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THE α-COMPLETION OF A LATTICE ORDERED GROUP

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The main result is the existence and uniqueness of the α-completion $G^{i\alpha}$ of an arbitrary l-group G. $G^{i\alpha}$ is obtained by applying the (iterated) Cauchy construction machinery of [1] to Papangelou's notion of α-convergence [7]. We prove α-convergence to be the coarsest convex Hausdorff order closed l-convergence structure on G; it follows that $G^{i\alpha}$ is complete with respect to any l-Cauchy structure inducing such a convergence. This sweeping Cauchy completeness implies, in turn, that $G^{i\alpha}$ is both laterally and Dedekind MacNeille complete.

Following Papangelou [7], we shall say that a filter $\mathscr{F}$ of subsets of G *α-converges to x*, written $\mathscr{F} \to x$, providing the following condition is met: $\wedge(F \vee g) = \vee(F \wedge g) = g$ for all $F \in \mathscr{F}$ if and only if $g = x$.

Lemma 1.1. *For any $F \subseteq G$ and any $x, y \in G$, if $\vee(F \wedge x) = x$ and $\vee(F \wedge y) = y$ then $\vee(F \wedge (x \vee y)) = x \vee y$, and dually.*

Proof. Let $X = F \wedge (x \vee y)$ and consider an arbitrary $t \in G$ such that $X \leqq t$. For any $f \in F, f \wedge x \leqq f \wedge (x \vee y) \leqq t$, hence $x = \vee(F \wedge x) \leqq t$. Likewise $y \leqq t$, which, together with the fact that $X \leqq x \vee y$, proves $\vee X = x \vee y$. $\square$

Lemma 1.2. *$\mathscr{F} \to x$ if and only if $\mathscr{F}$ satisfies the following conditions and its lattice dual. For every $x < g \in G$ there is some $x \leqq y \in G$ and $F \in \mathscr{F}$ with $F \wedge g \leqq y < g$.*

Proof. Suppose $\mathscr{F} \to x$ and $x < g \in G$. By definition there is some $F \in \mathscr{F}$ such that either $\wedge(F \vee g) \neq g$ or $\vee(F \wedge g) \neq g$. But $\wedge(F \vee x) = x$ implies $g = g \vee x = g \vee \wedge(F \vee x) = \wedge(F \vee g \vee x) = \wedge(F \vee g)$. Therefore $F \wedge g \leqq y < g$ for some y. Furthermore, $y \geqq f \wedge g \geqq f \wedge x$ for all $f \in F$ implies $x = \vee(F \wedge x) \leqq y$. Hence the condition and, by a similar argument, its dual both hold.

Now suppose $\mathscr{F}$ is a filter which satisfies the condition and its dual. For any $K \in \mathscr{F}$ it must be the case that $\wedge(K \vee x) = x$, for if $K \vee x \geqq g > x$ for some g then there is some $x \leqq y \in G$ and $F \in \mathscr{F}$ with $F \wedge g \leqq y < g$. But for any $z \in$

$\in F \cap K$, $z \vee x \geqq g$ and $z \wedge g \leqq y$, hence $g = (z \vee x) \wedge g = (z \wedge g) \vee (x \wedge \wedge g) \leqq y \vee x = y < g$, a contradiction. Similarly, $\vee(K \wedge x) = x$. Now consider $x \neq t \in G$. If $t \vee x > x$ then by the condition there is some $F \in \mathscr{F}$ such that $\vee(F \wedge (t \vee x)) \neq t \vee x$. Since $\vee(F \wedge x) = x$, Lemma 1.1 implies $\vee(F \wedge t) \neq t$. Likewise, $t \wedge x < x$ implies that $\wedge(F \vee t) \neq t$ for some $F \in \mathscr{F}$. This completes the proof that $\mathscr{F} \to x$. □

The preceding Lemma makes clear the following properties of α-convergence.

Lemma 1.3. *For any* $x, g \in G$,

(a) $\dot{x} \to x$;

(b) $\mathscr{K} \supseteq \mathscr{F} \to x$ *implies* $\mathscr{K} \to x$;

(c) $\mathscr{F} \to x$ *implies* $\mathscr{F}^{\sim} \to x$, $\mathscr{F} \vee \mathscr{F} \to x$, $\mathscr{F} \wedge \mathscr{F} \to x$, $\mathrm{ocl}(\mathscr{F}) \to x$, $\mathscr{F}^{-1} \to x^{-1}$, $g\mathscr{F} \to gx$, *and* $\mathscr{F}g \to xg$.

