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PROJECTIVELY GENERATED CONVERGENCE OF SEQUENCES

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This paper is devoted to spaces in which the convergence of sequences is projectively generated by a class of mappings of the space in question into a fixed terminal space. Such spaces were introduced by J. Novák (cf. [16], [17]) who considered the special case in which the terminal space is the real line, and were further studied, e.g., in [13], [4], [7], [12]. We focus on the case when the terminal space is a subspace either of the real line or of the plane. The main results of the present paper were announced in [8] and [9].

0.

Let X be a set, $\mathscr{E}$ a class of closure spaces, and $\mathscr{F}$ a class of mappings $f : X \to Y$, $Y \in \mathscr{E}$. *The class $\mathscr{F}$ projectively generates a convergence of sequence in X*: a sequ.nce $\langle x_n \rangle \in X^N$ converges to a point $x \in X$ whenever for each $f \in \mathscr{F}$ the sequence $\langle f(x_n) \rangle$ converges to the point $f(x)$ in the range space of f. Usually, X is a closure space, $\mathscr{E}$ consists of a single space Y, and the convergence of sequences in X is projectively generated by $\mathscr{F}$. Two special cases are of interest. First, if we study topological invariants, then we choose $\mathscr{F}$ to be the set of all sequentially continuous mappings of X into Y (see e.g. [16], [13], [7], [11], [9]). Secondly, if some other invariants (e.g. uniform, measure-theoretical) are studied, then X and Y are equipped with additional structures and $\mathscr{F}$ is a corresponding subset of sequentially continuous mappings of X into Y (see e.g. [17], [14], [5], [11], [10]). In both cases the categorical point of view is useful (see [11], [9]).

Let Y be a terminal space. As far as we are interested in notions based on a projectively generated convergence of sequences, the closure operator in Y is not essential to the extent that it can be replaced by any other closure operator inducing in Y the same convergence of sequences. In what follows, all terminal spaces are assumed to be convergence spaces.

In notation and terminology we generally follow [7] and [12]. However, classes of functions are denoted by script letters, e.g., we write $\mathscr{C}(X)$ instead of $C(X)$.

Symbols E and F always denote convergence spaces. Let X and Y be closure spaces. Then $\mathscr{C}(X, Y)$, resp. $\mathscr{C}_s(X, Y)$, denotes the set of all continuous, resp. sequentially continuous, mappings of X into Y, and $\mathscr{C}(X, R)$, resp. $\mathscr{C}_s(X, R)$, is condensed to $\mathscr{C}(X)$, resp. $\mathscr{C}_s(X)$. Throughout the paper we make a blanket assumption that in all spaces every sequence converges to at most one point.

1.

In this section we develop a general theory of spaces in which the convergence of sequences is projectively generated by a class of mappings (cf. [9], where such spaces are investigated from the categorical point of view). In most cases we give generalizations of known statements in which the terminal space is a subspace of R. Since the proofs are either straightforward or obvious modifications of those for subspaces of R, they are left out. However, we indicate in parentheses the statement being generalized.

We start with definitions and some fundamental properties. Then we proceed by describing certain extremal embeddings. The final part of this section is devoted to the mutual relationship between various terminal spaces.

Definition 1.1. Let X be a closure space, E a convergence space, and $\mathscr{F} \subset E^X$. The space X is said to be *$\mathscr{F}$-sequentially regular* if the convergence of sequences in X is projectively generated by $\mathscr{F}$.

Remark 1.2. Let X be an $\mathscr{F}$-sequentially regular closure space. If $E = R$, then we obtain Definition 0.2 in [12]. Clearly $\mathscr{F} \subset \mathscr{C}_s(X, E)$. If $\mathscr{F} = \mathscr{C}_s(X, E)$, then we speak of *E-sequential regularity* and moreover, if $E = R$, then we omit the letter E (cf. [7]). Trivially, every E is E-sequentially regular.

Theorem 1.3. (Theorem 9 and Theorem 10 in [16].) *E-sequential regularity is a hereditary and a productive property.*

Theorem 1.4. (Theorem 11 in [16], Theorem 1.13 in [7].) *A convergence space is E-sequentially regular iff it is homeomorphic to a subspace of a convergence power E^m* (pointwise convergence) *for some cardinal m.*

Lemma 1.5. *The following statements are equivalent:*

(i) *Every E-sequentially regular closure space is F-sequentially regular.*

(ii) *E is F-sequentially regular.*

Corollary 1.6. *Let X be an E-sequentially regular closure space. If E is sequentially regular, then X is sequentially regular. If E is $\{0, 1\}$-sequentially regular, then X is $\{0, 1\}$-sequentially regular.*

Corollary 1.7. *If X is a $\{0, 1\}$-sequentially regular closure space, then X is E-sequentially regular for each E such that* card $(E) > 1$.

Corollary 1.8. *If X is a sequentially regular closure space, then it is E-sequentially regular for each E containing an arc.*

Definition 1.9. Let X be a closure space, E a convergence space, and $\mathcal{F} \subset E^X$. A sequence $\langle x_n \rangle$ of points of X is said to be *$\mathcal{F}$-fundamental* if for each $f \in \mathcal{F}$ there is a point $a_f \in E$ such that $\lim f(x_n) = a_f$. We say that X is *$\mathcal{F}$-sequentially complete* if every $\mathcal{F}$-fundamental sequence converges in X.

Remark 1.10. Let X be an $\mathcal{F}$-sequentially complete closure space. If $E = R$, then we obtain Definition 1.5 in [12]. If $\mathcal{F} = \mathcal{C}_s(X, E)$, then we speak of *$E$-sequential completeness* and moreover, if $E = R$, then we omit the letter E (cf. [7]). Trivially, every E is E-sequentially complete.

