Vítězslav Novák; Miroslav Novotný

Dimension theory for cyclically and cocyclically ordered sets

Czechoslovak Mathematical Journal, Vol. 33 (1983), No. 4, 647–653

Persistent URL: http://dml.cz/dmlcz/101917

Czechoslovak Mathematical Journal, 33 (108) 1983, Praha

DIMENSION THEORY FOR CYCLICALLY AND COCYCLICALLY ORDERED SETS

VÍTĚZSLAV NOVÁK and MIROSLAV NOVOTNÝ, Brno

(Received March 7, 1983)

In [6], a general dimension theory for algebraic structures is developed. In this note we show that this theory can be applied also to cyclically and cocyclically ordered sets; the dimension of these sets — with a suitable choice of basic classes — coincides with the characteristics w, W, d, D as introduced in [7].

1. K_4-CLASSES

Here we summarize some necessary concepts from [6].

1.1. Notation. Let $I \neq \emptyset$ be a set, $(G_i;\ i \in I)$ an indexed family of sets, i.e. a mapping assigning a set G_i to any $i \in I$. We put $\sum_{i \in I} G_i = \{(i, k);\ i \in I,\ k \in G_i\}$. Let $\mathscr{C}$ be a class; an indexed family $(G_i;\ i \in I)$ with $G_i \in \mathscr{C}$ is called a *family of elements in* $\mathscr{C}$. By $\mathscr{S}(\mathscr{C})$ we denote the class of all families of elements in $\mathscr{C}$.

1.2. Definition. Let $\mathscr{C}$ be a class, R a correspondence between $\mathscr{C}$ and $\mathscr{S}(\mathscr{C})$ with the property: $G, H \in \mathscr{C}$, $(H_i;\ i \in I) \in \mathscr{S}(\mathscr{C})$, $G\,R(H)$, $H\,R\,(H_i;\ i \in I) \Rightarrow G\,R\,(H_i;\ i \in I)$. Then the pair $(\mathscr{C}, R)$ is called a K_1-*class*.

1.3. Notation. Let $(\mathscr{C}, R)$ be a K_1-class, $\mathscr{L} \subseteq \mathscr{C}$. We denote $\mathscr{C}(\mathscr{L}, R) = \{G \in \mathscr{C};$ there exists $(G_i;\ i \in I) \in \mathscr{S}(\mathscr{L})$ with $G\,R\,(G_i;\ i \in I)\}$.

1.4. Definition. Let $(\mathscr{C}, R)$ be a K_1-class, $\mathscr{L} \subseteq \mathscr{C}$. For any $G \in \mathscr{C}(\mathscr{L}, R)$ put $(\mathscr{L}, R) - \dim G = \min \{\text{card } I;$ there exists $(G_i;\ i \in I) \in \mathscr{S}(\mathscr{L})$ with $G\,R\,(G_i;\ i \in I)\}$. The cardinal $(\mathscr{L}, R) - \dim G$ is called the $(\mathscr{L}, R)$-*dimension of* G.

1.5. Definition. Let $\mathscr{C}$ be a class, $T : \mathscr{S}(\mathscr{C}) \to \mathscr{C}$ a partial mapping with the property: $I = \{i_0\}$, $G_{i_0} = G \in \mathscr{C} \Rightarrow (G_i;\ i \in I) \in \text{dom } T$ and $\underset{i \in I}{T} G_i = G$. Let $\leqq$ be a (fixed) preorder on $\mathscr{C}$. Then the triple $(\mathscr{C}, T, \leqq)$ is called a K_2-*class*.

1.6. Remark. Any K_2-class $(\mathscr{C}, T, \leqq)$ is a K_1-class if we define a correspondence $R(T, \leqq)$ between $\mathscr{C}$ and $\mathscr{S}(\mathscr{C})$ by $G\, R(T, \leqq)\, (G_i;\ i \in I) \Leftrightarrow (G_i;\ i \in I) \in \operatorname{dom} T$ and $G \leqq \underset{i \in I}{T} G_i$.

