically induced by finite measures $\hat{\sigma}, \hat{\tau} : \Sigma \otimes \Sigma \to \mathbb{R}$ such that $\hat{\sigma} \leqq \hat{\tau}$ on $\Sigma \otimes \Sigma$, where $\Sigma \otimes \Sigma$ denotes the usual σ-algebra on $S \times S$ generated by $\Sigma \times \Sigma$. Thus, by definition, we have $\sigma(A, B) = \hat{\sigma}(A \times B)$ and $\tau(A, B) = \hat{\tau}(A \times B)$ for all $A, B \in \Sigma$. Now, standard measure theory confirms that every biadditive set function $v : \Sigma \times \Sigma \to \mathbb{R}$ satisfying $\sigma \leqq v \leqq \tau$ on $\Sigma \times \Sigma$ canonically gives rise to a measure $\hat{v} : \Sigma \otimes \Sigma \to \mathbb{R}$ satisfying $\hat{\sigma} \leqq \hat{v} \leqq \hat{\tau}$ on $\Sigma \otimes \Sigma$ as well as $v(A, B) = \hat{v}(A \times B)$ for all $A, B \in \Sigma$. We conclude that our main result contains a Ford-Fulkerson type theorem for finite measures on $\Sigma \otimes \Sigma$ as a special case.

3. AUXILIARY RESULTS ON SUBLINEAR OPERATORS

Our principal tool will be the subsequent easy consequence of the vector-valued version of the Mazur-Orlicz theorem [4; Th. 2.41]. We note that the elegant proof of the latter result due to Pták [6] immediately carries over to the case of X-valued mappings; see also Peressini [5; p. 79]. Lemma 1 can also be deduced from a suitable extension version of the Hahn-Banach theorem.

Lemma 1. *Let H be a linear subspace of some real vector space G, and let $u \in G$. Further, consider a sublinear mapping $\varrho : G \to X$, a linear mapping $\mu : H \to X$, and some $x \in X$. Then the following assertions are equivalent*:

(7) $$tx + \mu(v) \leqq \varrho(tu + v) \quad \text{for all real} \quad t \geqq 0 \quad \text{and all} \quad v \in H\,.$$

(8) *There exists a linear mapping $\xi : G \to X$ such that $\xi \leqq \varrho$ on G, $\xi = \mu$ on H, and $\xi(u) \geqq x$.*

In the following, let E be the space of all Σ-measurable simple functions $\phi : S \to \mathbb{R}$, and let F consist of all $\Sigma \circ \Sigma$-measurable simple functions $\phi : S \times S \to \mathbb{R}$, where $\Sigma \circ \Sigma$ denotes the canonical algebra on $S \times S$ generated by $\Sigma \times \Sigma$. Given a subset A of S or $S \times S$, χ_A will stand for the corresponding characteristic function. Now,

$$\theta(\phi)(a, b) := \max\{\phi(a) - \phi(b), 0\} \quad \text{for all} \quad \phi \in E \quad \text{and} \quad a, b \in S$$

defines a sublinear operator $\theta : E \to F$. Assertion (9) of the following lemma indicates the relevance of this operator to the theory of networks.

Lemma 2. The operator $\theta : E \to F$ has the following properties:

(9) $\theta(\chi_A) = \chi_{A \times \bar{A}}$ and $\theta(-\chi_A) = \chi_{\bar{A} \times A}$ for all $A \in \Sigma$.

(10) $\theta(\phi + t) = \theta(\phi)$ for all $\phi \in E$ and $t \in \mathbb{R}$.

(11) $\theta(\phi - \psi) = \theta(\phi) + \theta(-\psi)$ for all $\phi, \psi \in E_+$ satisfying $\phi\psi = 0$.

(12) $\theta(t_1\chi_{A_1} + \ldots + t_n\chi_{A_n}) = t_1\theta(\chi_{A_1}) + \ldots + t_n\theta(\chi_{A_n})$ for all real $t_1, \ldots, t_n \geqq 0$ and all $A_1, \ldots, A_n \in \Sigma$ satisfying $A_1 \subset \ldots \subset A_n$ or $A_1 \supset \ldots \supset A_n$.