Theorem 1.11. *Let X be an $\mathcal{F}$-sequentially complete closure space. If $\mathcal{F} \subset \mathcal{C}_s(X, E)$, then X is $\mathcal{F}$-sequentially regular.*

Theorem 1.12. (Lemma 1.16 and Lemma 1.17 in [7].) *E-sequential completeness is a productive property and a hereditary property with respect to sequentially closed subspaces.*

Theorem 1.13. (Theorem 1.19 in [7].) *A convergence space is E-sequentially complete iff it is homeomorphic to a (sequentially) closed subspace of a convergence power E^m for some cardinal m.*

Theorem 1.14. (Theorem 1.8 in [12].) *Let X be a closure space, E a convergence space, and $\mathcal{F} \subset E^X$. Let X be $\mathcal{F}$-sequentially regular. Then the following statements are equivalent:*

(i) *X is $\mathcal{F}$-sequentially complete.*

(ii) *If $\langle x_n \rangle$ and $\langle y_n \rangle$ are sequences of points of X such that for each $f \in \mathcal{F}$ there is a point $a_f \in E$ such that* $\lim f(x_n) = a_f$ *and* $\lim f(y_n) = a_f$ *in E, then there is a point x in X such that* $\lim x_n = x$ *and* $\lim y_n = x$ *in X.*

(iii) *X is sequentially closed in every closure space Y in which it is sequentially $\mathcal{F}$-embedded (i.e., for each $f \in \mathcal{F} (\subset \mathcal{C}_s(X, E))$ there is $g \in \mathcal{C}_s(Y, E)$ such that $g \mid X = f$).*

Lemma 1.15. *The following statements are equivalent:*

(i) *Every E-sequentially complete closure space is F-sequentially complete.*

(ii) *E is F-sequentially complete.*

Corollary 1.16. *Let X be an E-sequentially complete closure space. If E is sequentially complete, then X is sequentially complete. If E is $\{0, 1\}$-sequentially complete, then X is $\{0, 1\}$-sequentially complete.*

Corollary 1.17. *If X is a $\{0, 1\}$-sequentially complete closure space, then X is E-sequentially complete for each E such that* $\operatorname{card}(E) > 1$.

Corollary 1.18. *If X is a sequentially complete closure space, then it is E-sequentially complete for each E containing an arc.*

Let E be a convergence space, $(E)_b$ the class of all E-sequentially regular convergence spaces, and $(E)_e$ the class of all E-sequentially complete convergence spaces. Then (cf. [9]) $(E)_b$ is the smallest bireflective class of convergence spaces containing E and $(E)_e$ is the smallest epireflective class of convergence spaces containing E. If $E \subset R$ and $X \in (E)_b$, then (cf. Corollary 3.2 in [12]) the $\mathscr{C}_E$-*sequential envelope* $\sigma_E(X)$ of X is the epireflection of X into $(E)_e$. Accordingly, if E is a convergence space and $X \in (E)_b$, then the epireflection of X into $(E)_e$ will be denoted by $\sigma_E(X)$. Recall that $\sigma_E(X)$ is characterized by the following properties: X is a sequentially dense subspace of $\sigma_E(X)$, each mapping in $\mathscr{C}_s(X, E)$ can be (uniquely) continuously extended over $\sigma_E(X)$, and $\sigma_E(X)$ is E-sequentially complete.

Theorem 1.19. *Let E be a sequentially complete convergence space and $X \in (E)_b$. Put $\mathscr{C}_0 = \mathscr{C}_s(X, E) \circ \mathscr{C}_s(E, R)$. Then $\sigma_E(X)$ is a $\mathscr{C}_0$-sequential envelope of X.*

Proof. It follows immediately that X is a sequentially dense $\mathscr{C}_0$-embedded subspace of $\sigma_E(X)$. Clearly, $\overline{\mathscr{C}}_0 = \mathscr{C}_s(\sigma_E(X), E) \circ \mathscr{C}(E, R)$ is the class of the extended functions. Since E is sequentially complete and $\sigma_E(X)$ is E-sequentially complete, it can be shown that $\sigma_E(X)$ is $\overline{\mathscr{C}}_0$-sequentially regular and $\overline{\mathscr{C}}_0$-sequentially complete. The assertion follows from Theorem 2.2 in [12].

Note that each $E \subset R^n$ is sequentially complete (cf. Theorem 1.14 in [7]). V. Koutník pointed out in [13] that the sequential envelope can be constructed via the Čech-Stone compactification. His construction can be generalized as follows.

Theorem 1.20. (Theorem 11 in [13] and Theorem 8 in [4].) *Let $E \subset R^n$ and $X \in (E)_b$. Let $\tilde{X}$ be the underlying set of X equipped with the $\mathscr{C}_s(X, E)$-weak topology, and let $\sigma_E\tilde{X}$ be the E-compactification of $\tilde{X}$* (cf. [15]). *Let S be the smallest sequentially closed subset of $\sigma_E\tilde{X}$ containing $\tilde{X}$, equipped with the inherited convergence of sequences and the closure operator derived from this convergence. Then $S = \sigma_E(X)$.*

Corollary 1.21. *Let X be a 0-dimensional Fréchet completely regular space. Let $\beta_0 X$ be the epireflection of X into compact 0-dimensional spaces (i.e., the Banaschewski 0-dimensional compactification of X). Let S be the smallest sequentially closed subset of $\beta_0 X$ containing X, equipped with the inherited convergence of sequences and the derived closure operator. Then $S = \sigma_{\{0,1\}}(X)$.*

Symbol $E\ \mu\ F$ denotes that a closure space X is E-sequentially regular iff it is F-sequentially regular. Similarly, $E\ \varrho\ F$ denotes that a closure space X is E-sequentially complete iff it is F-sequentially complete. Clearly, μ and ϱ are equivalence relations for the class of all convergence spaces, and it follows from Theorem 1.11 that $E\ \varrho\ F$ always implies $E\ \mu\ F$. We focus on convergence spaces in $\mathsf{E}_n = \{E \subset R^n\colon \operatorname{card}(E) > 1\}$, $n \in N$. In E_1 the situation is relatively simple.