1.7. Definition. Let $(\mathscr{C}, T, \leqq)$ be a K_2-class with the following properties:

(1) if $I \neq \emptyset$ is a set, $K_i \neq \emptyset$ a set for any $i \in I$, $G_{i,k} \in \mathscr{C}$ for any $(i, k) \in \sum_{i \in I} K_i$ and $(G_{i,k};\ (i, k) \in \sum_{i \in I} K_i) \in \operatorname{dom} T$, $(G_{i,k};\ k \in K_i) \in \operatorname{dom} T$ for any $i \in I$, $(\underset{k \in K_i}{T} G_{i,k};\ i \in I) \in$ $\in \operatorname{dom} T$, then $\underset{(i,k) \in \sum_{i \in I} K_i}{T} G_{i,k} = \underset{i \in I}{T}(\underset{k \in K_i}{T} G_{i,k})$ (the associative rule)

(2) if $I \neq \emptyset$ is a set, $(G_i;\ i \in I) \in \operatorname{dom} T$, $(H_i;\ i \in I) \in \operatorname{dom} T$ and $G_i \leqq H_i$ for any $i \in I$, then $\underset{i \in I}{T} G_i \leqq \underset{i \in I}{T} H_i$.

Then $(\mathscr{C}, T, \leqq)$ is called a K_3-*class*. If $(\mathscr{C}, T, \leqq)$ is a K_3-class and $\operatorname{dom} T = \mathscr{S}(\mathscr{C})$, then $(\mathscr{C}, T, \leqq)$ is called a K_4-*class*.

1.8. Remark. If $(\mathscr{C}, T, \leqq)$ is a K_4-class and $\mathscr{L} \subseteq \mathscr{C}$, we write $\mathscr{C}(\mathscr{L}; T, \leqq)$ instead of $\mathscr{C}(\mathscr{L}, R(T, \leqq))$, and $(\mathscr{L}; T, \leqq) - \dim G$ instead of $(\mathscr{L}, R(T, \leqq)) - \dim G$. Thus, $\mathscr{C}(\mathscr{L}; T, \leqq) = \{G \in \mathscr{C};$ there exists $(G_i;\ i \in I) \in \mathscr{S}(\mathscr{L})$ with $G \leqq \underset{i \in I}{T} G_i\}$ and $(\mathscr{L}; T, \leqq) - \dim G = \min\{\operatorname{card} I;$ there exists $(G_i;\ i \in I) \in \mathscr{S}(\mathscr{L})$ with $G \leqq \underset{i \in I}{T} G_i\}$ for any $G \in \mathscr{C}(\mathscr{L}; T, \leqq)$.

1.9. Notation. Let $\mathscr{C}$ be the class of all pairs (G, C) where G is a set and C a ternary relation on G. Let us put $\bigcup_{i \in I} (G_i, C_i) = (\bigcup_{i \in I} G_i, \bigcup_{i \in I} C_i)$, $\bigcap_{i \in I} (G_i, C_i) = (\bigcap_{i \in I} G_i, \bigcap_{i \in I} C_i)$ for any $((G_i, C_i);\ i \in I) \in \mathscr{S}(\mathscr{C})$. Let $\mathrm{id}_{\mathscr{C}}$ be the identity relation $=$ on $\mathscr{C}$, thus $(G, C) =$ $= (H, D)$ means $G = H$, $C = D$.

1.10. Lemma. $(\mathscr{C}, \bigcup, \mathrm{id}_{\mathscr{C}})$, $(\mathscr{C}, \bigcap, \mathrm{id}_{\mathscr{C}})$ *are* K_4-*classes.*

Proof. The identity $\mathrm{id}_{\mathscr{C}}$ is a preorder on $\mathscr{C}$ and the properties from 1.5. and 1.7. are simple consequences of the properties of the set-theoretical operations $\bigcup$, $\bigcap$.

