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Ján Jakubík; Milan Kolibiar

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# ISOMETRIES OF MULTILATTICE GROUPS

JÁN JAKUBÍK, Košice and MILAN KOLIBIAR, Bratislava

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Isometries in abelian lattice ordered groups were studied by K. L. Swamy [10], [11] and W. B. Powell [9]; the non-abelian case was dealt with in the papers [4], [5]. J. Trias [12] developed the theory of isometries in Riesz spaces.

Multilattice groups were introduced by M. Benado [1]. A thorough investigation of multilattice groups was performed by McAllister [7], [8]. In the present paper it will be shown that the results on the relations between isometries and direct decompositions of lattice ordered groups [4] can be extended to hold for abelian distributive multilattice groups.

## 1. PRELIMINARIES

At first we recall some notions concerning multilattices and multilattice groups.

Let $P$ be a partially ordered set. If $a$ and $b$ are elements of $P$, then we denote by $U(a, b)$ and $L(a, b)$ the set of all upper bounds or the set of all lower bounds of the set $\{a, b\}$, respectively. Next we denote by $a \bigvee_m b$ the set of all minimal elements of the set $U(a, b)$; analogously, $a \bigwedge_m b$ is defined to be the set of all maximal elements of the set $L(a, b)$.

The partially ordered set $P$ is said to be a *multilattice* (Benado [1]) if it fulfils the conditions for each pair $a, b \in P$:

($m_1$) If $x \in U(a, b)$, then there is $x_1 \in a \bigvee_m b$ such that $x_1 \leqq x$.

($m_2$) If $y \in L(a, b)$, then there is $y_1 \in a \bigwedge_m b$ such that $y_1 \geqq y$.

A multilattice $P$ is called *distributive* if, whenever $a, b, c$ are elements of $P$ such that

$$(a \bigwedge_m b) \cap (a \bigwedge_m c) \neq \emptyset$$

and

$$(a \bigvee_m b) \cap (a \bigvee_m c) \neq \emptyset ,$$

then $b = c$. (See [1] and [7].)

For the basic notions and denotations concerning partially ordered groups and lattice ordered groups cf. Fuchs [3] and Conrad [2]. The group operation in partially ordered groups will be written additively.

Let $G$ be a partially ordered group such that (i) $G$ is directed, and (ii) the partially ordered set $(G; \leqq)$ is a multilattice. Then $G$ is called a *multilattice group*. All multilattice groups dealt with in this paper are assumed to be abelian.

If $G$ is a lattice ordered group and $x \in G$, then we can define the absolute value $|x|$ in several equivalent ways; e.g., we can put

$$|x| = 2z - x\,, \tag{1}$$

where $z = x \vee 0$.

Now let $G$ be a multilattice group and let $x \in G$. Using an analogy with (1) we define

$$|x| = \{2z - x : z \in x \bigvee_m 0\}\,. \tag{1'}$$

Hence $|x|$ is a nonempty set for each $x \in G$. If $|x| = \{y\}$ is a one-element set, then we write also $|x| = y$. In the case $x \geqq 0$ $(x \leqq 0)$ we have $|x| = x$ $(|x| = -x)$.

Let $f$ be a one-to-one mapping of $G$ onto $G$ such that the relation

$$|f(x) - f(y)| = |x - y| \tag{$\alpha$}$$

is valid for each $x \in G$ and $y \in G$. Then $f$ is said to be an *isometry of* $G$.

If $f$ is an isometry of $G$ and $f(0) = 0$, then $f$ will be called a 0-*isometry*. Let $a \in G$; the mapping $f_a$ of $G$ onto $G$ defined by $f_a(x) = x + a$ for each $x \in G$ is a translation on $G$. Every translation is an isometry on $G$. Each isometry can be uniquely represented as a composition of a 0-isometry and a translation. Hence for determining all isometries of $G$ it suffices to find all 0-isometries.

## 2. REGULAR QUADRUPLES

Let $G$ be a multilattice group. A quadruple $\{a, b, u, v\}$ of elements of $G$ is said to be *regular* if $u \in a \bigwedge_m b$, $v \in a \bigvee_m b$ and $v - a = b - u$.