Theorem 1.22. (Theorem 1.12 in [7].) (i) *In E_1 there are precisely two μ-equivalence classes: $\{E \in \mathsf{E}_1 : E$ contains an interval$\}$ nad $\{E \in \mathsf{E}_1 : E$ does not contain any interval$\}$.*

(ii) For $E, F \in \mathsf{E}_1$ we have $E\ \mu\ F$ iff $E\ \varrho\ F$.

For convergence spaces in E_1 the following problem has not been solved yet.

Problem 1.23. Is there a $\{0, 1\}$-sequentially regular closure space which is sequentially complete but fails to be $\{0, 1\}$-sequentially complete?

In E_n, $n > 1$, the situation is quite different. To show that, we shall use de Groot's results from [3]. It follows from his Theorem 1, Lemmas 3 and 4 that there is a family $\{P_\alpha : \alpha \in 2^{2^\omega}\}$ of connected and locally connected subsets of R^2 such that for every subset $P \subset P_\alpha$ with $\operatorname{card}(P_\alpha - P) < 2^\omega$ the only continuous maps on P into P_β are trivial (i.e., constants or, if $\alpha = \beta$, also the identity embedding).

It follows directly from the properties of $\{P_\alpha\}$ that the class of P_α-sequentially regular spaces are pairwise incomparable because no P_β, $\beta \neq \alpha$, is P_α-sequentially regular. If $P \subset P_\alpha$, $\operatorname{card}(P_\alpha - P) < 2^\omega$, then P_α equals to $\sigma_{P_\alpha}(P)$. Hence, if $x \in P_\alpha$ and we put $E = P_\alpha$, $F = P_\alpha + (P_\alpha - (x))$ (of course, F is embeddable into R^2), then $E\ \mu\ F$ but E non $\varrho\ F$ because $P_\alpha - (x)$ is F-sequentially complete but not E-sequentially complete.

2.

Now we shall consider general $\mathscr{C}_0$-sequential envelopes (cf. [17], [12]). Let X be a sequentially regular convergence space. Denote $\mathbf{R}(X) = \{\mathscr{C}_0 \subset \mathscr{C}(X)\colon X$ is $\mathscr{C}_0$-sequentially regular$\}$. For $\mathscr{C}_1, \mathscr{C}_2 \in \mathbf{R}(X)$ put $\mathscr{C}_1 \sim \mathscr{C}_2$ whenever the $\mathscr{C}_1$-sequential envelope $\sigma_1(X)$ and the $\mathscr{C}_2$-sequential envelope $\sigma_2(X)$ of X are equivalent, i.e., there is a homeomorphism of $\sigma_1(X)$ onto $\sigma_2(X)$ leaving X pointwise fixed. We write $\sigma_1(X) = \sigma_2(X)$ in this case. It is easy to see that $\sim$ is an equivalence relation; by $[\mathscr{C}_0]$ we denote the equivalence class containing $\mathscr{C}_0$. Our aim is to study the properties of $[\mathscr{C}_0]$. They provide vital information about $\sigma_0(X)$ even when $\mathscr{C}_0$ is a general class.

Let X be a $\mathscr{C}_0$-sequentially regular convergence space. Denote $l\mathscr{C}_0 = \mathscr{C}(\sigma_0(X)) \mid X$ and denote by $r\mathscr{C}_0$ the smallest subring of $\mathscr{C}(X)$ containing $\mathscr{C}_0$ and all constants. Clearly $rl\mathscr{C}_0 = l\mathscr{C}_0$. Sets $\mathbf{R}(X)$ and $[\mathscr{C}_0]$ are partially ordered by inclusion.

Theorem 2.1. *Let X be a $\mathscr{C}_0$-sequentially regular convergence space. Then*:

(i) *$l\mathscr{C}_0$ is the largest element in $[\mathscr{C}_0]$.*

(ii) *If $\mathscr{C}_\alpha \in [\mathscr{C}_0]$, $\alpha \in A$, then $\bigcup_{\alpha \in A} \mathscr{C}_\alpha \in [\mathscr{C}_0]$.*

(iii) *$r\mathscr{C}_0 \in [\mathscr{C}_0]$.*

(iv) *There is a class $\mathscr{C}_1 \subset \mathscr{C}^*(X)$ such that $\mathscr{C}_0 \sim \mathscr{C}_1$.*

Proof. It is easy to see that if $\mathscr{C}_1 \subset \mathscr{C}(X)$ and X is $\mathscr{C}_1$-embedded in $\sigma_0(X)$, then $\mathscr{C}_1 \subset l\mathscr{C}_0$. Conditions (i), (ii), and (iii) follow from Corollary 2.4 in [12]. Condition (iv) follows from the fact that for $\mathscr{C}_1 = \mathscr{C}^*(\sigma_0(X)) \mid X$ the space X is $\mathscr{C}_1$-embedded in $\sigma_0(X)$ and, by Lemma 1.7 (condition (iii)) in [12], $\sigma_0(X)$ has the property p with respect to $\mathscr{C}^*(\sigma_0(X))$. Theorem 2.2 in [12] gives now the result.