1.11. Notation. Let $\mathscr{C}$ be the same as in 1.9. We put $\underset{i \in I}{\times} (G_i, C_i) = (\underset{i \in I}{\times} G_i, \underset{i \in I}{\times} C_i)$ for any $((G_i, C_i);\ i \in I) \in \mathscr{S}(\mathscr{C})$; here $\underset{i \in I}{\times} G_i$ denotes the cartesian product of sets and $\underset{i \in I}{\times} C_i$ the direct (cartesian) product of ternary relations, thus $(x, y, z) \in \underset{i \in I}{\times} C_i$ for $x, y, z \in \underset{i \in I}{\times} G_i$ means $(x(i), y(i), z(i)) \in C_i$ for any $i \in I$. For (G, C), $(H, D) \in \mathscr{C}$ put $(G, C)\ i\ (H, D)$ iff there exists an isomorphism of (G, C) into (H, D).

1.12. Lemma. $(C, \times, i)$ *is a* K_4-*class.*

Proof is trivial. i is evidently a preorder on $\mathscr{C}$ and $\times$ has the properties from 1.5. and 1.7.

2.1. Definition. Let G be a set, C a ternary relation on G. C is called a *cyclic order* on G, iff it is:

(i) asymmetric, i.e. $(x, y, z) \in C \Rightarrow (z, y, x) \bar{\in} C$

(ii) transitive, i.e. $(x, y, z) \in C$, $(x, z, u) \in C \Rightarrow (x, y, u) \in C$

(iii) cyclic, i.e. $(x, y, z) \in C \Rightarrow (y, z, x) \in C$.

C is called a *cocyclic order* on G, iff it is cyclic,

(iv) reflexive, i.e. $x, y, z \in G$, card $\{x, y, z\} \leqq 2 \Rightarrow (x, y, z) \in C$

(v) complete, i.e. $x, y, z \in G$, $x \neq y \neq z \neq x \Rightarrow (x, y, z) \in C$ or $(z, y, x) \in C$

and satisfies the condition

(vi) $x, y, z, u \in G$, pairwise distinct, $(x, y, z) \in C \Rightarrow (x, y, u) \in C$ or $(x, u, z) \in C$.

If G is a set and C a cyclic (cocyclic) order on G, then the pair (G, C) is called a *cyclically (cocyclically) ordered set.*

If C is a ternary relation on a set G, then we denote by $\mathrm{Co}_G\, C$ or, briefly, Co C the complement of C in G^3, i.e. Co $C = G^3 - C$.

2.2. Lemma. *Let G be a set, C a ternary relation on G. C is a cyclic order on G iff* Co C *is a cocyclic order on G.*

Proof. [7], Theorem 3.2.

2.3. Lemma. *Let $(G, <)$ be an ordered set. Put for any $x, y, z \in G (x, y, z) \in C_<$ iff either $x < y < z$ or $y < z < x$ or $z < x < y$. Then $C_<$ is a cyclic order on G.*

Proof. [5], Theorem 3.5.

Let $(G, <)$ be an ordered set. We call $<$ a *linear order in G*, iff there exists a subset $H \subseteq G$ such that $< \subseteq H^2$ and $<$ is a linear order on H.

2.4. Lemma. *Let (G, C) be a cyclically ordered set. Then there exists a family $(<_i;\ i \in I)$ of linear orders in G such that $C = \bigcup_{i \in I} C_{<_i}$.*

Proof. [7], Theorem 1.9. and Corollary 1.10.

2.5. Lemma. *Let (G, C) be a cocyclically ordered set. Then there exists a family $(<_i;\ i \in I)$ of linear orders in G such that $C = \bigcap_{i \in I} \mathrm{Co}\, C_{<_i}$.*

Proof follows from 2.2. and 2.4.; see also [7], Theorem 3.5. and Corollary 3.6.

2.6. Definition. Let (G, C) be a cyclically ordered set. Put $w(G, C) = \min \{\mathrm{card}\, I;$ there exists a family $(<_i;\ I \in I)$ of orders on G such that $C = \bigcup_{i \in I} C_{<_i}\}$, $W(G, C) =$ $= \min \{\mathrm{card}\, I;$ there exists a family $(<_i;\ i \in I)$ of linear orders in G such that $C =$

$= \bigcup_{i \in I} C_{<_i}\}$. The cardinal $w(G, C)$ is called the *width*, the cardinal $W(G, C)$ the *strong width* of (G, C).

