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ON GENERICITY OF COMPLETE CONTROLLABILITY IN THE SPACE OF LINEAR PARAMETRIZED CONTROL SYSTEMS

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In the control theory of differential equations the property of complete controllability is of fundamental importance. There is an important question related to this notion, which is of interest also from the physical view-point: Can any control system be approximated by a control system which is completely controllable? We will attempt to answer the above question for the space of linear control systems which differentiably depend on some parameters.

First, let us consider a linear control system represented by a system of linear differential equations

$$\dot{x} = Ax + Bu\,, \tag{1}$$

where $x \in R^n$, $u \in R^m$, $A \in M(n, n)$, $B \in M(n, m)$, $M(i, j)$ denotes the set of all $i \times j$ matrices. Since the control system (1) is uniquely determined by the pair of matrices A, B we shall denote it by (A, B). The set of all such control systems is denoted by $F(n, m)$.

A system $(A, B) \in F(n, m)$ is called *completely controllable* if for every $x_0, x_1 \in R^n$ there is a bounded, measurable control function $u(t)$ on an interval $[0, t_1]$ such that the solution $x(t)$ of the initial value problem $\dot{x} = Ax + Bu$, $x(0) = x_0$ satisfies the condition $x(t_1) = x_1$. Let us denote the set of all completely controllable systems of the type (1) by $F_c(n, m)$.

It is well known (cf. [5], [6]) that $(A, B) \in F_c(n, m)$ if and only if rank $(B, AB, \ldots$ $\ldots, A^{n-1}B) = n$ and so we have a nice pure by algebraic criterion for the complete controllability. Using this very useful criterion, it is possible to prove that the property of complete controllability is stable in the following sense: If we endow the set $F(n, m)$ with the topology induced by the metric $d[(A, B), (A', B')] = |A - A'|_1 + |B - B'|_2$, where $|\cdot|_1$ is the Euclidean norm in the space $M(n, n)$, $|\cdot|_2$ is the Euclidean norm in $M(n, m)$, then the set $F_c(n, m)$ is open and dense in $F(n, m)$ with respect to this topology (see [6, Theorem 11, p. 111]. This stability is very important for the description of real physical systems. We remark that J. D. Dauer [3] generalized this result to the linear time-dependent control systems $\dot{x} = A(t)\,x + B(t)\,u$ endowed with the topology induced by the L^p norm, supposing $A(t)$, $B(t)$ to be Lebesgue integrable on an interval I and $\int_I |A(s)|_1^p \,\mathrm{d}s < \infty$, $\int_I |B(s)|_2^p \,\mathrm{d}s < \infty$.

Many real processes depend on some parameters and in many cases this dependence is at least continuous. There is a question whether any C^r $(r \geqq 1)$ curve in the space of all control systems, or more generally any C^r parametrized control system, can be approximated by a C^r curve or a C^r parametrized control system, respectively, so that for all values of the parameter the corresponding control system is completely controllable. This problem was solved by the author for parametrized control systems with a 1-dimensional and particularly with a 2-dimensional parameter (see [7]). It was shown that if we require a C^r approximation $(r \geqq 1)$, then the answer to the above question is negative in general and it depends on the dimension of the parameter as well as on the number of columns of the matrix B. In this paper we solve this problem for a class of linear control systems depending on more-dimensional parameter.

Let us consider a parametrized linear control system

$$\dot{x} = A(\mu)\,x + B(\mu, \nu)\,u\,, \tag{2}$$

where $x \in R^n$, $u \in R^m$, $\mu \in P$, $\nu \in Q$, P, Q are compact manifolds, $\dim P = 1$, $\dim Q = q$, $A \in C^r(P, M(n, n))$, $B \in C^r(P \times Q, M(n, m))$. We remark that the compactness of P and Q is supposed because we are going to apply the transversality theorems.

Since the parametrized control system (2) is uniquely determined by the pair of C^r mappings $A : P \to M(n, n)$, $B : P \times Q \to M(n, m)$, we shall denote it by $(A, B)_r$. Denote by $F^r_{P\,Q}(n, m)$ the set of all parametrized control systems of the type (2) with the topology of uniform convergence of all partial derivatives up to the order r.