**2.1. Lemma.** *Let $a, b \in G$, $v \in a \bigvee_m b$. Put $u = a + b - v$. Then $\{a, b, u, v\}$ is a regular quadruple.*

Proof. It suffices to verify that $u \in a \bigwedge_m b$. We have $0 \leqq v - a = b - u$, hence $b \geqq u$, and analogously $a \geqq u$. There exists $u_1 \in a \bigwedge_m b$ with $u \leqq u_1$. Let $u', a', b' \in G$ such that $u_1 = u + u'$, $a = u_1 + a'$, $b = u_1 + b'$. Then $u' \geqq 0$, $a' \geqq 0$ and $b' \geqq 0$. Because of $a - u = u' + a'$, $b - u = u' + b'$ we obtain $v - u' = b + a'$, $v - u' = a + b'$, hence $v - u' \in U(a, b)$. Therefore $u' = 0$ and thus $u = u_1 \in a \bigwedge_m b$.

The assertion dual to 2.1 can be proved analogously.

**2.2. Lemma.** *Let $\{a, b, u, v\}$ be a regular quadruple. Let $a_1 \in [u, a]$, $b_1 = b + a_1 - u$. Then $\{a_1, b, u, b_1\}$ and $\{a, b_1, a_1, v\}$ are regular quadruples.*

Proof. From 2.1 it follows that $\{a, b_1, a_1, v\}$ is a regular quadruple. Next, from the assertion dual to 2.1 we infer that $\{a_1, b, u, b_1\}$ is a regular quadruple.

**2.3. Lemma.** *Let $\{a, b, u, v\}$ be a regular quadruple in $G$, $0 \leqq p \leqq a - u$, $x \in [u + p, b + p]$. Put $x - (u + p) = q$. Then $\{a, x, u + p, a + q\}$, $\{b, x, u + q, b + p\}$, $\{u + p, u + q, u, x\}$ and $\{a + q, b + p, x, v\}$ are regular quadruples.*

Proof. This is a consequence of 2.2.

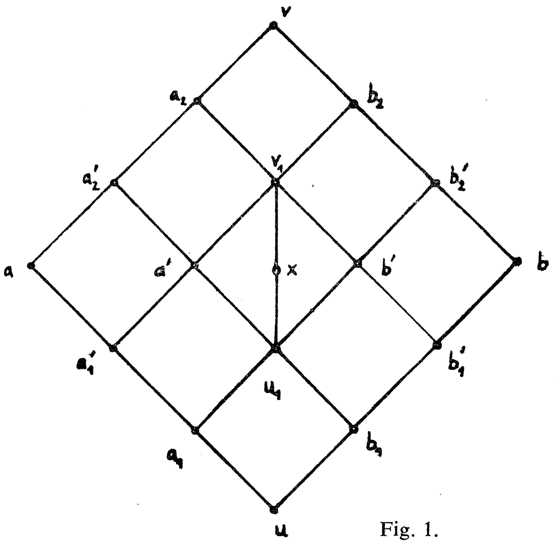

Fig. 1.

Again, let $\{a, b, u, v\}$ be a regular quadruple in $G$. Assume that $x \in [u, v]$. Let us apply the following construction (cf. Fig. 1).

We choose $a_1 \in a \wedge_m x$ with $a_1 \geqq u$. Denote $b_2' = b + (a_1 - u)$. In view of 2.2, $\{a_1, b, u, b_2'\}$ and $\{a, b_2', a_1, v\}$ are regular quadruples.

Choose $u_1 \in b_2' \wedge_m x$ with $u_1 \geqq a_1$. Denote $b_1 = u + (u_1 - a_1)$, $a_2' = a + (u_1 - a_1)$. According to 2.3, the quadruples

$$\{a_1, b_1, u, u_1\}, \ \{a, u_1, a_1, a_2'\}, \ \{u_1, b, b_1, b_2'\}, \ \{a_2', b_2', u_1, v\}$$

are regular.

**2.4. Lemma.** $u_1 \in a_2' \wedge_m x$.

Proof. We have $u_1 \leqq x$ and $u_1 \leqq a_2'$. Hence there is $z \in a_2' \wedge_m x$ with $z \geqq u_1$. If $z > u_1$, then we should have

$$a_1 < a_1 + (z - u_1) \leqq u_1 + (z - u_1) = z \leqq x ,$$
$$a_1 + (z - u_1) \leqq a_1 + (a_2' - u_1) = a_1 + (a - a_1) = a ,$$

hence $a_1 \notin a \wedge_m x$, which is a contradiction. Therefore $z = u_1$, completing the proof of the lemma.