If $w(G, C) = 1$, i.e. $C = C_<$ for a suitable order $<$ on G, then we shall say that the cyclic order C is *generated by an order*; if $W(G, C) = 1$, then C is said to be *generated by a linear order*.

2.7. Definition. Let (G, C) be a cocyclically ordered set. Put $d(G, C) = \min \{\text{card } I;$ there exists a family $(<_i;\ i \in I)$ of orders on G such that $C = \bigcap_{i \in I} \text{Co } C_{<_i}\}$, $D(G, C) = \min \{\text{card } I;$ there exists a family $(<_i;\ i \in I)$ of linear orders in G such that $C = \bigcap_{i \in I} \text{Co } C_{<_i}\}$.

If $d(G, C) = 1$, then we shall say that C is *generated by an order*; if $D(G, C) = 1$, then C is said to be *generated by a linear order*.

For the properties of characteristics $w(G, C)$, $W(G, C)$, $d(G, C)$, $D(G, C)$ see [7]. Here we recall only the following one:

2.8. Lemma. *Let (G, C) be a cocyclically ordered set. Then $d(G, C) = w(G, \text{Co } C)$, $D(G, C) = W(G, \text{Co } C)$.*

Proof. [7], Theorem 3.8.

3. DIMENSION THEORY

3.1. Notation. We denote by $\mathscr{Cy}$ the class of all cyclically ordered sets, by $\mathscr{Co}\,\mathscr{Cy}$ the class of all cocyclically ordered sets. Further, let ℓ denote the class of all cyclically ordered sets generated by an order, $\mathscr{L}$ the class of all cyclically ordered sets generated by a linear order, $\mathscr{Co}\,\ell$ the class of all cocyclically ordered sets generated by an order and $\mathscr{Co}\,\mathscr{L}$ the class of all cocyclically ordered sets generated by a linear order.

3.2. Theorem. $\mathscr{Cy}(\ell; \bigcup, \text{id}_{\mathscr{C}}) = \mathscr{Cy}$, $\mathscr{Cy}(\mathscr{L}; \bigcup, \text{id}_{\mathscr{C}}) = \mathscr{Cy}$, $\mathscr{Co}\,\mathscr{Cy}(\mathscr{Co}\,\ell, \bigcap, \text{id}_{\mathscr{C}}) = \mathscr{Co}\,\mathscr{Cy}$, $\mathscr{Co}\,\mathscr{Cy}(\mathscr{Co}\,\mathscr{L}; \bigcap, \text{id}_{\mathscr{C}}) = \mathscr{Co}\,\mathscr{Cy}$.

Proof. Clearly, $\mathscr{Cy}(\mathscr{L}; \bigcup, \text{id}_{\mathscr{C}}) \subseteq \mathscr{Cy}(\ell; \bigcup, \text{id}_{\mathscr{C}}) \subseteq \mathscr{Cy}$. On the other hand, if $(G, C) \in \mathscr{Cy}$, then by 2.4. there exists a family $(<_i;\ i \in I)$ of linear orders in G with $C = \bigcup_{i \in I} C_{<_i}$. Thus, $(G, C) = \bigcup_{i \in I} (G, C_{<_i})$ and $(G, C_{<_i}) \in \mathscr{L}$ for any $i \in I$, i.e. $((G, C_{<_i});\ i \in I) \in \mathscr{S}(\mathscr{L})$. This implies $(G, C) \in \mathscr{Cy}(\mathscr{L}; \bigcup, \text{id}_{\mathscr{C}})$ and we have shown $\mathscr{Cy} \subseteq \mathscr{Cy}(\mathscr{L}; \bigcup, \text{id}_{\mathscr{C}})$. The identities $\mathscr{Co}\,\mathscr{Cy}(\mathscr{Co}\,\ell; \bigcap, \text{id}_{\mathscr{C}}) = \mathscr{Co}\,\mathscr{Cy}(\mathscr{Co}\,\mathscr{L}; \bigcap, \text{id}_{\mathscr{C}}) = \mathscr{Co}\,\mathscr{Cy}$ follow analogously from 2.5.