Now, let us give a very simple example. Consider the parametrized control system

$$\dot{x} = A(\mu)\,x + B(\mu)\,u\,, \tag{3}$$

where $A(\mu) = \operatorname{diag}(\lambda_1(\mu), \lambda_2(\mu))$, $B(\mu) = \operatorname{transp}(b_1(\mu), b_2(\mu))$, λ_i, b_i $(i = 1, 2)$ are C^r functions on the interval $I = [-1, 1]$. The system (3) is completely controllable for μ fixed if and only if $\Delta(\mu) = \det(B(\mu), A(\mu)\,B(\mu)) = b_1(\mu)\,b_2(\mu)\,(\lambda_2(\mu) - \lambda_1(\mu)) \neq 0$. If for instance $b_1(\mu) = \mu$ on I then $\Delta(0) = 0$ and for all $\tilde{b}_1$ from a sufficiently small C^r neighbourhood of b_1 there is a $\tilde{\mu} \in (-1, 1)$ such that $\Delta(\tilde{\mu}) = \det(\tilde{B}(\tilde{\mu}), A(\tilde{\mu})\,\tilde{B}(\tilde{\mu})) = 0$, where $\tilde{B}(\mu) = \operatorname{transp}(\tilde{b}_1(\mu), b_2(\mu))$. This means that it is not possible to avoid such values $\mu_0 \in I$ of the parameter μ by a small perturbation of the mappings A and B for which $\Delta(\mu_0) = 0$, i.e. for each $(\tilde{A}, \tilde{B})_r$ from a sufficiently small neighbourhood of the system $(A, B)_r$ there is a $\mu_0 \in I$ such that the system $(A(\mu_0), B(\mu_0))$ is not completely controllable. Nevertheless, if

$$B(\mu) = \begin{bmatrix} b_{11}(\mu) & b_{12}(\mu) & b_{13}(\mu) \\ b_{21}(\mu) & b_{22}(\mu) & b_{23}(\mu) \end{bmatrix},$$

then it is possible to prove that there is a small C^r perturbation $\tilde{B}$ of B and C^r perturbations $\tilde{\lambda}_1, \tilde{\lambda}_2$ of λ_1, λ_2, respectively, such that if $\tilde{B}(\mu) = (\tilde{b}_{ij}(\mu))$, then

$$D(\mu) = \det \begin{bmatrix} \tilde{b}_{11}(\mu) & \tilde{b}_{12}(\mu) \\ \tilde{b}_{21}(\mu) & \tilde{b}_{22}(\mu) \end{bmatrix} = 0$$

only for μ from a subset $I_0 \subset I$ consisting of isolated points; $\tilde{\lambda}_2(\mu) - \tilde{\lambda}_1(\mu)$ has only isolated zero points on I. Furthermore, $\det(\tilde{b}_3(\mu), \tilde{A}(\mu)\, \tilde{b}_3(\mu)) = 0$ only on a subset $I_1 \subset I$ consisting of isolated points such that $I_0 \cap I_1 = \emptyset$, where $\tilde{b}_3 = \operatorname{transp}(\tilde{b}_{13}, \tilde{b}_{23})$, $\tilde{A}(\mu) = \operatorname{diag}(\tilde{\lambda}_1(\mu), \tilde{\lambda}_2(\mu))$. Then $\operatorname{rank}(\tilde{B}(\mu), \tilde{A}(\mu)\, \tilde{B}(\mu)) = 2$ for all $\mu \in [-1, 1]$, which means that the system $(\tilde{A}(\mu), \tilde{B}(\mu))$ is completely controllable for all $\mu \in$ $\in [-1, 1]$.

Since in the following considerations we shall use some results from the theory of transversal mappings, let us make some preliminary remarks.

Let X and Y be C^r manifolds $(r \geqq 1)$ and let $W \subset Y$ be a C^r submanifold of Y. We say that a mapping $f : X \to Y$ *intersects* W *transversally at a point* $x \in X$ $(f \mathrel{\bar{\cap}}_x W)$ if either

(a) $f(x) \notin W$ or

(b) $f(x) \in W$ and

$T_{f(x)}Y = T_{f(x)}W \oplus \mathrm{d}f(x)\,(T_xX)$, where $T_{f(x)}Y$, $T_{f(x)}W$, T_xX are the tangent spaces to Y, W and X, respectively, at $f(x)$ and x, respectively, and $\mathrm{d}f(x)$ is the differential of f at x. We say that f intersects W transversally on a subset A of X if $f \mathrel{\bar{\cap}}_x W$ for all $x \in A$, and we write in this case $f \mathrel{\bar{\cap}} W$ on A.