The further steps of our construction are dual to the previous ones with the distinction that we consider the regular quadruple $\{a'_2, b'_2, u_1, v\}$ instead of $\{a, b, u, v\}$.

We choose $a_2 \in a'_2 \vee_m x$ with $a_2 \leqq v$. Denote $b' = b'_2 + (a_2 - v)$. In view of 2.2, $\{a'_2, b', u_1, a_2\}$ and $\{a_2, b'_2, b', v\}$ are regular quadruples. Put $b'_1 = b + (a_2 - v)$; then according to 2.3 the quadruples $\{u_1, b'_1, b_1, b'\}$ and $\{b', b, b'_1, b'_2\}$ are regular as well.

Now choose $v_1 \in b' \vee_m x$ with $v_1 \leqq a_2$. Denote $b_2 = v + (v_1 - a_2)$, $a' = a'_2 + (v_1 - a_2)$. In view of 2.3, all the quadruples $\{a', b', u_1, v_1\}$, $\{a'_2, v_1, a', a_2\}$, $\{v_1, b'_2, b', b_2\}$ and $\{a_2, b_2, v_1, v\}$ are regular. Put $a'_1 = a + (v_1 - a_2)$. Then according to 2.2, the quadruples $\{a'_1, u_1, a_1, a'\}$ and $\{a, a', a'_1, a'_2\}$ are regular as well. By an argument dual to that applied in the proof of 2.4 we obtain

$$v_1 \in a' \vee_m x .$$

We shall prove that the equivalence

$$(*) \qquad a'_2 = a_2 \Leftrightarrow a' = b'$$

is valid.

Let $a'_2 = a_2$ hold. Since $\{a'_2, v_1, a', a_2\}$ is a regular quadruple, we infer that $a' = v_1$, hence $x \leqq a'$ and thus $a' \wedge_m x = \{x\}$. In view of 2.4 we have $u_1 \in a' \wedge_m x$, hence $u_1 = x$. From the fact that $\{a', b', u_1, v_1\}$ is a regular quadruple we obtain that $u_1 = b'$. Thus $b' \vee_m x = x \vee_m x = \{x\}$; because of $v_1 \in b' \vee_m x$ we have $v_1 = x$ and so $u_1 = v_1$, impying $a' = b'$.

Conversely, assume that $a' = b'$. Then $a' \vee_m b' = \{a'\}$ hence $v_1 = a'$. Since $\{a'_2, v_1, a', a_2\}$ is regular, we infer that $a_2 = a'_2$ holds.

Similarly we can verify that the relation $a' = b'$ is equivalent to each of the following relations: $b'_2 = b_2$; $a_1 = a'_1$; $b_1 = b'_1$.

**2.5. Lemma.** *If $a_2 < a'_2$, then $G$ fails to be distributive.*

Proof. This follows from $(*)$ and from the definition of distributivity (cf. Sec. 1).

**2.6. Lemma.** *Assume that $(G; \geqq)$ is distributive. Let $\{a, b, u, v\}$ be a regular quadruple in $G$, $x \in [u, v]$ and $a_1 \in a \wedge_m x$, $a_1 \geqq u$. Then there are elements $b_1 \in [u, b]$, $a_2 \in [a, v]$ and $b_2 \in [b, v]$ such that $\{a, x, a_1, a_2\}$, $\{b, x, b_1, b_2\}$, $\{a_1, b_1, u, x\}$ and $\{a_2, b_2, x, v\}$ are regular quadruples.*

Proof. Let $a_2$, $b_1$ and $b_2$ be as in the construction above. In view of 2.5 we have $a_2 = a'_2$; similarly, the relations $b'_2 = b_2$, $a_1 = a'_1$ and $b_1 = b'_1$ hold. Hence all the quadruples involved in the assertion of the lemma are regular.

In this section we assume that $G$ is a distributive multilattice group and $f$ is an isometry on $G$.