Remark 2.2. Note that $[\mathscr{C}_0]$ need not contain the smallest element. Consider the following example. Let $X =]0, 1[$ and let $\mathscr{C}_0$ consist of the identity mapping on X. Then X is $\mathscr{C}_0$-sequentially regular and $\sigma_0(X) = [0, 1]$. If $\mathscr{C}_1$ consists of the function sin restricted to X, then $\mathscr{C}_0 \sim \mathscr{C}_1$, but $\mathscr{C}_0$ and $\mathscr{C}_1$ are incomparable minimal elements of $[\mathscr{C}_0]$. Moreover, rings $r\mathscr{C}_0 \in [\mathscr{C}_0]$ and $r\mathscr{C}_1 \in [\mathscr{C}_0]$ are incomparable. Since their intersection contains only constants, there is no ring in $[\mathscr{C}_0]$ which is smaller than both $r\mathscr{C}_0$ and $r\mathscr{C}_1$.

Remark 2.3. Let X be a sequentially regular convergence space. In [7] it was proved that $\mathscr{C}(X) \sim \mathscr{C}^*(X)$. Condition (iv) in Theorem 2.1 generalizes this fundamental feature of $\sigma(X)$ to $\mathscr{C}_0$-sequential envelopes. Namely, if we study properties of $\sigma_0(X)$, then it suffices to consider $\mathbf{R}^*(X) = \{\mathscr{C}_0 \subset \mathscr{C}^*(X) : X \text{ is } \mathscr{C}_0\text{-sequentially regular}\}$ and the restriction of $\sim$ to $\mathbf{R}^*(X)$. Clearly, if $\mathscr{C}_0 \in \mathbf{R}^*(X)$, then the ring $l^*\mathscr{C}_0 = \mathscr{C}^*(\sigma_0(X)) \mid X$ is the largest element in the restricted equivalence class containing $\mathscr{C}_0$.

Problem 2.4. Characterize rings $\mathscr{C}_0 \in \mathbf{R}^*(X)$ such that $\mathscr{C}_0 = l^*\mathscr{C}_0$.

Our next objective is the interplay between closure operators for $\mathscr{C}^*(X)$ and the equivalence relation $\sim$.

Remark 2.5. Let w be a closure operator for $\mathscr{C}^*(X)$ and $\overline{w}$ its topological modification (i.e., the finest of all topological (idempotent) closure operators for $\mathscr{C}^*(X)$ coarser than w). Since $\overline{w}$ can be obtained from w via a transfinite construction using only iterations and unions of w-closures, it follows (cf. condition (ii) in Theorem 2.1) that $\overline{w}\mathscr{C}_0 \sim \mathscr{C}_0$ whenever $w\mathscr{C}_0 \sim \mathscr{C}_0$. Consequently, we restrict ourselves to topological closure operators.

Let X be a sequentially regular convergence space. Let A be a countable subset of X, ε a positive real number, and $f \in \mathscr{C}^*(X)$. Sets $O_f(A, \varepsilon) = \{g \in \mathscr{C}^*(X): \sup_{x \in A} |f(x) - g(x)| < \varepsilon\}$ form a fundamental family of neighborhoods for a topology u for $\mathscr{C}^*(X)$. Clearly, u is coarser than the metric topology, but it is finer than the topology of pointwise convergence.

Theorem 2.6. *If $\mathscr{C}_0 \in \mathbf{R}^*(X)$, then $\mathscr{C}_0 \sim u\mathscr{C}_0$.*

Proof. Let (S, σ) be the $\mathscr{C}_0$-sequential envelope of X. First, we have to prove that X is $u\mathscr{C}_0$-embedded in (S, σ). By Corollary 6 in [6], it suffices to show that if A and B are countable subsets of X such that $\mathrm{cl}\, f[A] \cap \mathrm{cl}\, f[B] = \emptyset$ for some $f \in u\mathscr{C}_0$, then $\sigma^{\omega_1} A \cap \sigma^{\omega_1} B = \emptyset$. Put $\varepsilon = (1/3)\, \mathrm{d}(f[A], f[B])$. Then there is a function $h \in \mathscr{C}_0$ such that $h \in O_f(A \cup B, \varepsilon)$ and hence $\mathrm{cl}\, h[A] \cap \mathrm{cl}\, h[B] = \emptyset$. Since h can be continuously extended over S, we have $\sigma^{\omega_1} A \cap \sigma^{\omega_1} B = \emptyset$. Secondly, by Theorem 2.2 in [12], it suffices to prove that (S, σ) has the property p with respect to $\overline{u\mathscr{C}_0}(S) = \{f \in \mathscr{C}(S): f \mid X \in u\mathscr{C}_0\}$. But this follows from $\overline{\mathscr{C}}_0(S) \subset \overline{u\mathscr{C}_0}(S)$ and the fact that (S, σ) has the property p with respect to $\overline{\mathscr{C}}_0(S)$.

Corollary 2.7. (i) *If $\mathscr{C}_0 \in \mathbf{R}^*(X)$ and $\mathscr{C}_0 \subset \mathscr{C}_1 \subset u\mathscr{C}_0$, then $\mathscr{C}_0 \sim \mathscr{C}_1$.*
(ii) *If $\mathscr{C}_1, \mathscr{C}_2 \in \mathbf{R}^*(X)$, then $u\mathscr{C}_1 \sim u\mathscr{C}_2$ iff $\mathscr{C}_1 \sim \mathscr{C}_2$.*

Remark 2.8. Let X be a sequentially regular convergence space. It can be shown that for each $\mathscr{C}_0 \in \mathbf{R}^*(X)$ we have $rur\mathscr{C}_0 = ur\mathscr{C}_0$. Let $\tilde{X}$ be the completely regular modification of X(cf. [13]). Then $\mathscr{C}^*(X) = \mathscr{C}^*(\tilde{X})$. Since $ur\mathscr{C}_0$ is ring containing constants and it is closed in the metric topology, it follows from Lemma 16.2 in [2] that $ur\mathscr{C}_0$ is a sublattice of $\mathscr{C}^*$.