3.3. Theorem. *For any $(G, C) \in \mathscr{Cy}$ it holds $(\ell; \bigcup, \text{id}_{\mathscr{C}}) - \dim (G, C) = w(G, C)$, $(\mathscr{L}; \bigcup, \text{id}_{\mathscr{C}}) - \dim (G, C) = W(G, C)$.*

Proof. If $w(G, C) = m$, then there exists a family $(<_i;\ i \in I)$ of orders on G with $\bigcup_{i \in I} C_{<_i} = C$ and $\operatorname{card} I = m$. Then $((G, C_{<_i});\ i \in I) \in \mathscr{S}(\ell)$ and $(G, C) = \bigcup_{i \in I} (G, C_{<_i})$ which implies $(\ell; \bigcup, \mathrm{id}_{\mathscr{C}}) - \dim (G, C) \leqq w(G, C)$. On the other hand, there exists a set $J \neq \emptyset$ with $\operatorname{card} J = (\ell; \bigcup, \mathrm{id}_{\mathscr{C}}) - \dim (G, C)$ and $(G_j, C_j) \in \ell$ for any $j \in J$ such that $(G, C) = (\bigcup_{j \in J} G_j, \bigcup_{j \in J} C_j)$. This implies $C = \bigcup_{j \in J} C_j$. Further, for any $j \in J$, there exists an order $<_j$ on G_j such that $C_j = C_{<_j}$. Clearly, $<_j$ is an order on G, and if we denote by D_j the cyclic order on G generated by $<_j$, then $D_j = C_j$ for any $j \in J$. Thus, $C = \bigcup_{j \in J} D_j$ and this implies $w(G, C) \leqq (\ell; \bigcup, \mathrm{id}_{\mathscr{C}}) - \dim (G, C)$. Analogously we can prove $(\mathscr{L}; \bigcup, \mathrm{id}_{\mathscr{C}}) - \dim (G, C) = W(G, C)$.

3.4. Theorem. *For any* $(G, C) \in \mathscr{C}o\,\mathscr{C}y$ *it holds* $(\mathscr{C}o\,\ell; \bigcap, \mathrm{id}_{\mathscr{C}}) - \dim (G, C) = d(G, C)$, $(\mathscr{C}o\,\mathscr{L}; \bigcap, \mathrm{id}_{\mathscr{C}}) - \dim (G, C) = D(G, C)$.