Let $x, y, z \in G$, $t = z + y$. The relation $z \in 0 \bigvee_m (x - y)$ is equivalent to $t \in x \bigvee_m y$, whence

$$(3.1) \qquad |x - y| = \{2t - x - y : t \in x \bigvee_m y\} .$$

By using 2.1 and the assertion dual to 2.1 we obtain also

$$(3.2) \qquad |x - y| = \{x + y - 2z : z \in x \bigwedge_m y\} .$$

**3.1. Lemma.** *Let $a, b, x \in G$, $a \leqq x \leqq b$. Assume that $f(a) \leqq f(b)$. Then $f(a) \leqq f(x) \leqq f(b)$.*

Proof. We have $|b - x| = b - x$, $|x - a| = x - a$, hence in view of ($\alpha$) (cf. Sec. 1) $|f(b) - f(x)|$ and $|f(x) - f(a)|$ are one-element sets. Choose $u \in f(a) \bigwedge_m f(x)$, $v \in f(b) \bigwedge_m f(x)$. In view of (3.1) and (3.2) we obtain

$$|f(b) - f(x)| = 2v - f(b) - f(x) ,$$
$$|f(x) - f(a)| = f(x) + f(a) - 2u .$$

Because of

$$|b - a| = |b - x| + |x - b|$$

we obtain

$$|f(b) - f(a)| = |f(b) - f(x)| + |f(x) - f(b)| ,$$

hence

$$f(b) - f(a) = 2v - f(b) - f(x) + f(x) + f(a) - 2u =$$
$$= (v - f(b)) + (v - u) + (f(a) - u) \geqq v - u .$$

We evidently have $v - u \geqq f(b) - f(a)$. Thus $v - u = f(b) - f(a)$ and (since $v - f(b) \geqq 0$, $f(a) - u \geqq 0$) we get $v = f(b)$, $u = f(a)$. Hence $f(a) \leqq f(x) \leqq f(b)$.

Analogously we can verify

**3.1.1. Lemma.** *Let $a, b, x \in G$, $a \leqq x \leqq b$. Assume that $f(a) \geqq f(b)$. Then $f(a) \geqq f(x) \geqq f(b)$.*

**3.2. Lemma.** *Let $x, y \in G$, $x \geqq 0 \geqq y$.* If *$f(x) \geqq 0$, then $f(x) = x$. If $f(x) \leqq 0$, then $f(x) = -x$. If $f(y) \geqq 0$ ($f(y) \leqq 0$), then $f(y) = -y$ ($f(y) = y$).*

Proof. Let $f(x) \geqq 0$. Then $x = |x| = |f(x)| = f(x)$. The other assertions can be verified analogously.

**3.3. Lemma.** *Let $x, y \in G$, $x \geqq y$, $u' \in f(x) \bigwedge_m f(y)$, $u = f^{-1}(u')$. Then $u \in [y, x]$.*

Proof. From (3.2) we infer $f(x) + f(y) - 2u' \in |f(x) - f(y)|$. Since $|f(x) - f(y)| = |x - y| = x - y$, we have card $|f(x) - f(y)| = 1$, whence

$$|f(x) - f(y)| = f(x) + f(y) - 2u' = f(x) - f(u) + f(y) - f(u) = \\ = |f(x) - f(u)| + |f(y) - f(u)| .$$

Both $|f(x) - f(u)|$ and $|f(u) - f(y)|$ are one-element sets. Hence

$$|x - y| = |x - u| + |u - y|$$

and both $|x - u|$ and $|u - y|$ are one-element sets. Choose $u_1 \in y \wedge_m u$, $v_1 \in x \wedge_m u$. Then

$$|x - u| = 2v_1 - x - u\,, \quad |u - y| = u + y - 2u_1\,,$$

whence

$$x - y = |x - y| = (v_1 - u_1) + (v_1 - x) + (y - u_1) \geqq v_1 - u_1\,.$$

Because of $u_1 \leqq y \leqq x \leqq v_1$ we have $v_1 - u_1 \geqq x - y$, therefore $x - y = v_1 - u_1$ and thus $v_1 = x$, $u_1 = y$. Hence $y \leqq u \leqq x$.

Similarly we obtain:

**3.4. Lemma.** *Let $x, y \in G$, $y \leqq x$, $v' \in f(x) \vee_m f(y)$, $v = f^{-1}(v')$. Then $v \in [y, x]$.*

**3.5. Lemma.** *Let $x, y, u, v$ be as in* 3.3 *and* 3.4. *Then $y \in u \wedge_m v$, $x \in u \vee_m v$.*

Proof. Let $u_1 \in u \wedge_m v$, $y \leqq u_1$. Since $y \leqq u_1 \leqq u$ and $f(y) \geqq f(u)$, according to 3.1.1 we have $f(y) \geqq f(u_1)$. On the other hand, from 3.1 and from the relations $y \leqq u_1 \leqq v$, $f(y) \leqq f(v)$ we obtain $f(y) \leqq f(u_1)$. Thus $f(u_1) = y$, hence $y \in u \wedge_m v$. Analogously we can prove that $x \in u \vee_m v$.

