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ON THE CONGRUENCE LATTICE OF STABLE ALGEBRAS WITH DEFINABILITY OF COMPACT CONGRUENCES*)

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1. INTRODUCTION

The congruence lattice of an arbitrary algebra has been characterized as an algebraic lattice. This follows by a theorem of G. Birkhoff and O. Frink [3] and by a theorem of G. Grätzer and E. T. Schmidt [6] (see also the paper [9] by P. Pudlák for a short and elegant proof). Much work has been done in this general direction for congruence lattices (see [1], [8], [10]).

These considerations address ourselves to investigate, instead, global properties of congruence lattice of algebras which are models of a first order theory. In particular, we are interested in the congruence lattice size of models of any first order theory T. To this end we defined in a previous paper [13], for every cardinal λ, a cardinal $C_T(\lambda)$ which is the supremum of the cardinalities of the congruence lattices of all models of T which have cardinality λ. We showed that a theory T with infinite models in a countable language without relation symbols falls into one of the following cases.

Case 1. $\operatorname{ded}(\lambda) \leqq C_T(\lambda) \leqq 2^\lambda$ for every infinite cardinal λ, where $\operatorname{ded}(\lambda)$ is the supremum of the cardinals μ such that there is a linearly ordered set of cardinality λ with μ Dedekind cuts (Cf. [7]).

Case 2. $C_T(\lambda) = \lambda$ for every infinite cardinal λ.

Case 3. There exists a positive integer n such that $C_T(\lambda) = n$ for every infinite cardinal λ.

In agreement with the above cases, a theory will be called of kind 1, 2 or 3 respectively.

In this paper we give a sufficient condition for a theory T in order that $C_T(\lambda) = 2^\lambda$ for every infinite cardinal λ. This happens when, for every positive integer k, there exists a congruence of a model of T which has a minimal set of generators with

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cardinality k. As a consequence $C_T(\lambda) = 2^\lambda$ for every infinite cardinal λ when T is a stable theory of kind 1 and with Definability of Compact Congruences (DCC).

We recall that there are some examples [13] of $\aleph_0$-categorical theories T for which $C_T(\lambda) = \text{ded}(\lambda)$ for every infinite cardinal λ. Moreover, we observe that the above results are interesting only when the General Continuum Hypothesis fails.

Finally, we give an example which shows that the main theorem cannot be improved.

2. PRELIMINARIES

We assume that the reader is familiar with the basic concepts of Universal Algebra [6] and of Model-Theory [4]. The non-explained notation will be standard.

We consider algebras and theories in a countable type without relations. We denote by the same typographical symbol A an algebra and its basic set. For any algebra A, $\text{Con}(A)$ is the lattice of its congruences and $\text{Comp}(A)$ is the join-semilattice of its compact congruences. The join operation in $\text{Con}(A)$ is denoted, as usual, by $\vee$; the context will show when such a symbol means logical disjunction. If $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ are elements of an algebra A, $\theta_A(a_1, \ldots, a_n; b_1, \ldots, b_n)$ will denote the smallest congruence of A which contains the n pairs $(a_1, b_1), \ldots, (a_n, b_n)$.

A first order formula in $2n + 2$ free variables $\psi(u_1, \ldots, u_n; v_1, \ldots, v_n; x, y)$ is called a *weak n-congruence formula* (Cf. [2], [11], [12]) if it is positive and the universal quantification of the formula

$$\left(\bigwedge_{1 \leqq i \leqq n} (u_i = v_i) \wedge \psi(u_1, \ldots, u_n; v_1, \ldots, v_n; x, y)\right) \rightarrow (x = y)$$

is logically valid. The following fact is well-known under the name of Mal'cev's Lemma (see [5], [6], [12]): for every algebra A and every $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$, $(c, d) \in \theta_A(a_1, \ldots, a_n; b_1, \ldots, b_n)$ if and only if there exists a weak n-congruence formula ψ in the language of A such that $A \sqsubset \psi(a_1, \ldots, a_n; b_1, \ldots, b_n; c, d)$.

We denote by Γ_n the set of weak congruence formulas for $n \geqq 1$; Γ_0 is the set containing only the formula $x = y$.

A theory T has the Definability of Compact Congruences (DCC) if for every n there exists a formula $\Phi_n(u_1, \ldots, u_n; v_1, \ldots, v_n; x, y)$ such that for every model A of T and for every $a_1, \ldots, a_n, b_1, \ldots, b_n, c, d \in A$ it holds $A \sqsubset \Phi_n(a_1, \ldots, a_n; b_1, \ldots, b_n; c, d)$ if and only if $(c, d) \in \theta_A(a_1, \ldots, a_n; b_1, \ldots, b_n)$. An easy compactness argument shows that if a theory T has the DCC, then Φ_n can be chosen in Γ_n (see [2], [12]). We say that an algebra A has the DCC if $\text{Th}(A)$ has such a property.