The next example shows that if v is the topology of pointwise convergence for $\mathscr{C}^*$, then $\mathscr{C}_0 \sim v\mathscr{C}_0$ need not hold.

Example 2.9. Let $X =]0, 1[$ and let $\mathscr{C}_0$ be the set of all polynomials on X. It is easy to see that $\sigma_0(X) = [0, 1]$. Since $v\mathscr{C}_0$ contains functions behaving in the neighborhood of 0 like $\sin(1/x)$, $\mathscr{C}_0$ and $v\mathscr{C}_0$ are not equivalent.

Remark 2.10. Let X be a sequentially regular convergence space. Trivially, the discrete topology is the finest topology for $\mathscr{C}^*(X)$ such that $\mathrm{cl}\mathscr{C}_0 \sim \mathscr{C}_0$ whenever $\mathscr{C}_0 \in \mathbf{R}^*(X)$. Let $\mathscr{C}_0 \in \mathbf{R}^*(X)$ and let $\{t_\alpha\}$ be the set of all topologies for $\mathscr{C}^*(X)$ such that $t_\alpha\mathscr{C}_0 \sim \mathscr{C}_0$. Then $\sup t_\alpha$ (defined by $\sup t_\alpha \mathscr{F} = \bigcup t_\alpha \mathscr{F}$, $\mathscr{F} \subset \mathscr{C}^*(X)$) is a closure operator for $\mathscr{C}^*(X)$ and its topological modification is the coarsest topology for $\mathscr{C}^*(X)$ such that $\mathrm{cl}\, \mathscr{C}_0 \sim \mathscr{C}_0$. In the same way it can be shown that there is the coarsest topology t for $\mathscr{C}^*(X)$ such that $t\mathscr{C}_0 \sim \mathscr{C}_0$ for all $\mathscr{C}_0$ in $\mathbf{R}^*(X)$. It follows from Theorem 2.6 that t is coarser than u. Note that for $X =]0, 1[$ the topology of pointwise convergence for $\mathscr{C}^*(X)$ is strictly coarser than t (cf. Example 2.9). It might be interesting to study the properties of t more closely.

Let $\mathbf{A}$ be a ring of sets and $\sigma(\mathbf{A})$ the generated σ-ring, both considered as convergence spaces. Let $\mathscr{P}_0$ be the set of all Dirac measures on $\mathbf{A}$ and $\mathscr{P}$ the set of all probability measures on $\mathbf{A}$. Then (cf. [17]) $\mathscr{P}_0 \subset \mathscr{P} \subset \mathscr{C}^*(\mathbf{A})$ and $\sigma(\mathbf{A})$ is both the $\mathscr{P}_0$-sequential and the $\mathscr{P}$-sequential envelope of $\mathbf{A}$.

Problem 2.11. Find a topology for $\mathscr{C}^*(A)$ such that $\mathscr{P} \subset \mathrm{cl}\, \mathscr{P}_0$ and $\mathrm{cl}\, \mathscr{P}_0 \sim \mathscr{P}_0$.

Finally, we give a necessary and sufficient condition for two classes $\mathscr{C}_1, \mathscr{C}_2 \in \mathbf{R}(X)$ to be equivalent. Recall that the $\mathscr{C}_i$-sequential envelope $\sigma_i(X)$ of X is characterized by the following three proporties (cf. [12]): X is sequentially dense in $\sigma_i(X)$, each $f \in \mathscr{C}_i$ can be (uniquely) continuously extended over $\sigma_i(X)$ (i.e., X is $\mathscr{C}_i$-embedded in $\sigma_i(X)$), and $\sigma_i(X)$ is sequentially complete with respect to the extended functions. A natural way to give a necessary and sufficient condition comprising the three properties is in terms of separating countable subsets by functions (cf. [7], [6]).

Theorem 2.12. *Let X be a sequentially regular convergence space. For $i \in \{0, 1\}$ let $\mathscr{C}_i \in \mathbf{R}(X)$ and let (S_i, σ_i) be the $\mathscr{C}_i$-sequential envelope of X. Then $\mathscr{C}_1 \sim \mathscr{C}_2$ iff the following condition is satisfied:*

($\sim$) *Let $i, j \in \{0, 1\}$, $i \neq j$. Let $a, b \in S_i$, let A and B be countable subsets of X such that $a \in \sigma_i^{\omega_1} A$ and $b \in \sigma_i^{\omega_1} B$, and let $\operatorname{cl} f[A] \cap \operatorname{cl} f[B] = \emptyset$ for some $f \in \mathscr{C}_j$. Then there are subsets $A' \subset A$, $B' \subset B$ such that $a \in \sigma_i^{\omega_1} A'$ and $b \in \sigma_i^{\omega_1} B'$, and $\operatorname{cl} g[A'] \cap \operatorname{cl} g[B'] = \emptyset$ for some $g \in \mathscr{C}_i$.*

Proof. 1. Let $\mathscr{C}_1 \sim \mathscr{C}_2$. Then there is a homeomorphism $h: (S_1, \sigma_1) \to (S_2, \sigma_2)$ leaving X pointwise fixed. Clearly, it suffices to prove that condition ($\sim$) holds for $i = 1$, $j = 2$. Suppose that the assumptions of ($\sim$) are satisfied. Then $h(a) \in \sigma_2^{\omega_1} A$ and $h(b) \in \sigma_2^{\omega_1} B$. If $\bar{f}$ is the continuous extension of f over S_2, then $\bar{f}(h(a)) \neq \bar{f}(h(b))$ and hence $a \neq b$. Thus there is a function $\bar{g} \in \mathscr{C}(S_1)$ such that $g = \bar{g} \mid X \in \mathscr{C}_1$ and $\bar{g}(a) \neq \bar{g}(b)$. It is easy to see that there are subsets $A' \subset A$ and $B' \subset B$ such that $a \in \sigma_1^{\omega_1} A'$ and $b \in \sigma_1^{\omega_1} B'$, and $\operatorname{cl} g[A'] \cap \operatorname{cl} g[B'] = \emptyset$.