Proof. Analogously as in the proof of 3.3. we easily see that $(\mathscr{C}o\,\ell; \bigcap, \mathrm{id}_{\mathscr{C}}) - \dim (G, C) \leqq d(G, C)$, $(\mathscr{C}o\,\mathscr{L}; \bigcap, \mathrm{id}_{\mathscr{C}}) - \dim (G, C) \leqq D(G, C)$. On the other hand, let $((G_i, C_i);\ i \in I) \in \mathscr{S}(\mathscr{C}o\,\ell)$ be such a family that $(G, C) = (\bigcap_{i \in I} G_i, \bigcap_{i \in I} C_i)$ and $\operatorname{card} I = (\mathscr{C}o\,\ell; \bigcap, \mathrm{id}_{\mathscr{C}}) - \dim (G, C)$. Then $C = \bigcap_{i \in I} C_i$; furthermore, for any $i \in I$, there exists an order $<_i$ on G_i such that $C_i = \mathrm{Co}_{G_i} C_{<_i}$. Put $D_i = \mathrm{Co}_G C_{<_i \cap G^2}$ for any $i \in I$. Clearly, $<_i \cap G^2$ is an order on G, hence $C_{<_i \cap G^2}$ is a cyclic order on G and D_i is a cocyclic order on G generated by an order, i.e. $(G, D_i) \in \mathscr{C}o\,\ell$ for any $i \in I$. Let $x, y, z \in G$, $(x, y, z) \bar{\in} \bigcap_{i \in I} D_i$. Then there exists $i_0 \in I$ such that $(x, y, z) \bar{\in} D_i$, i.e. $(x, y, z) \in C_{<_{i_0} \cap G^2}$. This implies $x <_{i_0} y$, $y <_{i_0} z$ or $y <_{i_0} z$, $z <_{i_0} x$ or $z <_{i_0} x$, $x <_{i_0} y$. Thus $(x, y, z) \in C_{<_{i_0}}$ and, hence, $(x, y, z) \bar{\in} \bigcap_{i \in I} \mathrm{Co}_{G_i} C_{<_i}$. We have proved $\bigcap_{i \in I} \mathrm{Co}_{G_i} C_{<_i} \subseteq \bigcap_{i \in I} D_i$. Suppose that $x, y, z \in G$, $(x, y, z) \bar{\in} \bigcap_{i \in I} \mathrm{Co}_{G_i} C_{<_i}$. Then $(x, y, z) \in G^3$ and there exists $i_0 \in I$ such that $(x, y, z) \in C_{<_{i_0}}$. Thus either $x <_{i_0} y$, $y <_{i_0} z$ or $y <_{i_0} z$, $z <_{i_0} x$ or $z <_{i_0} x$, $x <_{i_0} y$. As $x, y, z \in G$, we have $(x, y, z) \in C_{<_{i_0} \cap G^2}$ so that $(x, y, z) \bar{\in} \bigcap_{i \in I} \mathrm{Co}_G C_{<_i \cap G^2} = \bigcap_{i \in I} D_i$ and we have proved $\bigcap_{i \in I} D_i \subseteq \bigcap_{i \in I} \mathrm{Co}_{G_i} C_{<_i}$. Thus $\bigcap_{i \in I} D_i = \bigcap_{i \in I} \mathrm{Co}_{G_i} C_{<_i}$, which implies $(G, C) = \bigcap_{i \in I} (G, D_i)$ where D_i is a cocyclic order on G generated by an order on G. Hence $d(G, C) \leqq (\mathscr{C}o\,\ell; \bigcap, \mathrm{id}_{\mathscr{C}}) - \dim (G, C)$. For the second assertion the proof is similar.

3.5. Theorem. $\mathscr{C}o\,\mathscr{C}y(\mathscr{C}o\,\ell; \mathsf{X}, i) = \mathscr{C}o\,\mathscr{C}y$, $\mathscr{C}o\,\mathscr{C}y(\mathscr{C}o\,\mathscr{L}; \mathsf{X}, i) = \mathscr{C}o\,\mathscr{C}y$.