If $\Psi(u_1, \ldots, u_r)$ denotes a set of first order formulas with free variables among $u_1, \ldots, u_r$, then $A \sqsubset \Psi(a_1, \ldots, a_r)$ means that $A \sqsubset \psi(a_1, \ldots, a_r)$ for every $\psi \in \Psi$; analogously $A \sqsubset \exists u_1 \ldots \exists u_r \Psi$ means that there exist $a_1, \ldots, a_n \in A$ such that $A \sqsubset \sqsubset \Psi(a_1, \ldots, a_r)$.

In this section $\mathscr{L}$ is a given first order language for an algebraic countable type τ. For each positive integer k, Ψ_k will denote the set of formulas $\psi(u_1, \ldots u_k; v_1, \ldots, v_k)$ in $\mathscr{L}$ which are logically equivalent to

$$(3.1) \qquad \bigwedge_{1 \leqq j \leqq k} \neg \varphi(u_1, \ldots, u_{j-1}, u_{j+1}, \ldots, u_k; v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_k; u_j, v_j)$$

for some formula φ of Γ_{k-1}.

From definition of sets Γ_k we can easily get

Proposition 3.1. *For every algebra A of type τ, for all positive integers k, h with $k \leqq h$ and for all $a_1, \ldots, a_h, b_1, \ldots, b_h \in A$ it follows:*

$$(3.2) \qquad A \sqsubset \Psi_k(a_1, \ldots, a_k;\ b_1, \ldots, b_k) \quad \textit{iff} \quad (a_j, b_j) \notin \bigvee_{\substack{1 \leqq i \leqq k \\ i \neq j}} \theta_A(a_i, b_i)\,.$$

If $A \sqsubset \Psi_h(a_1, \ldots, a_h;\ b_1, \ldots, b_h)$ and $\{i_1, \ldots, i_k\}$ is a subset of $\{1, \ldots, h\}$ of k elements, then

$$(3.3) \qquad A \sqsubset \Psi_k(a_{i_1}, \ldots, a_{i_k};\ b_{i_1}, \ldots, b_{i_k})\,.$$

For any algebra A denote by $C_k(A)$ the set of congruences R of A such that each set of generators for R has a subset of generators for R of cardinality less than k. It is obvious that:

$$C_1(A) \subseteq C_2(A) \subseteq \ldots \subseteq C_k(A) \subseteq \ldots \subseteq \mathrm{Comp}(A)\,.$$

Lemma 3.2. *For every positive integer k and for every algebra A of type τ the following are equivalent*

i) $C_k(A) \neq \mathrm{Comp}(A)$,

ii) $A \sqsubset \exists u_1 \ldots \exists u_k \exists v_1 \ldots \exists v_k \Psi_k$.

Proof. (i) → (ii). Let R be a compact congruence with $R \notin C_k(A)$. Then there exist $a_1, \ldots, a_h, b_1, \ldots, b_h \in A$ with $h \geqq k$, such that $\{(a_1, b_1), \ldots, (a_h, b_h)\}$ is a minimal set of generators for R. Therefore, $A \sqsubset \Psi_h(a_1, \ldots, a_h;\ b_1, \ldots, b_h)$. Hence, we get (ii) by (3.3).

(ii) → (i) follows by (3.2).

The following theorem is the main contribution of this paper. We denote by $P(\lambda)$ the semilattice with respect to set-theoretical union of the power-set of all ordinals less than λ.

Theorem 3.3. *Suppose T is a theory in a countable language without relation symbols. If, for every positive integer k, there exists a model A of T such that $C_k(A) \neq \mathrm{Comp}(A)$, then, for every infinite cardinal λ there exists a model B of T with the following properties. B is of cardinality λ and there exists an embedding of* *the semilattice* $P(\lambda)$ *into* $\mathrm{Con}(B)$ *which preserves the join operation and the minimum. Hence, the cardinality of* $\mathrm{Con}(B)$ *is* 2^λ.

Proof. Let λ be an infinite cardinal. Consider the language $\mathscr{L}'$ obtained by adding to the language $\mathscr{L}$ of T sets of new constants $C = \{c_\nu : \nu < \lambda\}$, $D = \{d_\nu : \nu < \lambda\}$. Write in $\mathscr{L}'$ the following theory

$$(3.4) \qquad T' = T \cup \{\psi(c_{\nu_1}, \ldots, c_{\nu_k}; d_{\nu_1}, \ldots, d_{\nu_k}) : k \in \mathbb{N},\ k \geqq 1,\ \psi \in \Psi_k,\ 0 \leqq \nu_1 \leqq \ldots \leqq \nu_k < \lambda\}.$$

Let T_0 be a finite subset of T'. From (3.3) we get that T_0 is consistent whenever the set

$$(3.5) \qquad T \cup \Psi_k(c_1, \ldots, c_k;\ d_1, \ldots, d_k)$$

is consistent, where k is an integer greater than the number of constants which appear in T_0. But, by hypothesis and Lemma 3.2, (3.5) is consistent. Therefore, by the Compactness Theorem, T' has a model, say B', of cardinality λ. Observe that T has infinite models and that the language $\mathscr{L}'$ has cardinality λ.