In the above consideration, $v'$ was an arbitrary element of the set $f(x) \vee_m f(y)$. Now assume that $\{f(x), f(y), u', v'\}$ is a regular quadruple. Such an element $v'$ does exist (cf. the dual of 2.1). Under this assumption we have:

**3.6. Lemma.** *$\{u, v, y, x\}$ is a regular quadruple.*

Proof. In view of 3.5 we have to verify that $x - v = u - y$. In fact, $x - v = |x - v| = |f(x) - f(v)| = f(v) - f(x) = f(y) - f(u) = |f(y) - f(u)| = |y - u| = u - y$.

**3.7. Lemma.** *Let $x, y, u, v$ be as in* 3.6. *Let $z \in [y, x]$ and assume that $f(z) \leqq f(y)$. Then $z \leqq u$.*

Proof. From 2.6 it follows that there are elements $u_1 \in [y, u]$ and $v_1 \in [y, v]$ such that $z \in u_1 \vee_m v_1$. In view of 3.1.1 and 3.1 we have $f(v_1) \geqq f(y)$ (since $f(v) \geqq f(y)$) and at the same time, $f(v_1) \leqq f(y)$ (since $f(z) \leqq f(y)$; thus $f(v_1) = f(y)$. Therefore $v_1 = y$ and thus $z = u_1 \leqq u$.

Analogously we obtain:

**3.8. Lemma.** *Let $x$, $y$, $u$, $v$ be as in 3.6. Let $z \in [y, x]$ and assume that $f(z) \geqq f(y)$. Then $z \leqq v$.*

**3.9. Lemma.** *Let $x, y \in G$, $x \geqq y$. Then both $f(x) \wedge_m f(y)$ and $f(x) \vee_m f(y)$ are one-elements sets.*

Proof. Let us apply the above denotations. Let $u'' \in f(x) \wedge_m f(y)$. In view of 3.3 we have $f^{-1}(u'') \in [y, x]$ and thus, according to 3.7, $f^{-1}(u'') \leqq f^{-1}(u')$. But the roles of $u'$ and $u''$ can be interchanged, whence $f^{-1}(u') \leqq f^{-1}(u'')$. Therefore $u'' = u'$ and hence card $(f(x) \wedge_m f(y)) = 1$. In view of the assertion dual to 2.1 we infer that $f(x) \vee_m f(y)$ is a one-element set as well.

**3.10. Corollary.** *Let $x, y \in G$, $x \geqq y$. Then the elements $u$, $v$ from 3.3 and 3.4 are uniquely determined.*

Now let $0 \leqq x \in G$; put $y = 0$. Let $u$, $v$ be as above.

In view of 3.10 we denote $u = x_u$, $v = x_v$. Since $\{u_x, v_x, 0, x\}$ is a regular quadruple (cf. 3.6) we have $x = x_u + x_v$.

## 4. THE SETS $A$ AND $B$

Again, let $G$ be a distributive multilattice and let $f$ be an isometry of $G$ with $f(0) = 0$. We denote

$$A_1 = \{x \in G : x \geqq 0 \text{ and } f(x) \geqq 0\},$$
$$B_1 = \{x \in G : x \geqq 0 \text{ and } f(x) \leqq 0\}.$$

**4.1. Lemma.** *The set $A_1$ is closed with respect to the operation $+$.*

Proof. Let $a_1, a_2 \in A_1$, $x = a_1 + a_2$, $u = x_u$, $v = x_v$. In view of 3.8 we have

$$a_1 \leqq v, \quad a_2 \leqq v.$$

Because of $x \leqq 2v$ and $x = u + v$, the relation $u \leqq v$ is valid. According to 3.5, $0 \in u \wedge_m v$; hence $u = 0$. Therefore $0 = f(u) \leqq f(x)$, yielding $x \in A_1$.

Analogously we can verify

**4.2. Lemma.** *The set $B_1$ is closed with respect to the operation $+$.*

**4.3. Lemma.** *Let $a \in A_1$, $b \in B_1$, $x = a + b$. Then $x_u = a$ and $x_v = b$.*

Proof. From 3.7 and 3.8 we obtain $0 \leqq a \leqq x_u$, $0 \leqq b \leqq x_v$, hence

$$x = a + b \leqq x_u + x_v = x.$$

Thus we must have $a = x_u$, $b = x_v$.