For instace, if $X = R^1$, $W = R^1 \times \{0\}$, $Y = R^2$ and $f : R^1 \to R^2$, $f(x) = (x, x^2)$, then $f \mathrel{\bar{\cap}} W$ on $R^1 \setminus \{0\}$, because $f(x) \notin W$ for all $x \in R^1 \setminus \{0\}$, but f does not intersect W transversally at 0. Indeed, $T_0X = R^1$, $T_{f(x)}W = R^1 \times \{0\}$, $T_{f(x)}Y = R^2$, $\mathrm{d}f(0) =$ $= (1, 0)$, $T_{f(0)}W \oplus \mathrm{d}f(0)\,(T_0X) = (R^1 \times \{0\}) \oplus (1, 0)\, R^1 \neq R^2$. Notice that f can be perturbed by a sufficiently small C^1 perturbation to a mapping which is transversal to W on R^1.

In our paper we shall use Abraham's transversality theorem.

Abraham's theorem (cf. [1, Theorem 19.1]). *Assume that*

(1) *X is a compact C^r manifold,* $\dim X = d$;

(2) *W is a closed submanifold of a C^r manifold Y,* $\operatorname{codim} W = q < \infty$;

(3) *L is a C^r manifold and $\varrho : L \to C^r(X, Y)$ is a C^r representation, i.e. the mapping $ev_\varrho : L \times X \to Y$, $ev_\varrho(a, x) = \varrho(a)(x)$ for $a \in L$, $x \in X$ is a C^r map;*

(4) *L satisfies the second axiom of countability;*

(5) $r > \max(0, d - q)$;

(6) $ev_\varrho \mathrel{\bar{\cap}} W$.

Then the set $L_W = \{a \in L \mid \varrho(a) \mathrel{\bar{\cap}} W \text{ on } X\}$ is open and dense in L.

Now, let us turn our attention back to control systems. We shall use the necessary and sufficient condition for the complete controllability formulated by R. Triggiani [9], which is different from Kalman's rank condition. Let us formulate this condition as

Lemma 1. *Let* $\lambda_1, \lambda_2, \ldots, \lambda_s$ *be distinct eigenvalues of a normal matrix* A *with multiplies* $r_1, r_2, \ldots, r_s$, $\sum_{j=1}^{s} r_j = n$. *Denote by* $\{x_{jk}\}$ *the basis of the corresponding eigenvectors* $j = 1, 2, \ldots, s$; $k = 1, 2, \ldots, r_j$. *Let* $b_1, b_2, \ldots, b_m$ *be the column vectors of the matrix* B *and let*

(4)
$$B_j = \begin{bmatrix} (b_1, x_{j1}), \ldots, (b_m, x_{j1}) \\ (b_1, x_{j2}), \ldots, (b_m, x_{j2}) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ (b_1, x_{jr_j}), \ldots, (b_m, x_{jrj}) \end{bmatrix}, \quad j = 1, 2, \ldots, s,$$

where $(\cdot, \cdot)$ *denotes the scalar product. Then the linear control system* (1) *is completely controllable if and only if* rank $B_j = r_j$ *for* $j = 1, 2, \ldots, s$.

Let $S \subset M(n, n)$ be the set of all symmetric matrices. Denote by $S^r_{P,Q}(n, m)$ the set of all parametrized linear control systems $(A, B)_r \in F^r_{P,Q}(n, m)$ such that $A \in C^r(P, S)$. The set $S^r_{P,Q}(n, m)$ is a linear subspace of $F^r_{P,Q}(n, m)$.

A property $G(A, B)$ of the system $(A, B)_r \in S^r_{P,Q}(n, m)$ is called *generic in* $S^r_{P,Q}(n, m)$ if the set $\{(A, B)_r \in S^r_{P,Q}(n, m) \mid G(A, B)\}$ contains a residual set, i.e. a set which is a countable intersection of open dense subsets of $S^r_{P,Q}(n, m)$. Since the space $S^r_{P,Q}(n, m)$ is complete, every residual subset of $S^r_{P,Q}(n, m)$ is dense in $S^r_{P,Q}(n, m)$ by the Baire category theorem.