Proof. It suffices to show $\mathscr{C}o\,\mathscr{C}y \subseteq \mathscr{C}o\,\mathscr{C}y(\mathscr{C}o\,\mathscr{L}; \mathsf{X}, i)$, because the inclusion $\mathscr{C}o\,\mathscr{C}y(\mathscr{C}o\,\mathscr{L}; \mathsf{X}, i) \subseteq \mathscr{C}o\,\mathscr{C}y(\mathscr{C}o\,\ell; \mathsf{X}, i) \subseteq \mathscr{C}o\,\mathscr{C}y$ is trivial. Let $(G, C) \in \mathscr{C}o\,\mathscr{C}y$. By 2.5. there exists a family $(<_i;\ i \in I)$ of linear orders in G such that $C = \bigcap_{i \in I} \mathrm{Co}\, C_{<_i}$. For any $i \in I$, put $G_i = G$, $C_i = \mathrm{Co}_G C_{<_i}$. Then $((G_i, C_i);\ i \in I) \in \mathscr{S}(\mathscr{C}o\,\mathscr{L})$. Furthermore, $\mathsf{X}_{i\in I}(G_i, C_i) = (\mathsf{X}_{i\in I} G_i, \mathsf{X}_{i\in I} \mathrm{Co}_G C_{<_i})$. For any $x \in G$ and any $i \in I$, put $f(x)(i) = x$. Thus, f is a mapping of G into $\mathsf{X}_{i\in I} G_i$; clearly, f is an injection. Let $x, y, z \in G$, $(x, y, z) \in C$. Then $(x, y, z) \in \mathrm{Co}_G C_{<_i}$ for any $i \in I$, i.e. $(f(x)(i), f(y)(i), f(z)(i)) \in C_i$ for any $i \in I$. Hence $(f(x), f(y), f(z)) \in \mathsf{X}_{i\in I} C_i$. If $(f(x), f(y), f(z)) \in \mathsf{X}_{i\in I} C_i$, then we have $(x, y, z) = (f(x)(i), f(y)(i), f(z)(i)) \in C_i$ for any $i \in I$, i.e. $(x, y, z) \in \bigcap_{i\in I} \mathrm{Co}_G C_{<_i} = C$. We have proved that f is an isomorphism of (G, C) into $\mathsf{X}_{i\in I}(G_i, C_i)$ so that $(G, C)\, i\, \mathsf{X}_{i\in I}(G_i, C_i)$ and $(G, C) \in \mathscr{C}o\,\mathscr{C}y(\mathscr{C}o\,\mathscr{L}; \mathsf{X}, i)$. Thus, $\mathscr{C}o\,\mathscr{C}y \subseteq \mathscr{C}o\,\mathscr{C}y(\mathscr{C}o\,\mathscr{L}; \mathsf{X}, i)$.

3.6. Theorem. *For any* $(G, C) \in \mathscr{C}o\,\mathscr{C}y$ *it holds* $(\mathscr{C}o\,\ell; \mathsf{X}, i) - \dim(G, C) = d(G, C)$, $(\mathscr{C}o\,\mathscr{L}; \mathsf{X}, i) - \dim(G, C) = D(G, C)$.