From 4.1, 4.2 and 4.3 we obtain:

**4.4. Lemma.** *Let* $x, y \in G$, $x \geqq 0$, $y \geqq 0$. *Then* $(x + y)_u = x_u + y_u$, $(x + y)_v = x_v + y_v$.

**4.5. Lemma.** *Let* $x$ *and* $y$ *be as in* 4.4. *Then the following conditions are equivalent*: (i) $y \leqq x$; (ii) $y_u \leqq x_u$ *and* $y_v \leqq x_v$.

Proof. The implication (ii) $\Rightarrow$ (i) is obvious. The implication (i) $\Rightarrow$ (ii) follows from 3.3, 3.4, 3.7 and 3.8.

**4.6. Lemma.** *The partially ordered semigroup* $G^+ = \{g \in G : g \geqq 0\}$ *is a direct product of the partially ordered semigroups* $A_1$ *and* $B_1$.

Proof. This is a consequence of 4.4 and 4.5.

Put $A = A_1 - A_1$, $B = B_1 - B_1$. From 4.6 and Thm. 2.3 [6] we infer:

**4.7. Lemma.** *The partially ordered group* $G$ *is a direct product of partially ordered groups* $A$ *and* $B$.

**4.7.1. Remark.** For $g \in G$ we denote by $g_A$ and $g_B$ the component of $g$ in the direct factor $A$ and $B$, respectively. If $0 \leqq x \in G$ and $u, v$ are is in 3.10 (with $y = 0$), then according to the definition of $A_1$ and $B_1$ we have

$$x_A = u\,, \quad x_B = v\,.$$

**4.8. Lemma.** *Let* $\{a, b, u, v\}$ *be a regular quadruple. Assume that* $f(a) \leqq f(u)$, $f(a) \leqq f(v)$. *Then* $\{f(u), f(v), f(a), f(b)\}$ *is a regular quadruple.*

Proof. From 3.1 we obtain (by considering the isometry $f^{-1}$) that $f(a) \in f(u) \bigwedge_m \bigwedge_m f(b)$ holds. In view of 3.10, $f(u) \bigwedge_m f(b)$ is a one-element set, hence $f(u) \bigwedge_m f(b) = \{f(a)\}$. Also (see 3.10), $f(u) \bigvee_m f(b)$ is a one-element set; let us write $f(u) \bigvee_m \bigwedge_m f(b) = \{f(v_1)\}$. Then the quadruple $\{f(u), f(v), f(a), f(v_1)\}$ must be regular. Now from 3.6 it follows that $\{a, v_1, u, v\}$ is a regular quadruple, thus $v_1 = b$.

**4.9. Lemma.** *For each* $x \in G$ *we have* $f(x) = x_A - x_B$.

Proof. Chose $u \in 0 \bigwedge_m x$. According to the dual of 2.1 there exists $v \in 0 \bigvee_m x$ such that $\{0, x, u, v\}$ is a regular quadruple. Then $x = u + v$, hence

$$x_A = u_A + v_A\,, \quad x_B = u_B + v_B\,.$$

In view of 3.3, 3.4 and 3.2 there exist $u_1 \in [u, 0]$, $v_1 \in [0, v]$ such that

$$0 = f(0) \leqq f(v_1) = v_1\,, \quad f(v_1) \geqq f(v)\,,$$
$$0 \leqq f(u_1) = -u_1\,, \quad f(u_1) \geqq f(u)\,.$$

(Cf. Fig. 2; dashed lines denote the fact that the corresponding interval is reversed under $f$ (e.g. $u_1 < 0$ and $f(u_1) > f(0)$.)

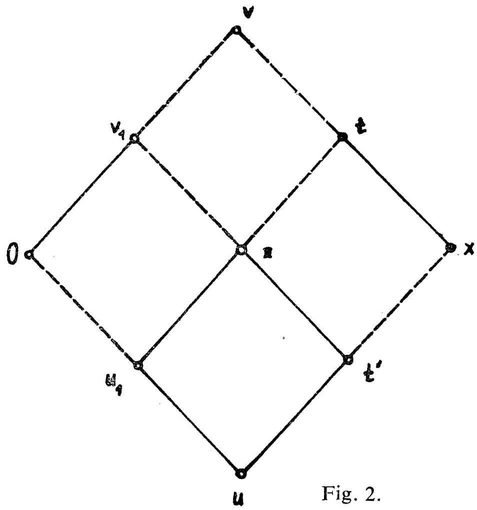


Fig. 2.