The implicit function theorem implies that if the matrix $A(\mu_0)$ has only simple eigenvalues $\lambda^0_1, \lambda^0_2, \ldots, \lambda^0_n$ then there is a neighbourhood $U(\mu_0)$ of μ_0 and unique C^r functions $\lambda_i : U(\mu_0) \to C$ such that $\lambda_i(\mu_0) = \lambda^0_i$, $i = 1, 2, \ldots, n$ and $\lambda_1(\mu), \lambda_2(\mu), \ldots$ $\ldots, \lambda_n(\mu)$ are eigenvalues of $A(\mu)$ for all $\mu \in U(\mu_0)$. This implies that if $U(\mu_0)$ is sufficiently small then there exist C^r functions $x_j(\mu)$, $j = 1, 2, \ldots, n$ such that for all $\mu \in U(\mu_0)$ $\{x_j(\mu)\}^n_{j=1}$ is the basis of eigenvectors corresponding to the eigenvalues $\{\lambda_j(\mu)\}^n_{j=1}$. Define the mappings $B_{j\mu_0} : U(\mu_0) \times Q \to M(1, m)$, $j = 1, 2, \ldots, n$,

(5)
$$B_{j\mu_0}(\mu, \nu) = [(b_1(\mu, \nu), x_j(\mu)), \ldots, (b_m(\mu, \nu), x_j(\mu))],$$

which are of the class C^r.

By Lemma 1 the system $(A(\mu), B(\mu, \nu))$ is completely controllable for $(\mu, \nu) \in U(\mu_0) \times Q$ if and only if

(6)
$$\operatorname{rank} B_{j\mu_0} = 1 \quad \text{for} \quad j = 1, 2, \ldots, n.$$

Let us formulate this assertion in terms of manifolds. We can identify $M(i, j)$ with R^{ij}, which is an ij-dimensional smooth manifold. Denote by $S_k(i, j)$ the set of all matrices in $M(i, j)$ of the rank $p - k$, where $p = \min(i, j)$. By [7] (see also [8]) $S_k(i, j)$ is a submanifold of $M(i, j)$ and

(7)
$$\operatorname{codim} S_k(i, j) = (i - p + k)(j - p + k).$$

If M_p denotes the set of all $i \times j$ matrices with rank p then this set is described by a finite number of algebraic equations, i.e. M_p is an algebraic manifold. By the above considerations $M_p = \bigcup_{k=1}^{p} S_k(i, j)$. Such a decomposition of an algebraic manifold into a finite sum of smooth or C^r manifolds is called a stratification.

Now, we can rewrite the condition (6) into the following one:

$$(8) \qquad B_{j\mu_0}(U(\mu_0) \times Q) \cap S_1(1, m) = \emptyset\,, \quad j = 1, 2, \ldots, n\,.$$

Unfortunately, all mappings $B_{j\mu_0}$, $j = 1, 2, \ldots, n$ are well defined only for such points $\mu_0 \in P$ for which the matrix $A(\mu_0)$ has only simple eigenvalues. This is the reason why we suppose A to be defined on P with $\dim P = 1$ and B on $P \times Q$. If $\dim P = 1$ then the set of all such points μ_0 for which the matrix $A(\mu_0)$ has eigenvalues of multiplicity $\geqq 2$ consists generically, i.e. for an open dense subset of $C^r(P, S)$, of isolated points (see [2]). Define the following mappings

$$(9) \qquad \varrho_{j\mu_0} : C^r(P \times Q, M(n, m)) \to C^r(U(\mu_0) \times Q, M(1, m))\,, \quad j = 1, 2, \ldots, n\,,$$
$$\varrho_{j\mu_0}(B) = B_{j\mu_0}\,, \quad \text{where } B_{j\mu_0} \text{ is defined by (5)}\,.$$

Lemma 2. *Let $r > \max(0, 1 + s - m)$, where $s = \dim Q$. Then the sets $H_{j\mu_0} = \{B \in C^r(P \times Q, M(n, m)) \mid \varrho_{j\mu_0}(B) \pitchfork S_1(1, m)\}$, $j = 1, 2, \ldots, n$ are open and dense in $C^r(P \times Q, M(n, m))$.*

We can prove more than Lemma 2 assertes. Define the mapping $\varrho_{\mu_0} : C^r(P \times Q, M(n, m)) \to C^r(U(\mu_0) \times Q, M(n, m))$ as follows:

$$\varrho_{\mu_0}(B) = \hat{B}_{\mu_0}\,, \quad \text{where} \quad \hat{B}_{\mu_0}(\mu, v) = \operatorname{transp}(B_{1\mu_0}(\mu, v), \ldots, B_{n\mu_0}(\mu, v))\,.$$

Then the following lemma holds.