2. Suppose that condition ($\sim$) holds. Using Corollary 6 in [6] it can be shown that for $i \neq j$ the space X is $\mathscr{C}_j$-embedded in S_i. Since S_i is the $\mathscr{C}(S_i)| X$-sequential envelope of X (cf. Corollary 2.4 in [12]), for $i \neq j$ we have $\mathscr{C}_j \subset \mathscr{C}(S_i) X$. It follows from Theorem 6 in [17] that there are continuous mappings $\varphi_1: (S_1, \sigma) \to (S_2, \sigma_2)$ and $\varphi_2: (S_2, \sigma_2) \to (S_1, \sigma_1)$ such that $\varphi_1(x) = \varphi_2(x) = x$ for each $x \in X$. The extension of identity principle (cf. [11]) implies that $\varphi_2 \circ \varphi_1$ is the identity mapping on (S_1, σ_1). Consequently, φ_1 is one-to-one, φ_2 is onto, and $\varphi_2 \mid \varphi_1[S_1] = \varphi_1^{-1}$. In the same way it can be proved that φ_1 is onto. Then $\varphi_2 = \varphi_1^{-1}$ and hence $\mathscr{C}_1 \sim \mathscr{C}_2$.

Condition ($\sim$) is a clumsy one. Remark 2.13 indicates why it is so.

Remark 2.13. Let X be a sequentially regular convergence space and $\mathscr{C}_1, \mathscr{C}_2 \in R(X)$. Consider the following conditions.

(i) Let $i, j \in \{0, 1\}$, $i \neq j$. Let A and B be countably infinite subsets of X. If $\operatorname{cl} f[A] \cap \operatorname{cl} f[B] = \emptyset$ for some $f \in \mathscr{C}_i$, then $\operatorname{cl} g[A] \cap \operatorname{cl} g[B] = \emptyset$ for some $g \in \mathscr{C}_j$.

(ii) Let $i, j \in \{0, 1\}$, $i \neq j$. Let A and B be countably infinite subsets of X. If $\operatorname{cl} f[A] \cap$ $\cap \operatorname{cl} f[B] = \emptyset$ for some $f \in \mathscr{C}_i$, then there are infinite subsets $A' \subset A$ and $B' \subset B$ such that $\operatorname{cl} g[A'] \cap \operatorname{cl} g[B'] = \emptyset$ for some $g \in \mathscr{C}_j$.

Condition (i) clearly implies $\mathscr{C}_1 \sim \mathscr{C}_2$, but Example 2.14 shows that $\mathscr{C}_1 \sim \mathscr{C}_2$ does not imply (i). On the other hand, condition (ii) follows from $\mathscr{C}_1 \sim \mathscr{C}_2$, but Example 2.15 shows that $\mathscr{C}_1 \sim \mathscr{C}_2$ does not follow from (ii).

Example 2.14. Let X be the space of rational numbers. Let $\mathscr{C}_1$ be the set of all uniformly continuous functions on X. Let $\mathscr{C}_2$ be the set of all continuous functions f_r, $r \in X$, defined as follows: $f_r(r) = 1$; $f_r(x) = 0$ for $x \in X - [r - 1, r + 1]$; and f_r is linear on $[r - 1, r]$ and $[r, r + 1]$. It is easy to verify that X is $\mathscr{C}_i$-sequentially regular and the real line is the $\mathscr{C}_i$-sequential envelope of X. Hence $\mathscr{C}_1 \sim \mathscr{C}_2$. It can be shown that condition (i) does not hold.

Example 2.15. Let X be the subset of the plane consisting of points $(1/m, 1/n)$, there m and n are natural numbers. Let $\mathscr{C}_1$ be the set of all functions f of the form: for each n, $f((1/m, 1/n)) = 1/n$ for all but finitely many m. Let $\mathscr{C}_2$ be the set of all functions f_ξ, $\xi = \langle \xi(n) \rangle \in \{0, 1\}^N$, of the form: for each n, $f_\xi((1/m, 1/n)) = \xi(n)$ for all but finitely many m. The discrete space X is $\mathscr{C}_i$-sequentially regular. Put $S_1 = X \cup (\bigcup_{n \in N} ((0, 1/n))) \cup ((0, 0))$. Consider the following convergence of sequences in S_1: for each n, $(0, 1/n) = \lim_{i \to \infty} (1/m_i, 1/n)$ whenever $\langle m_i \rangle$ is a subsequence of $\langle m \rangle$; $(0, 0) = \lim (0, 1/n_i)$ whenever $\langle n_i \rangle$ is a subsequence of $\langle n \rangle$; and for each $s \in S_1$ the constant sequence $\langle s \rangle$ converges to s. Let σ_1 be the induced closure operator. Put $S_2 = S_1 - ((0, 0))$ and let (S_2, σ_2) be the corresponding subspace. It is easy to verify that (S_i, σ_i) is the $\mathscr{C}_i$-sequential envelope of X. Hence $\mathscr{C}_1$ non $\sim \mathscr{C}_2$. It can be shown that condition (ii) holds.