Proof. From the proof of 3.5. it follows that $(\mathscr{C}o\,\ell; \mathsf{X}, i) - \dim(G, C) \leqq d(G, C)$, $(\mathscr{C}o\,\mathscr{L}; \mathsf{X}, i) - \dim(G, C) \leqq D(G, C)$. On the other hand, let $((G_i, C_i);\ i \in I) \in \mathscr{S}(\mathscr{C}o\,\ell)$ be such a family that $\operatorname{card} I = (\mathscr{C}o\,\ell; \mathsf{X}, i) - \dim(G, C)$ and there is an isomorphism f of (G, C) into $(\mathsf{X}_{i\in I} G_i, \mathsf{X}_{i\in I} C_i)$. For any $i \in I$, there exists an order $<_i$ on G_i such that $C_i = \mathrm{Co}_{G_i} C_{<_i}$. For $i \in I$ and $x, y \in G$, put $x \prec_i y$ iff $f(x)(i) <_i f(y)(i)$. We show that $\prec_i$ is an order on G. Indeed, if $x \in G$ is arbitrary, then $x \prec_i x$ is equivalent to $f(x)(i) <_i f(x)(i)$ which never holds. Hence, $\prec_i$ is irreflexive. If $x, y, z \in G$ and $x \prec_i y$, $y \prec_i z$, then $f(x)(i) <_i f(y)(i)$, $f(y)(i) <_i f(z)(i)$ which implies $f(x)(i) <_i f(z)(i)$, thus $x \prec_i z$. Hence, $\prec_i$ is transitive. Further, we prove $C = \bigcap_{i\in I} \mathrm{Co}_G C_{\prec_i}$. Indeed, suppose $x, y, z \in G$. If $(x, y, z) \in C$, then $(f(x)(i), f(y)(i), f(z)(i)) \in \mathrm{Co}_{G_i} C_{<_i}$ for any $i \in I$. Suppose that there exists $i_0 \in I$ such that $(x, y, z) \bar{\in} \mathrm{Co}_G C_{\prec_{i0}}$. Thus, $(x, y, z) \in C_{\prec_{i0}}$ which means either $x \prec_{i_0} y$, $y \prec_{i_0} z$ or $y \prec_{i_0} z$, $z \prec_{i_0} x$ or $z \prec_{i_0} x$, $x \prec_{i_0} y$. By definition of $\prec_{i_0}$, this means either $f(x)(i_0) <_{i_0} f(y)(i_0)$, $f(y)(i_0) <_{i_0} f(z(i_0)$ or $f(y)(i_0) <_{i_0} f(z)(i_0)$, $f(z)(i_0) <_{i_0} f(x)(i_0)$ or $f(z)(i_0) <_{i_0} f(x)(i_0)$, $f(x)(i_0) <_{i_0} f(y)(i_0)$. Hence, $(f(x)(i_0), f(y)(i_0), f(z)(i_0)) \in C_{<_{i0}}$ which is a contradiction for $(f(x)(i_0), f(y)(i_0), f(z)(i_0)) \in \mathrm{Co}_{G_{i0}} C_{<_{i0}}$. Thus, $C \subseteq \bigcap_{i\in I} \mathrm{Co}_G C_{\prec_i}$. Assume that there exists $(x, y, z) \in \bigcap_{i\in I} \mathrm{Co}_G C_{\prec_i} - C$. Then $(f(x), f(y), f(z)) \bar{\in} \mathsf{X}_{i\in I} \mathrm{Co}_{G_i} C_{<_i}$; hence, there exists $i_0 \in I$ such that $(f(x)(i_0), f(y)(i_0), f(z)(i_0)) \in C_{<_{i0}}$. This means either $f(x)(i_0) <_{i_0} f(y)(i_0)$, $f(y)(i_0) <_{i_0} f(z)(i_0)$ or $f(y)(i_0) <_{i_0} f(z)(i_0)$, $f(z)(i_0) <_{i_0} f(x)(i_0)$ or $f(z)(i_0) <_{i_0} f(x)(i_0)$, $f(x)(i_0) <_{i_0} f(y)(i_0)$. By definition of $\prec_{i_0}$, we obtain either $x \prec_{i_0} y$, $y \prec_{i_0} z$ or $y \prec_{i_0} z$, $z \prec_{i_0} x$ or $z \prec_{i_0} x$, $x \prec_{i_0} y$, i.e. $(x, y, z) \in C_{\prec_{i0}}$ which contradicts our hypothesis. Thus, $\bigcap_{i\in I} \mathrm{Co}_G C_{\prec_i} = C$ and we have proved $d(G, C) \leqq \operatorname{card} I = (\mathscr{C}o\,\ell; \mathsf{X}, i) - \dim(G, C)$. If each $<_i$ is a linear order in G_i, then $\prec_i$ is a linear order in G which proves the second assertion of the theorem.

References

[1] *G. Birkhoff:* Generalized arithmetic. Duke Math. Journ. *9* (1942), 283—302.

[2] *E. Čech:* Bodové množiny. (Point sets.) Academia Praha, 1966.

[3] *N. Megiddo:* Partial and complete cyclic orders. Bull. Am. Math. Soc. *82* (1976), 274—276.

[4] *G. Müller:* Lineare und zyklische Ordnung. Praxis Math. *16* (1974), 261—269.

[5] *V. Novák:* Cyclically ordered sets. Czech. Math. Journ. *32* (*107*) (1982), 460—473.

[6] *V. Novák, M. Novotný:* Abstrakte Dimension von Strukturen. Ztschr. math. Logik und Grundl. d. Math. *20* (1974), 207—220.

[7] *V. Novák, M. Novotný:* On determination of a cyclic order. Czech. Math. Journ., *33* (*108*) (1983), 555—563.

Addresses of authors': V. Novák, 662 95 Brno, Janáčkovo nám. 2a, ČSSR (Přírodovědecká fakulta UJEP); M. Novotný, 603 00 Brno, Mendlovo nám. 1, ČSSR (MÚ ČSAV).