Lemma 3. *Let $r > \max(0, 1 + s - q_k)$, where $s = \dim Q$, $q_k = (n - p + k) \,.\, (m - p + k)$, $k = 1, 2, \ldots, p$, $p = \min(n, m)$. Then the set $H_{\mu_0} = \{B \in C^r(P \times Q, M(n, m)) \mid \varrho_{\mu_0}(B) \pitchfork S_k(n, m),\ k = 1, 2, \ldots, p\}$ is open and dense.*

Proof. We shall use Abraham's transversality theorem. Let us put $\varrho = \varrho_{\mu_0}$, $L = C^r(P \times Q, M(n, m))$, $X = U(\mu_0) \times Q$, $Y = M(n, m)$, $W = W_k = S_k(n, m)$. It is clear that the assumptions (1)–(4) of Abraham's theorem are fulfilled. Since $d = \dim X = 1 + s$, $q_k = \operatorname{codim} W_k = (n - p + k)(m - p + k)$ and by the assumption $r > \max(0, 1 + s - q_k)$, we have $r > \max(0, d - q_k) = \max(0, 1 + s - q_k)$. Therefore the assumption (5) is also fulfilled. The main problem is to prove that the assumption (6) is fulfilled. Define the mapping $ev_\varrho : C^r(P \times Q, M(n, m)) \times U(\mu_0) \times Q \to M(n, m)$, $ev_\varrho(B, \mu, v) = \hat{B}_{\mu_0}(\mu, v)$, where $\hat{B}_{\mu_0}(\mu, v)$ is defined as above. We have to prove that $ev_\varrho \pitchfork S_k(n, m)$ for all $k = 1, 2, \ldots, p$. We shall prove that $ev_\varrho \pitchfork N$ for any submanifold $N \subset M(n, m)$. We have to prove that if $g = \hat{B}_{\mu_0}(\mu, v) \in N$ then

$$(10) \qquad T_g M(n, m) = T_g N \oplus (\mathrm{d}\, ev_\varrho)(B, \mu, v)\,(T_{(B,\mu,v)}(C^r(P \times Q, M(n, m)) \times U(\mu_0) \times Q))\,.$$

We can identify $T_g M(n, m)$, $T_g N$, $T_{(B,\mu,\nu)}(C^r(P \times Q, M(n, m)) \times U(\mu_0) \times Q))$ with $M(n, m)$, N and $C^r(P \times Q, M(n, m)) \times U(\mu_0) \times Q$, respectively. Then for $(\tilde{B}, \tilde{\mu}, \tilde{\nu}) \in$ $\in C^r(P \times Q, M(n, m)) \times U(\mu_0) \times Q$ we have

$$(\mathrm{d}\, ev_\varrho)(B, \mu, \nu)(\tilde{B}, \tilde{\mu}, \tilde{\nu}) = \mathrm{d}/\mathrm{d}s\{ev_\varrho(B + s\tilde{B}, \mu, \nu)\}_{s=0} +$$
$$+ \mathrm{d}\tilde{\varrho}(\mu, \nu)(\tilde{\mu}, \tilde{\nu}) = \mathrm{d}/\mathrm{d}s\{\hat{B}_{\mu_0}(\mu, \nu) + s\hat{\tilde{B}}_{\mu_0}(\mu, \nu)\}_{s=0} +$$
$$+ \mathrm{d}\tilde{\varrho}(\mu, \nu)(\tilde{\mu}, \tilde{\nu}) = \hat{\tilde{B}}_{\mu_0}(\mu, \nu) + \mathrm{d}\tilde{\varrho}(\mu, \nu)(\tilde{\mu}, \tilde{\nu}),$$

where $\tilde{\varrho} : U(\mu_0) \times Q \to M(n, m)$, $\tilde{\varrho}(\mu, \nu) = \hat{B}_{\mu_0}(\mu, \nu)$. To prove (10) it suffices to prove that for every $C \in M(n, m)$ there is a $\tilde{B} \in C^r(P \times Q, M(n, m))$ such that