Lemma 1.4. $\mathscr{F} \to x$ *and* $\mathscr{K} \to x$ *imply* $\mathscr{F} \cap \mathscr{K} \to x$.

Proof. Consider $x < g \in G$ and choose $x \leqq y \in G$, $F \in \mathscr{F}$ such that $F \wedge g \leqq \leqq y < g$. Then find $x \leqq z \in G$, $K \in \mathscr{K}$ such that $K \wedge gy^{-1}x \leqq z < gy^{-1}x$. Then $(F \cup K) \wedge g \leqq zx^{-1}y < g$. This is so because $k \wedge g \leqq kx^{-1}y \wedge g \leqq zx^{-1}y$ for all $k \in K$, and because $f \wedge g \leqq y \leqq zx^{-1}y$ for all $f \in F$. A dual argument and Lemma 1.2 complete the proof. □

Lemma 1.5. $\mathscr{F} \to x$ *implies* $\cap\mathscr{F} \subseteq \{x\}$.

Proof. $y \in \cap F$ implies $\vee(F \wedge y) = \wedge(F \vee y) = y$ for all $F \in \mathscr{F}$, so $y = x$. □

Lemma 1.6. $\mathscr{F} \to 1$ *implies* $\mathscr{F}^2 \to 1$.

Proof. Consider $1 < g \in G$. Find $1 \leqq y \in G$ and $K \in \mathscr{F}$ such that $K \wedge g \leqq y < < g$, then find $1 \leqq z \in G$ and $F \in \mathscr{F}$ such that $F \wedge gy^{-1} \leqq z < gy^{-1}$ and $F \subseteq K$. We claim $FF \wedge g \leqq zy < g$. To establish this claim consider $f_1, f_2 \in F$ and arbitrary prime P, the objective being to prove $P(f_1f_2 \wedge g) \leqq Pzy$. If $Pg = Pzy$ then we are done, and if $Pf_1 \leqq P$ then $P(f_1f_2 \wedge g) \leqq P(f_2 \wedge g) \leqq Py \leqq Pzy$ since $f_2 \in K$. Therefore suppose $Pz < Pgy^{-1}$ and $Pf_1 > P$. From this and $F \wedge gy^{-1} \leqq z$ follows $Pf_1 \leqq Pz < Pgy^{-1} < Pf_1gy^{-1}$, hence $(f_1^{-1}Pf_1)\,y < (f_1^{-1}Pf_1)\,g$. Since $K \wedge g \leqq y < g$, we get $(f_1^{-1}Pf_1)f_2 \leqq (f_1^{-1}Pf_1)\,y$ or $Pf_1f_2 \leqq Pf_1y$. Then $Pf_1y \leqq \leqq Pzy$ since $Pf_1 \leqq Pz$, yielding $P(f_1f_2 \wedge g) \leqq Pf_1f_2 \leqq Pzy$. This proves the claim, and by a dual argument and Lemma 1.2, the proposition. □

Lemma 1.7. $\mathscr{K}\mathscr{K}^{-1} \to 1$ *and* $\mathscr{F} \to 1$ *imply* $\mathscr{K}\mathscr{F}\mathscr{K}^{-1} \to 1$.

Proof. Consider $1 < g \in G$. First find $L \in \mathscr{K}$ and $a \geqq 1$ with