Now, for every $J \in P(\lambda)$ define the following congruence relation on the reduct B of B' to the language $\mathscr{L}$.

$$(3.6) \qquad \Phi_J = \bigvee_{\nu \in J} \theta_B(c_\nu, d_\nu).$$

The map $m : J \mapsto \Phi_J$ from $P(\lambda)$ into $\mathrm{Con}(B)$ preserves clearly the join operation and the minimum. In order to prove that m is injective, it is sufficient the following

Claim. *If* $\nu \notin J$, *then* $(c_\nu, d_\nu) \notin \Phi_J$.

Assume $\nu \notin J$ and $(c_\nu, d_\nu) \in \Phi_J$. Then, there exists a finite subset of J, say $\mu_1, \ldots, \mu_{k-1}$, such that $(c_\nu, d_\nu) \in \theta_B(c_{\mu_1}, \ldots, c_{\mu_{k-1}};\ d_{\mu_1}, \ldots, d_{\mu_{k-1}})$. Hence, by (3.2) there exists $\psi \in \Psi_k$ such that

$$(3.7) \qquad B \sqsubset \neg\psi(c_{\mu_1}, \ldots, c_{\mu_{k-1}}, c_\nu;\ d_{\mu_1}, \ldots, d_{\mu_{k-1}}, d_\nu).$$

Rearrange the elements $\mu_1, \ldots, \mu_{k-1}, \nu$ in a sequence $\nu_1, \ldots, \nu_k$ such that $\nu_1 < \nu_2 < \ldots < \nu_k$. By (3.3) and by (3.7) we get that

$$(3.8) \qquad B \text{ non} \sqsubset \Psi_k(c_{\nu_1}, \ldots, c_{\nu_k};\ d_{\nu_1}, \ldots, d_{\nu_k})$$

Since B' is a model of T', (3.8) is a contradiction.

In order to derive a Corollary for stable theories, we need the following

Proposition 3.4. *Let* A *be a stable algebra with* DCC, *then, for each integer* k, *the following are equivalent*

(i) $C_k(A) = \mathrm{Con}(A)$,

(ii) $C_k(A) = \mathrm{Comp}(A)$.

Proof. (i) → (ii) since $C_k(A) \subseteq \mathrm{Comp}(A) \subseteq \mathrm{Con}(A)$.

(ii) → (i). Suppose $\Phi \in \mathrm{Con}(A)$. If, by contradiction, $\Phi \notin \mathrm{Comp}(A)$, then there exists a chain of compact congruences, say $\{\varrho_n : n \in \mathbb{N}\}$, such that

$$(3.9) \qquad \varrho_1 < \varrho_2 < \ldots < \varrho_n < \varrho_{n+1} < \ldots < \Phi .$$

Hence, by (ii) and by DCC, for every n, there are elements $a_1^n, \ldots, a_k^n, b_1^n, \ldots, b_k^n \in A$ and a formula $\varphi(u_1, \ldots, u_k; v_1, \ldots, v_k; x, y)$ such that

$$(3.10) \qquad \varrho_n = \{(c, d) : A \sqsubset \varphi(a_1^n, \ldots, a_k^n; b_1^n, \ldots, b_k^n; c, d)\} .$$

Then, (3.9) and (3.10) contradict the stability of Th(A) (Cf. [4], 7.1.33).

Corollary 3.5. *If T is a stable theory with* DCC *in a countable language, then the following are equivalent*

(i) *T is of kind* 1.

(ii) *For every infinite cardinal λ there exists a model B of T such that $|B| = \lambda$, $|\mathrm{Con}(B)| = 2^\lambda$.*

(iii) *$C_T(\lambda) = 2^\lambda$ for every infinite cardinal λ.*

Proof. (i) → (ii). Let A be an infinite model of T such that $|\mathrm{Con}(A)| > |A|$. Then, $\mathrm{Con}(A) \neq \mathrm{Comp}(A)$. Therefore, by Proposition 3.4, $C_k(A) \neq \mathrm{Comp}(A)$ for every positive integer k. Hence, (ii) follows by Theorem 3.3.

(ii) → (iii) and (iii) → (i) are trivial.

Remark 3.6. Let A be an algebra such that $C_k(A) \neq \mathrm{Comp}(A)$ for every positive integer k. Then Th(A) satisfies the hypothesis of Theorem 3.3, but not necessarily Con(A) has cardinality $2^{|A|}$. In fact, the next example shows that this does not happen even under the assumption of DCC for Th(A).

Example 3.7. Let Z be the ring of the integers. Then:

(3.11) Th(Z) has DCC.

(3.12) $C_k(Z) \neq \mathrm{Comp}(Z)$ for every integer k.

(3.13) $|\mathrm{Con}(Z)| = \aleph_0$.

The formula $\exists z_1 \ldots \exists z_k(\sum_{i=1}^{k} z_i(v_i - u_i) = y - x)$ defines the k-generated congruences in the class of commutative rings with unity. Therefore, we have (3.11).

Let $p_1, \ldots, p_k$ be the first k prime numbers. Define $q_j = \prod_{i \neq j} p_i$ for $j = 1, \ldots, k$. Then the ideal generated by $H = \{q_1, \ldots, q_k\}$ is Z. But any proper subset of H generates a proper ideal of Z. This proves (3.12).

Finally, (3.13) is obvious since Z is a principal ring.

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