Proof. (9) and (10) are obvious. In order to prove (11), fix an $A \in \Sigma$ such that $\phi = 0$ on $\bar{A}$ and $\psi = 0$ on A. Then the desired identity can be easily checked pointwise on $A \times A$, $\bar{A} \times \bar{A}$, $A \times \bar{A}$, and $\bar{A} \times A$. For the proof of (12) we may assume that $S = A_1 \supset \ldots \supset A_n = \emptyset$. Then, given $a, b \in S$, there is a greatest integer k such that $a \in A_k$ and a smallest integer l such that $b \in \bar{A}_l$. By means of (9) we conclude that the functions from the left and right hand side of (12) both assume at the point (a, b) the value $t_l + \ldots + t_k$ if $l \leqq k$, resp. 0 if $l > k$.

Next, let G denote the space of all $\Sigma \circ \Sigma$-measurable simple functions $\psi : S \times S \to$ $\to E$ and define

$$\varrho(\psi) := \int_{S \times S} \theta(\psi(a, b))\,(a, b)\,\mathrm{d}\lambda(a, b) \quad \text{for all} \quad \psi \in G\,, \tag{13}$$

where $\lambda : \Sigma \circ \Sigma \to X$ is a given additive set function, and the X-valued integral is understood in the usual elementary sense.

Lemma 3. *The operator $\varrho : G \to X$ is sublinear provided that $\lambda(A \times B) \geqq 0$ holds for all disjoint $A, B \in \Sigma$.*

Proof. First note that any $\psi \in G$ can be written in the form

$$\psi = \sum_{i=1}^{n} \phi_i \chi_{A_i \times B_i} \quad \text{with} \quad \phi_i \in E\,, \quad A_i \times B_i \in \Sigma \times \Sigma \quad \text{pairwise disjoint}\,.$$

And for every representation of this type we have

$$\varrho(\psi) = \sum_{i=1}^{n} \int_{A_i \times B_i} \theta(\phi_i)\,\mathrm{d}\lambda\,.$$

Hence it suffices to show that

$$\int_{A \times B} \theta(\phi + \tilde{\phi})\,\mathrm{d}\lambda \leqq \int_{A \times B} \theta(\phi)\,\mathrm{d}\lambda + \int_{A \times B} \theta(\tilde{\phi})\,\mathrm{d}\lambda$$

holds for all $\phi, \tilde{\phi} \in E$ and $A, B \in \Sigma$. Now, for suitable $s_j, t_j \in \mathbb{R}$ and pairwise disjoint $C_j \in \Sigma$ we have $C_1 \cup \ldots \cup C_m = S$,

$$\phi = \sum_{j=1}^{m} s_j \chi_{C_j} \quad \text{and} \quad \tilde{\phi} = \sum_{j=1}^{m} t_j \chi_{C_j}\,,$$

and therefore

$$\int_{A \times B} \theta(\phi + \tilde{\phi})\,\mathrm{d}\lambda = \sum_{j,k=1}^{m} \max\{s_j + t_j - s_k - t_k, 0\}\,\lambda((A \cap C_j) \times (B \cap C_k))\,.$$

On the right hand side, the scalar factor vanishes for $j = k$, whereas for $j \neq k$ our assumption on λ implies $\lambda((A \cap C_j) \times (B \cap C_k)) \geqq 0$. Hence an obvious estimation finishes the proof.

4. PROOF OF THE THEOREM

First suppose that there exists some flow v in N. Then every $A \in \Sigma$ with $A \subset R$ satisfies

$$\tau(A, \bar{A}) - \sigma(\bar{A}, A) \geqq v(A, \bar{A}) - v(\bar{A}, A) = v(A, S) - v(S, A) = 0 .$$

And in the case $\bar{A} \subset R$ we have similarly

$$\tau(A, \bar{A}) - \sigma(\bar{A}, A) \geqq v(A, \bar{A}) - v(\bar{A}, A) = v(S, \bar{A}) - v(\bar{A}, S) = 0 ,$$

so that condition (3) turns out to be necessary. For the remainder of the proof, we assume that (3) is fulfilled. We claim that every cut $A \in \Sigma$ satisfies $\sigma(Q, \bar{Q}) - \tau(\bar{Q}, Q) \leqq g(A)$. Indeed, writing $A = Q \cup B$ with some $B \subset R$, we obtain

$$\tau(A, \bar{A}) + \tau(\bar{Q}, Q) = \tau(Q, \bar{A}) + \tau(\bar{A}, Q) + \tau(B, \bar{B}) \quad \text{and}$$
$$\sigma(\bar{A}, A) + \sigma(Q, \bar{Q}) = \sigma(Q, \bar{A}) + \sigma(\bar{A}, Q) + \sigma(\bar{B}, B) ,$$