$$\hat{\tilde{B}}_{\mu_0}(\mu, \nu) = C\,. \tag{11}$$

We remark that $\hat{\tilde{B}}_{\mu_0}(\mu, \nu) = \operatorname{transp}(\tilde{B}_{1\mu_0}(\mu, \nu), \ldots, \tilde{B}_{n\mu_0}(\mu, \nu))$, $\tilde{B}_{j\mu_0}(\mu, \nu) = [(\tilde{b}_1(\mu, \nu), x_j(\mu)), \ldots, (b_m(\mu, \nu), x_j(\mu))]$, $j = 1, 2, \ldots, n$. Let $C = (c_{ij})$, $x_j(\mu) = (x_{1j}(\mu), \ldots, x_{nj}(\mu))$, $\tilde{b}_i(\mu, \nu) = (\tilde{b}_{1i}(\mu, \nu), \ldots, \tilde{b}_{ni}(\mu, \nu))$, $i, j = 1, 2, \ldots, n$. Then we can rewrite (11) into n systems of linear algebraic equations:

$$X(\mu)\,\tilde{B}_{j\mu_0}(\mu, \nu) = c_j\,, \quad j = 1, 2, \ldots, n\,, \tag{12}$$

where $X(\mu) = (x_{ij}(\mu))$ and c_j, $j = 1, 2, \ldots, n$ denotes the j-th column of the matrix C. Since the vectors $x_j(\mu)$, $j = 1, 2, \ldots, n$ are linearly independent for all $\mu \in U(\mu_0)$, the equations (12) have unique solutions $B_{j\mu_0}(\mu, \nu)$, $j = 1, 2, \ldots, n$. Now, it suffices to take the mapping $\tilde{B} \in C^r(P \times Q, M(n, m))$ such that $\tilde{B}(\mu, \nu) =$ $= \operatorname{transp}(\tilde{B}_{1\mu_0}(\mu, \nu), \ldots, \tilde{B}_{n\mu_0}(\mu, \nu))$. We have proved that also the assumption (6) of Abraham's theorem is fulfilled and so by this theorem the sets $H^k_{\mu_0} = \{B \in C^r(P \times$ $\times\, Q, M(n, m)) \mid \varrho_{\mu_0}(B) \pitchfork S_k(n, m)\}$, $k = 1, 2, \ldots, p$ are open and dense and therefore the set $H_{\mu_0} = \bigcap_{k=1}^{n} H^k_{\mu_0}$ is open and dense.

Now, taking $k = 1$, $n = 1$ in Lemma 3, we easily get the assertion of Lemma 2.

Corollary of Lemma 2. *If $m > 1 + s$ and $B \in H^1_{\mu_0} = \{B \in C^r(P \times Q, M(n, m)) \mid$ $\mid \varrho_{j\mu_0}(B) \pitchfork S_1(n, m)\}$, then $[\varrho_{j\mu_0}(B)]^{-1}(S_1(1, m)) = \emptyset$ for all $j = 1, 2, \ldots, n$ and thus for $(\mu, \nu) \in U(\mu_0) \times Q$ the system $(A(\mu), B(\mu, \nu))$ is completely controllable if $A(\mu_0)$ has only simple eigenvalues.*

Proof. Since $\varrho_{j\mu_0}(B) \pitchfork S_1(1, m)$, we have by [1, Corollary 17.2] that $\operatorname{codim}[\varrho_{j\mu_0}]^{-1}(S_1(1, m)) = \operatorname{codim} S_1(1, m) = m$. Therefore $\dim[\varrho_j\mu_0]^{-1}$ $(S_1(1, m)) = \dim X - m = 1 + s - m < 0$ and so $[\varrho_{j\mu_0}]^{-1}(S_1(1, m)) = \emptyset$ for $j = 1, 2, \ldots, n$.

By the above corollary, the points $(\mu, \nu) \in P \times Q$ for which the system $(A(\mu),$ $B(\mu, \nu))$ is not completely controllable are generically only such points for which the matrix $A(\mu)$ has an eigenvalue of multiplicity $\geqq 2$. But as we have remarked above, if $\dim P = 1$, then the set of such points $\mu_0 \in P$ for which the matrix $A(\mu_0)$ has an eigenvalue of multiplicity $\geqq 2$ generically consists of isolated points and by [2] the following is generically valid: If $\mu_0 \in P$ and $A(\mu_0)$ has eigenvalues of multiplicity $\geqq 2$ then this matrix has one eigenvalue of multiplicity 2 while the others are simple.