$$LL^{-1} \wedge g \leqq a < g\,. \tag{1}$$

Then find $K \in \mathscr{K}$ and $b \geqq 1$ such that $K \subseteq L$ and

$$(2) \qquad KK^{-1} \wedge a^{-1}g \leqq b < a^{-1}g\,.$$

Fix $k \in K$, and choose $F \in \mathscr{F}$ and $y \geqq 1$ such that

$$(3) \qquad kFk^{-1} \wedge a^{-1}gb^{-1} \leqq y < a^{-1}gb^{-1}\,.$$

We claim that $KFK^{-1} \wedge g \leqq ayb < g$. To establish this claim consider $k_1, k_2 \in \in K$, $f \in F$ and an arbitrary prime P, the objective being to prove that $P(k_1fk_2^{-1} \wedge \wedge g) \leqq Payb$. If $Py = Payb$ we are done, so assume $Pg > Payb$. In this case it is necessary to marshall three facts. The first fact is that $Pk_1k^{-1} \leqq Pa$. This follows from (1) and the observation that $Pa < Pg$, since $Pa = Pg$ implies $Payb \geqq Pa = = Pg$, contrary to assumption. The second fact is that $Pakfk^{-1} \leqq Pay$. This follows from (3) since $(a^{-1}Pa)\,y < (a^{-1}Pa)\,a^{-1}gb^{-1}$. The third fact is that $Paykk_2^{-1} \leqq \leqq Payb$. To support this conclusion observe that $y \geqq 1$ implies $Pg \leqq Paya^{-1}g$, which, together with the assumption that $Payb < Pg$, implies by (2) that $(y^{-1}a^{-1}Pay)\,kk_2^{-1} \leqq (y^{-1}a^{-1}Pay)\,b$. It remains to combine these three facts as follows. The first two facts yield $Pk_1fk^{-1} = Pk_1k^{-1}kfk^{-1} \leqq Pakfk^{-1} = Pay$. Then the third fact gives $Pk_1fk_2^{-1} = Pk_1fk^{-1}kk_2^{-1} \leqq Paykk_2^{-1} \leqq Payb$, completing the proof of the claim. A dual argument completes the proof of the Lemma. □

The preceding lemmas, when applied to Theorem 1.14 and Corollary 2.20 of [1], prove the first theorem. In this theorem we use the more standard term "positive universal formula" for what is called a "disjunctive formula" in [1].

Theorem 1.8. *On any l-group G, α-convergence is an order closed convex Hausdorff strongly normal l-convergence structure. Therefore G^α is an l-group in which G is order dense. G and G^α satisfy the same positive universal formulas and so generate the same variety of l-groups.*

The purpose of the next several propositions is to show that α convergence has properties C_1, C_2, and C_3 of [1]. The following notation will be useful for that purpose. If $G \leqq H$, call an element $s \in H$ *small with respect to G* if there is a filter $\mathscr{F}$ such that $\mathscr{F} \to 1$ in G and yet $\vee(F \wedge s) = \wedge(F \vee s) = s$ for all $F \in \mathscr{F}$.

Lemma 1.9. *Suppose $G \leqq H$ and S is the set of elements of H small with respect to G. Then S is a convex l-subgroup of H such that $S \cap G = 1$.*

Proof. Clearly $1 \in S$, and $x \in S$ implies $x^{-1} \in S$. Suppose $1 \leqq x \leqq s \in S$ and let $\mathscr{F}$ be the filter on G corresponding to s. For $F \in \mathscr{F}$, $x = x \wedge s = x \wedge \vee(F \wedge \wedge s) = \vee(F \wedge s \wedge x) = \vee(F \wedge x)$. Therefore $x = \vee(\dot{K} \wedge x)$ for all $K \in \mathscr{F} \cap \dot{1}$. Since $x = \wedge(K \vee x)$ is clear for all $K \in \mathscr{F} \cap \dot{1}$ and since $\mathscr{F} \wedge \dot{1} \to 1$ in G, $x \in S$. Now suppose $1 \leqq s_i \in S$ with corresponding filter $\mathscr{F}_i$ on G, $i = 1, 2$. For $F_i \in \mathscr{F}_i$, $s_1s_2 = [\vee(F_1 \wedge s_1)]\,[\vee(F_2 \wedge s_2)] = \vee(F_1F_2 \wedge s_1F_2 \wedge F_1s_2 \wedge s_1s_2) \leqq \leqq \vee(F_1F_2 \wedge s_1s_2) \leqq s_1s_2$. Similarly, $\wedge(F_1F_2 \vee s_1s_2) = s_1s_2$, proving $s_1s_2 \in S$.

A standard argument now shows S to be a convex l-subgroup. That $G \cap S = 1$ is direct result of the definition of α-convergence.