3.

It was shown in [7] that a sequentially regular convergence space X has at most two topologically different $\mathscr{C}_E$-sequential envelopes, $E \in \mathsf{E}_1$. Namely, either $\sigma_E(X) = \sigma_{\{0,1\}}(X)$ (whenever X is $\{0, 1\}$-sequentially regular and E does not contain an interval) or $\sigma_E(X) = \sigma(X)$ (whener E contains an interval). In this section we construct a $\{0, 1\}$-sequentially regular convergence space X such that $\sigma_{\{0,1\}}(X) \neq \sigma(X)$. This is a solution of Problem 2.5 in [7].

The space X is the usual topological space $N \cup \mathscr{N}$, where $\mathscr{N}$ is an infinite maximal almost disjoint family of infinite subsets of N (i.e., $\mathscr{N}$ is infinite and it is maximal with respect to: card $(M) = \omega$ for $M \in \mathscr{N}$, card $(M_1 \cap M_2) < \omega$ for different $M_1, M_2 \in \mathscr{N}$ — for basic properties see, e.g., [1]), N is an open discrete subspace, and for $M \in \mathscr{N}$ sets $\{M\} \cup (M - F)$, F is a finite subset of N (here $\{M\}$ is a singleton and $M - F$ is a subset of N), form a local base at M. For technical reasons, we shall consider the elements of $\mathscr{N}$ as one-to-one sequences. Thus X is Fréchet, completely regular, separable, pseudocompact, scattered, locally compact, locally countable, not normal, not realcompact, and not countably compact. Also, our space X is 0-dimensional but not strongly 0-dimensional. An example of $\mathcal{N}$ such that $N \cup \mathcal{N}$ is not strongly 0-dimensional was constructed in [18]; our example is stronger in the sense that the Banaschewski 0-dimensional compactification of $N \cup \mathcal{N}$ equals to one-point compactification of $N \cup \mathcal{N}$ while the Čech-Stone compactification of $N \cup \mathcal{N}$ has the remainder of cardinality at least continuum.

Our task is to find a family $\mathcal{N}$ such that $\sigma_{\{0,1\}}(N \cup \mathcal{N}) \neq \sigma(N \cup \mathcal{N})$. It suffices to construct $\mathcal{N}$ such that the topological modification $T\sigma_{\{0,1\}}(N \cup \mathcal{N})$ of $\sigma_{\{0,1\}}(N \cup \mathcal{N})$ is the one-point compactification of $N \cup \mathcal{N}$ and card $(\beta(N \cup \mathcal{N}) - - (N \cup \mathcal{N})) > 1$. Indeed, then $T\sigma_{\{0,1\}}(N \cup \mathcal{N})$ is compact and the topological modification $T\sigma(N \cup \mathcal{N})$ of $\sigma(N \cup \mathcal{N})$ is either not compact or $T\sigma(N \cup \mathcal{N}) = = \beta(N \cup \mathcal{N})$. In both cases $\sigma_{\{0,1\}}(N \cup \mathcal{N}) \neq \sigma(N \cup \mathcal{N})$.

We do not know whether $N \cup \mathcal{N} = \sigma(N \cup \mathcal{N})$, i.e., whether $N \cup \mathcal{N}$ is sequentially complete (cf. Problem 1.23).

In our construction we modify some of the ideas used by S. Mrówka in [15], proof of Theorem 2.5 (there is a family $\mathcal{N}$ such that card$(\beta(N \cup \mathcal{N}) - (N \cup \mathcal{N})) = 1$).

Example 3.1. The construction of X is done in three steps. The corresponding spaces $T \cup \mathcal{T}_i$, $i = 1, 2, 3$, are equipped with the same type of topology as $N \cup \mathcal{N}$. The underlying sets of all three spaces can be visualized as subsets of the closed unit cube in R^3, where points of T have positive third coordinates and the elements of $\mathcal{T}_i$ are indexed by the points of the base of the cube. Sometimes it is convenient to identify an element of $\mathcal{T}_i$ with its index.

Denote by T the subset $\{(k/n, l/n, 1/n): k, l = 0, 1, 2, \ldots, n = 1, 2, 3, \ldots\}$ of R^3 and by S the closed unit square $[0, 1] \times [0, 1]$. For $x \in S$ denote by T_x the set of points of T contained in the closed pyramid (turned upside down) with $S \times (1)$ as its base and $(x) \times (0)$ as its top vertex. Then $\mathcal{T}_1 = \{T_x: x \in S\}$ is an almost disjoint family (because T_x, considered as a one-to-one sequence, converges in R^3 to $(x) \times \times (0)$). Let f be a continuous function on the space $T \cup \mathcal{T}_1$. If we identify $\mathcal{T}_1$ with S, then the restriction $f \mid S$ is of the first Baire class on S equipped with its usual topology (we can continuously extend f to $\bar{f}$ over $(\bigcup_{n=1}^{\infty} (S \times (1/n))) \cup S$, e.g., linearly in both directions, and $f \mid S$ is the pointwise limit of continuous functions $\bar{f} |(S \times (1/n))$, $n = 1, 2, 3, \ldots$).