Consider the elements $u_1, 0, v_1$. Put $z = u_1 + v_1$. According to 4.8, $\{0, z, u_1, v_1\}$ is a regular quadruple and

$$f(u_1) \leqq f(z), \quad f(v_1) \leqq f(z).$$

Put $t = v + u_1$. In view of 2.6 we have $z \leqq t$, and clearly $t \leqq v$. Since $f(z) \geqq f(v)$, from 3.1 it follows

$$f(z) \geqq f(t) \geqq f(v).$$

Next we put $t' = u + v_1$. In view of 2.6 we have

$$u \leqq t' \leqq z.$$

Because of $f(u) \leqq f(z)$, by using 3.1 qe get

$$f(u) \leqq f(t') \leqq f(z).$$

From 2.6 it follows that $\{z, x, t', t\}$ is a regular quadruple. In view of the dual to 4.8 we obtain that

$$f(t') \geqq f(x), \quad f(x) \leqq f(t).$$

By applying the above inequalities we infer

$$f(x) = (f(x) - f(t')) + f(t') - (f(u)) + (f(u) - f(u_1)) + (f(u_1) - f(0)) =$$
$$= -|f(x) - f(t')| + |f(t') - f(u)| - |f(u) - f(u_1)| +$$
$$+ |f(u_1) - f(0)| = -|x - t'| + |t' - u| - |u - u_1| + |u_1 - 0| =$$

$$= -(x - t') + (t' - u) + (u - u_1) - (u_1 - 0) =$$
$$= -(v - v_1) + (v_1 - 0) + (u - u_1) - u_1 \cdot$$

According to 4.7.1 we have $v_1 = v_A$, hence $v - v_1 = v_B$. Similarly we have $u_1 = u_B$, hence $u - u_1 = u_A$. Thus

$$f(x) = -v_B + v_A + u_A - u_B = (u + v)_A - (u + v)_B = x_A - x_B \,.$$

Let $G = P \times Q$ be any direct decomposition of $G$. Then for arbitrary $x, y \in G$ we have

$$x \wedge_m y = (x_P \wedge_m y_P) + (x_Q \wedge_m y_Q)\,,$$

and analogously for $\vee_m$. From this we obtain

$$|x| = |x_P| + |x_Q|\,.$$

**4.10. Lemma.** *Let $G = P \times Q$. For each $x \in G$ define $g(x) = x_P - x_Q$. Then $g$ is an isometry of $G$ and $g(0) = 0$.*

Proof. Let $x, y \in G$. Then $g(x - y) = g(x) - g(y)$. Thus

$$|g(x) - g(y)| = |g(x - y)| = |(g(x - y))_P| + |(g(x - y))_Q| =$$
$$= |(x - y)_P| + |(x - y)_Q| = |x - y|\,.$$

Clearly $g(0) = 0$.

Summarizing, we have

**4.11. Theorem.** *Let $G$ be a distributive abelian multilattice group. For each isometry $f$ on $G$ with $f(0) = 0$ there exist a direct decomposition $G = A \times B$ such that $f(x) = x_A - x_B$ is valid for each $x \in G$. Conversely, if $G = P \times Q$ is a direct decomposition of $G$ and if we put $g(x) = x_P - x_Q$ for each $x \in G$, then $g$ is an isometry on $G$ with $g(0) = 0$.*

The question wheter the assumption of distributivity or commutativity of $G$ in the above theorem can be cancelled remains open.

The first author announces the following correction to the paper [4] concer ing isometries of $l$-groups: In the assertion (**) of § 3 it should be assumed that $B_0(G)$ is the system of all abelian direct factors of $G$ and that $B \in B_0(G)$. Theorem 2.5, which is the main result of [4], remains unchanged. The author is indebted to A. M. W. Glass for this observation.

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*Authors' addresses:* J. Jakubík, 040 01 Košice, Švermova 9, ČSSR (Vysoká škola technická); M. Kolibiar, 842 15 Bratislava, Mlynská dolina, Matematický pavilon, ČSSR (Matematicko-fyzikálna fakulta UK).