so that the desired inequality is immediate after (3) and our basic assumption on σ and τ. Now the Dedekind completeness of X ensures the existence of $c := \inf \{g(A) : A \in \Sigma \text{ cut in } N\} \in X$. We proceed to the construction of some flow v in N satisfying $f(v) \geqq c$, which will establish the identity (4). In a natural way, the biadditive set function $\tau - \sigma$ on $\Sigma \times \Sigma$ induces an additive set function $\lambda : \Sigma \circ \Sigma \to X$ via the formula

$$\lambda\left(\bigcup_{i=1}^{n} A_i \times B_i\right) = \sum_{i=1}^{n} (\tau - \sigma)(A_i, B_i)$$

for every finite system of pairwise disjoint rectangles $A_i \times B_i \in \Sigma \times \Sigma$. Let $\varrho : G \to X$ denote the corresponding sublinear operator given by (13). We identify the elements of E with the constant functions in G so that $E \subset G$. Consider now the linear subspace $H := \{\eta \in E : \eta = 0 \text{ on } P \cup Q\}$ of G, the linear mapping $\mu : H \to X$ given by

$$\mu(\eta) := \int_S \eta \, d\sigma(S, \cdot) - \int_S \eta \, d\sigma(\cdot, S) \quad \text{for all} \quad \eta \in H ,$$

and finally $u := \chi_Q \in G$ and $x := c + \sigma(S, Q) - \sigma(Q, S) \in X$. We claim that in this situation condition (7) is fulfilled. To prove this assertion, let $t \geqq 0$ and $\eta \in H$ be arbitrarily given, and let ϕ and ψ denote the positive and negative part of $t\chi_Q + \eta$, respectively. Thus $t\chi_Q + \eta = \phi - \psi$ with $\phi, \psi \in E_+$ and $\phi\psi = 0$. We arrange all values of ϕ and ψ into an increasing sequence of real numbers $0 = s_0 < s_1 < \ldots < s_n$ and define

$$A_i := \{a \in S : \phi(a) \geqq s_i\}\,, \quad B_i := \{a \in S : \psi(a) \geqq s_i\}\,, \quad t_i := s_i - s_{i-1}$$

for $i = 1, \ldots, n$. Then it follows that

$$\phi = \sum_{i=1}^{n} t_i \chi_{A_i} \quad \text{and} \quad \psi = \sum_{i=1}^{n} t_i \chi_{B_i}\,.$$

For all $i = 1, \ldots, n$ we have $B_i \subset R$ and hence $\tau(\bar{B}_i, B_i) \geqq \sigma(B_i, \bar{B}_i)$ in view of (3). Further, note that $t = s_j$ for exactly one $j = 0, 1, \ldots, n$. If $i > j$, then certainly $A_i \subset R$. Using (3) again, we arrive at $\tau(A_i, \bar{A}_i) \geqq \sigma(\bar{A}_i, A_i)$ for $i = j + 1, \ldots, n$. In the remaining case $i \leqq j$, the set A_i turns out to be a cut in N which implies $\tau(A_i, \bar{A}_i) \geqq \sigma(\bar{A}_i, A_i) + c$ for $i = 1, \ldots, j$ according to the definition of c. We finally observe that $t = t_1 + \ldots + t_j$. Combining all these facts with the properties of θ from lemma 2, we conclude that

$$\begin{aligned}
\varrho(t\chi_Q + \eta) = \varrho(\phi - \psi) &= \int_{S \times S} \theta(\phi - \psi)\, \mathrm{d}\lambda = \int_{S \times S} (\theta(\phi) + \theta(-\psi))\, \mathrm{d}\lambda = \\
&= \int_{S \times S} \theta(\phi)\, \mathrm{d}\lambda + \int_{S \times S} \theta(t_1 + \ldots + t_n - \psi)\, \mathrm{d}\lambda = \\
&= \sum_{i=1}^{n} t_i \lambda(A_i \times \bar{A}_i) + \sum_{i=1}^{n} t_i \lambda(\bar{B}_i \times B_i) \geqq \\
&\geqq tc + \sum_{i=1}^{n} t_i (\sigma(\bar{A}_i, A_i) - \sigma(A_i, \bar{A}_i) + \sigma(B_i, \bar{B}_i) - \sigma(\bar{B}_i, B_i)) = \\
&= tc + \sum_{i=1}^{n} t_i (\sigma(S, A_i) - \sigma(A_i, S) + \sigma(B_i, S) - \sigma(S, B_i)) = \\
&= tc + \int_S (\phi - \psi)\, \mathrm{d}\sigma(S, \cdot) - \int_S (\phi - \psi)\, \mathrm{d}\sigma(\cdot, S) = tx + \mu(\eta)\,.
\end{aligned}$$