For $x \in S$ let $\mathcal{T}_x$ be a maximal almost disjoint subfamily of one-to-one sequences of points of T (i.e., countably infinite subsets of T) converging in R^3 to $(x) \times (0)$; we may suppose that $T_x \in \mathcal{T}_x$ and card $(\mathcal{T}_x) = 2^\omega$. It follows from the Bolzano-Weierstrass theorem that $\bigcup_{x \in S} \mathcal{T}_x$ is a maximal almost disjoint family in T. Denote $\mathcal{T}'_x = \mathcal{T}_x - (T_x)$ and $S(r) = (r) \times [0, 1]$. Let φ be a bijection of $\mathcal{T}_1$ onto $\bigcup_{x \in S} \mathcal{T}'_x$ such that for each $r \in [0, 1]$ φ maps $\{T_x: x \in S(r)\}$ onto $\bigcup_{x \in S(r)} \mathcal{T}'_x$. Then $\mathcal{T}_2 = = \{T_x \cup \varphi(T_x): x \in S\}$ is a maximal disjoint family in T which is coarser than $\mathcal{T}_1$.

Let f be a continuous function on the space $T \cup \mathscr{T}_2$. Again, if we identify $\mathscr{T}_2$ with S, then $f \mid S$ is of the first Baire class. According to a known property of measurable functions, for each $a \in R$ the set $f^{-1}(a) \cap \mathscr{T}_2$ is either countable or has cardinality 2^ω. If f ranges in $\{0, 1\}$, then $f^{-1}(a) \cap \mathscr{T}_2$ cannot be countably infinite (because $\{(T_x \cup \varphi(T_x)) \cap f^{-1}(a): a \in S \cap f^{-1}(a)\}$ is a maximal almost disjoint family in its union). In this case we prove even more.

Lemma. *Let f be a continuous function on $T \cap \mathscr{T}_2$ into $\{0, 1\}$. Then at least one of the following four statements is true.*

(i) card $(\mathscr{T}_2 \cap f^{-1}(0)) < \omega$;

(ii) card $(\mathscr{T}_2 \cap f^{-1}(1)) < \omega$;

(iii) card $(\{r \in [0, 1]: \text{card}\,(f[S(r)]) = 2\}) = 2^\omega$;

(iv) *There is a point $r_f \in [0, 1]$ such that* card $(S(r_f) \cap f^{-1}(0)) =$ card $(S(r_f) \cap \cap f^{-1}(1)) = 2^\omega$.

Proof. Suppose that statements (i), (ii), and (iii) are not true. Since $C_i = \{r \in \in [0, 1]: f[S(r)] = i\}$, $i \in \{0, 1\}$, are open subsets of $[0, 1]$ (recall that $f \mid S$ is of the first Baire class and hence $f^{-1}(i) \cap S$ is both G_δ and F_σ) and card $(S(r) \cap f^{-1}(i)) = 2^\omega$ whenever $r \in \text{cl}\, C_i$, the required point r_f is any point of cl $C_0 \cap$ cl C_1 (this intersection is not void provided both C_0 and C_1 are not void) or any point r such that card $(S(r) \cap f^{-1}(i)) = 2^\omega$ and card $(f[S(r)]) = 2$ provided $C_{1-i} = \emptyset$.

Since card $(\mathscr{C}(T \cup \mathscr{T}_2, \{0, 1\})) \leqq 2^\omega$, we can assign to each f of $\mathscr{F} = \{f \in \mathscr{C}(T \cup \cup \mathscr{T}_2, \{0, 1\}): \text{card}\,(\mathscr{T}_2 \cap f^{-1}(0)) = \text{card}\,(\mathscr{T}_2 \cap f^{-1}(1)) = 2^\omega\}$ points s_f, a_f, $b_f \in \in [0, 1]$ such that $f((s_f, a_f)) = 0$, $f((s_f, b_f)) = 1$, and for $f \neq g$ we have $b_f \neq b_g \neq \neq a_f \neq a_g$. Denote $A = \{(s_f, a_f) \in S: f \in \mathscr{F}\}$, $B = \{(s_f, b_f) \in S: f \in \mathscr{F}\}$, and $\mathscr{T}_3 = = \{T_x \cup \varphi(T_x): x \in (S - (A \cup B))\} \cup \{T_{(s_f, a_f)} \cup \varphi(T_{(s_f, a_f)}) \cup T_{(s_f, b_f)} \cup \varphi(T_{(s_f, b_f)}): f \in \in \mathscr{F}\}$. Then $\mathscr{T}_3$ is a maximal almost disjoint family in T. Let f be a continuous function on the locally compact space $T \cup \mathscr{T}_3$ into $\{0, 1\}$. Then f can be continuously extended over the one-point compactification of $T \cup \mathscr{T}_3$ (since $\mathscr{T}_3$ is coarser than $\mathscr{T}_2$ and the corresponding canonical mapping of $T \cup \mathscr{T}_2$ onto $T \cup \mathscr{T}_3$ is a quotient, for each $f \in \mathscr{C}(T \cup \mathscr{T}_3, \{0, 1\})$ we have eithr card $(\mathscr{T}_3 \cap f^{-1}(0)) < \omega$ or card $(\mathscr{T}_3 \cap \cap f^{-1}(1)) < \omega$). On the other hand, there is a continuous function f on $T \cup \mathscr{T}_3$ such that $f((k/n, l/n, 1/n)) = k/n$ and hence $f(M) = r$ for each $M \in \mathscr{T}_3$ such that $T_{(r,s)} \subset M \subset T$. Since in every $S(r)$ there "remains" a continuum of points by which the elements of $\mathscr{T}_3$ are indexed, card $(\beta(T \cup \mathscr{T}_3) - (T \cup \mathscr{T}_3)) \geqq 2^\omega$. Denote $X = = T \cup \mathscr{T}_3$. It follows from the construction that X has the desired properties.

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Added in proof. Recently, Problem 1.23 has been solved positively by L. Mišík, Jr. provided $\varkappa = 2^{\omega}$, a set-theoretical assumption weaker than Martin's axiom. The solution will appear in Czechoslovak Math. J.

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