Let G be the set of $(A, B)_r \in S^r_{P,Q}(n, m)$ such that the following conditions are satisfied:

(I) $A(\mu)$ has only simple eigenvalues for all $\mu \in P$ except for a finite set $I(A) = \{\mu_1, \mu_2, \ldots, \mu_{s(A)}\}$, where the matrices $A(\mu_k)$, $k = 1, 2, \ldots, s(A)$ have one eigenvalue of the multiplicity 2 while the others are only simple;

(II) for all $\mu_0 \in P \setminus I(A)$, $\varrho_j \mu_0(B) \bar{\cap}\, S_1(1, m)$ for all $j = 1, 2, \ldots, n$, where the mapping $\varrho_{j\mu_0}$ is defined by (9).

(III) Let

$$\varrho_{1\mu_k} : C^r(P \times Q, M(n, m)) \to C^r(Q, M(2, m)),$$

$$\varrho_{i\mu_k} : C^r(P \times Q, M(n, m)) \to C^r(Q, M(1, m)), \quad j = 3, 4, \ldots, n,$$

$$k = 1, 2, \ldots, s(A),$$

be defined as follows:

$$\varrho_{1\mu_k}(B)(v) = \begin{bmatrix} (b_1(\mu_k, v),\ x_1(\mu_k)), \ldots, (b_m(\mu_k, v),\ x_1(\mu_k)) \\ (b_1(\mu_k, v),\ x_2(\mu_k)), \ldots, (b_m(\mu_k, v),\ x_2(\mu_k)) \end{bmatrix},$$

$$\varrho_{i\mu_k}(B)(v) = [(b_1(\mu_k, v),\ x_i(\mu_k)), \ldots, (b_m(\mu_k, v),\ x_i(\mu_k))],$$

where $\{x_i(\mu_k)\}_{i=1}^n$ is a basis of eingenvectors. Then $\varrho_{1\mu_k}(B) \bar{\cap}\, S_1(2, m)$, $\varrho_{1\mu_k}(B) \bar{\cap}\, S_2(2, m)$ and $\varrho_{i\mu_k}(B) \bar{\cap}\, S_1(1, m)$ for $i = 3, 4, \ldots, n$; $k = 1, 2, \ldots, s(A)$.

Theorem. *Let* $\dim P = 1$ *and* $m > 1 + \dim Q$. *Then there exists an open dense subset* G *of* $S^r_{P,Q}(n, m)$ *such that if* $(A, B)_r \in G$ *then* $(A(\mu), B(\mu, v)) \in F_c(n, m)$ *for all* $(\mu, v) \in P \times Q$, *i.e. it is completely controllable.*

Proof. Let G be the set defined as above. The openness of the set of $(A, B)_r$ satisfying the condition (I) was discussed above and the openness of the set of $(A,B)_r$ satisfying the conditions (II) and (III) follows from the openness of the transversality property, since all mappings $\varrho_{j\mu_0}(B)$ and $\varrho_{i\mu_k}(B)$, $i, j = 1, 2, \ldots, n$; $k = 1, 2, \ldots, s(A)$ are defined on compact sets (see [1, Theorem 18.2]). Now, we shall prove the density of G. Let $(A, B)_r \in S^r_{P,Q}(n, m)$. Then there exists $\tilde{A} \in C^r(P, S)$ sufficiently close to A which has the property (I). Fix such an $\tilde{A}$ and suppose that $I(\tilde{A}) = \{\mu_1, \mu_2, \ldots, \mu_s(\tilde{A})$. Let V_{kl} be the set of all such points $(\mu, v) \in P \times Q$ with distance from the set $Q_k = \{\mu_k\} \times Q$ less than $q_l = 1/l$ where l is an integer. Denote by W_{kl} the complement of the set V_{kl} in $P \times Q$. Let $W_{kl}(P)$ be the set of all $\mu \in P$ for which there exists a $v \in Q$ such that $(\mu, v) \in W_{kl}$. The sets W_{kl} and $W_{kl}(P)$ are compact and therefore there exist a finite number of points $\varepsilon_i = \varepsilon_i(k, l) \in W_{kl}(P)$, $i = 1, 2, \ldots, n_{kl}$ and their neighbourhoods $U(\varepsilon_i)$ which cover the set $W_{kl}(P)$ and such that the following mappings are well defined:

$$\varrho_{j\varepsilon_i} : C^r(P \times Q, M(n, m)) \to C^r(U(\varepsilon_i) \times Q, M(1, m)), \quad \varrho_{j\varepsilon_i}(B) = B_{j\varepsilon_i},$$

where $B_{j\varepsilon_i}$ is defined similarly as $B_{j\mu_0}$ (see (5)). Similarly as Lemma 2 it is possible to prove that the sets

$$H_{ikl} = \{B \in C^r(P \times Q, M(n, m)) \mid \varrho_{j\varepsilon_i}(B) \mathrel{\bar{\pitchfork}} S_1(1, m), \quad j = 1, 2, \ldots, n\},$$
$$i = 1, 2, \ldots, n_{kl}; \quad k = 1, 2, \ldots, s(\tilde{A}),$$
$$H_k = \{B \in C^r(P \times Q, M(n, m)) \mid \varrho_{1\mu_k}(B) \mathrel{\bar{\pitchfork}} S_1(2, m), \varrho_{1\mu_k}(B) \mathrel{\bar{\pitchfork}} S_2(2, m)\},$$
$$k = 1, 2, \ldots, s(\tilde{A})$$

and

$$K_k = \{B \in C^r(P \times Q, M(n, m)) \mid \varrho_{j\mu_k}(B) \mathrel{\bar{\pitchfork}} S_1(1, m), \quad j = 3, 4, \ldots, n\},$$
$$k = 1, 2, \ldots, s(\tilde{A})$$

are open and dense in $C^r(P \times Q, M(n, m))$. Therefore the sets $H = \bigcap_{k=1}^{s(\tilde{A})} H_k$, $K = \bigcap_{k=1}^{s(\tilde{A})} K_k$ and $\tilde{G}_l = \bigcap_{k=1}^{s(\tilde{A})} \bigcap_{i=1}^{n_{kl}} H_{ikl}$ are open and dense in $C^r(P \times Q, M(n, m))$. Then $G = \bigcap_{l=1}^{\infty} G_l$, where $G_l = \tilde{G}_l \cap H \cap K$, and therefore the set G is residual and thus dense in $C^r(P \times Q, M(n, m))$.

Let $(A, B)_r \in G$. Then for all $\mu_0 \in P \setminus I(A)$, $\varrho_{j\mu_0}(B) \mathrel{\bar{\pitchfork}} S_1(1, m)$ and by Corollary of Lemma 2, if $m > 1 + \dim Q$ then $[\varrho_{j\mu_0}(B)]^{-1}(S_1(1, m)) = \emptyset$ for all $j = 1, 2, \ldots, n$ and thus for $(\mu, \nu) \in (P \setminus I(A)) \times Q$ the system $(A(\mu), B(\mu, \nu))$ is completely controllable. If $(\mu_k, \nu) \in I(A) \times Q$ then by the property (III) $\varrho_{i\mu_k}(B) \mathrel{\bar{\pitchfork}} S_1(1, m)$ for $i = 3, 4, \ldots, n$ and therefore by the same argument as in the proof of Corollary of Lemma 2 we have $[\varrho_{i\mu_k}(B)]^{-1}(S_1(1, m)) = \emptyset$. By the property (III), $\varrho_{1\mu_k}(B) \mathrel{\bar{\pitchfork}} S_1(2, m)$, $\varrho_{1\mu_k}(B) \mathrel{\bar{\pitchfork}} S_2(2, m)$. Therefore if $m \geqq 2$ then by (7) $\operatorname{codim} S_1(2, m) = m - 1$, $\operatorname{codim} S_2(2, m) = m$, i.e. $\operatorname{codim} [\varrho_{1\mu_k}(B)]^{-1}(S_1(2, m)) = m - 1$ and $\operatorname{codim} [\varrho_{1\mu_k}(B)]^{-1}(S_2(2, m)) = m$. Then $\dim [\varrho_{1\mu_k}(B)]^{-1}(S_1(2, m)) = \dim Q - m + 1 = s - m + 1 < 0$, $\dim [\varrho_{1\mu_k}(B)]^{-1}(S_2(2, m)) = s - m < 0$ and therefore $[\varrho_{1\mu_k}(B)]^{-1}(S_j(2, m)) = \emptyset$ for $j = 1, 2$. The proof of Theorem is complete.

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