Hence lemma 1 supplies us with some linear mapping $\xi : G \to X$ with the properties stated in (8). We now define

$$\alpha(A, B) := \xi(\psi_{A,B}) \quad \text{for all} \quad A, B \in \Sigma\,,$$

where $\psi_{A,B} \in G$ denotes the function being constant to χ_A on $S \times B$ and constant to 0 on $S \times \bar{B}$. The mapping $\alpha : \Sigma \times \Sigma \to X$ is certainly biadditive. Furthermore, from $\xi \leqq \varrho$ on G we obtain the estimates

$$(14) \quad (\sigma - \tau)(\bar{A}, A \cap B) \leqq \alpha(A, B) \leqq (\tau - \sigma)(A, \bar{A} \cap B) \quad \text{for all} \quad A, B \in \Sigma\,.$$

In particular, these estimates imply that $\alpha(S, A) = 0$ for all $A \in \Sigma$ as well as

$$0 \leqq \alpha(A, B) \leqq (\tau - \sigma)(A, B) \quad \text{for all disjoint} \quad A, B \in \Sigma\,.$$

On the other hand, from $\xi(u) \geqq x$ and $\xi = \mu$ on H we conclude that

$$\alpha(Q, S) = \xi(\chi_Q) \geqq c + \sigma(S, Q) - \sigma(Q, S),$$
$$\alpha(A, S) = \xi(\chi_A) = \sigma(S, A) - \sigma(A, S) \quad \text{for all} \quad A \in \Sigma \quad \text{with} \quad A \subset R .$$

Thus $v := \alpha + \sigma$ turns out to be a flow in N satisfying $f(v) \geqq c$, which completes the proof of (4). It should be noted that this flow satisfies $v \leqq \tau$ on $\Sigma \times \Sigma$ whenever $\sigma \leqq \tau$ on $\Sigma \times \Sigma$, but because of $\alpha(A, A) = -\alpha(\bar{A}, A)$ for all $A \in \Sigma$, we cannot expect that the estimate $\sigma \leqq v$ will hold on $\Sigma \times \Sigma$ in general. Consequently, a certain modification of α and v has to be taken into account for the proof of (5) and (6). In order to give a unified approach to these assertions, let us assume that $\beta : \Sigma \times \times \Sigma \to X$ is a positive biadditive mapping satisfying $\tau - \sigma \leqq \beta$ on $\Sigma \times \Sigma$. From (14) we obtain $0, \hat{\alpha} \leqq \hat{\beta}$ on $\Sigma \circ \Sigma$, where $\hat{\alpha}$ and $\hat{\beta}$ denote the respective additive set functions on $\Sigma \circ \Sigma$. Hence the Dedekind completeness of X ensures the existence of

$$\gamma(V) := \sup \{\hat{\alpha}(U) : U \in \Sigma \circ \Sigma \text{ with } U \subset V\} \in X$$

for every $V \in \Sigma \circ \Sigma$. This definition yields an additive mapping $\gamma : \Sigma \circ \Sigma \to X$ satisfying $0 \leqq \gamma \leqq \hat{\beta}$ on $\Sigma \circ \Sigma$. Moreover, from $\alpha(A, B) \geqq 0$ for all disjoint $A, B \in \Sigma$ one easily deduces that $\gamma(A \times \bar{A}) = \alpha(A, \bar{A})$ holds for all $A \in \Sigma$. Now, the mapping $\tilde{v} : \Sigma \times \times \Sigma \to X$ given by

$$\tilde{v}(A, B) := \gamma(A \times B) + \sigma(A, B) \quad \text{for all} \quad A, B \in \Sigma$$

is biadditive and fulfills $\sigma \leqq \tilde{v} \leqq \sigma + \beta$ on $\Sigma \times \Sigma$. And from $\tilde{v}(A, \bar{A}) = v(A, \bar{A})$ for all $A \in \Sigma$, it is obvious that $\tilde{v}$ is a maximal flow again. Now, the assertions (5) and (6) are readily obtained by choosing β to be $\tau - \sigma$ and τ, respectively. Let us finally note that the present approach reveals a maximal flow $\tilde{v}$ in N satisfying $\sigma \leqq \leqq \tilde{v} \leqq \sigma + (\tau - \sigma)^+$ as soon as the positive part $(\tau - \sigma)^+$ exists in a reasonable sense.

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