Proposition 1.10. *If G is large in H then $\to$ on H reduces to $\to$ on G.*

Proof. Suppose $\mathscr{F} \to 1$ in H and that $G \in \mathscr{F}$. Because suprema and infima in G and H agree, $\mathscr{F} \to 1$ in G also. Now suppose $\mathscr{F}$ is a filter such that $\mathscr{F} \to 1$ in G. Because suprema and infima in G and H agree, $\wedge(F \vee 1) = \vee(F \wedge 1) = 1$ holds in H for all $F \in \mathscr{F}$. From Lemma 1.9 and the largeness of G in H it follows that $S = 1$, so that for each $1 \neq h \in H$ there is some $F \in \mathscr{F}$ such that either $\vee(F \wedge h) \neq h$ or $\wedge(F \vee h) \neq h$. That is, $\mathscr{F} \to 1$ in H. □

To say that $\to$ on G^α *meshes nicely* with $\to$ on G is to assert the following: for each $h \in G^\alpha$ and each filter $\mathscr{F}$ on G^α such that $G \in \mathscr{F}$, $\mathscr{F} \to h$ if and only if $h = [\mathscr{F}]$.

Proposition 1.11. *$\to$ on G^α meshes nicely with $\to$ on G.*

Proof. By Proposition 2.18 on [1] it is enough to show that $\mathscr{F} \to [\mathscr{F}]$ for each Cauchy filter $\mathscr{F}$ on G. Let $[\mathscr{F}] = h \in G^\alpha$; we must show $\mathscr{F}h^{-1} \to 1$ in G^α. To that end consider $1 < x \in G^\alpha$ and find $g \in G$ with $1 < g \leqq x$. Since $\mathscr{F}$ is Cauchy there is some $F \in \mathscr{F}$ and $1 \leqq y \in G$ such that $FF^{-1} \wedge g \leqq y < g$. Fix $f \in F$. Because $fF^{-1} \wedge g \in f\mathscr{F}^{-1} \wedge g$ and $[f\mathscr{F}^{-1} \wedge g] = fh^{-1} \wedge g$, Proposition 1.2 of [1] implies $fh^{-1} \wedge g \leqq y$. We claim $fh^{-1} \wedge x \leqq yg^{-1}x < x$. To establish this claim consider an arbitrary prime P of H. If $Py = Pg$ then $Pyg^{-1} = P$ so $P(fh^{-1} \wedge x) \leqq Px = Pyg^{-1}x$. If $Py < Pg$ then $P(fh^{-1} \wedge x) \leqq Pfh^{-1} \leqq Py \leqq Pyg^{-1}x$. This proves the claim and, since f was arbitrary, establishes $Fh^{-1} \wedge x \leqq yg^{-1}x < x$. Since $yg^{-1}x \geqq 1$, Lemma 1.2 together with a dual argument proves $\mathscr{F}h^{-1} \to 1$ in G^α. □

G^α enjoys the following important universal mapping property.

Theorem 1.12. *Every α-continuous l-homomorphism $\psi : G \to H$ has a unique α-continuous l-homomorphism $\psi^\wedge : G^\alpha \to H^\alpha$ extending ψ. In particular, every l-monomorphism ψ from G onto a large l-subgroup of H has a unique l-monomorphism $\psi^\wedge$ extending ψ.*

Proof. The first assertion is a straightforward application of Proposition 2.6 of [1]. Since Proposition 1.10 and 1.11 demonstrate that α-convergence has properties C1, C2, and C3, the second assertion can be deduced from Proposition 2.21 or [1].

Corollary 1.13. *If G is large in H then $G^\alpha \leqq H^\alpha$.*

Theorem 1.14. *G is large and α-dense in H if and only if H is l-isomorphic to an l-subgroup of G^α over G.*

Proof. Suppose G is large and α-dense in H. For each $h \in H$ there is some filter $\mathscr{F}$ on H such that $G \in \mathscr{F} \to h$. Since $\mathscr{F}\mathscr{F}^{-1}$, $\mathscr{F}^{-1}\mathscr{F} \to 1$, $\mathscr{F}$ can be considered a Cauchy filter on G. Define $\theta : H \to G^{\alpha}$ be declaring $h\theta = [\mathscr{F}]$. θ is well defined, since $G \in \mathscr{F} \to h$ and $G \in \mathscr{K} \to h$ imply $G \in \mathscr{F}\mathscr{K}^{-1} \to 1$ in H and, by Proposition 1.10, in G also, giving $[\mathscr{F}] = [\mathscr{K}]$. θ is clearly an l-homomorphism: $g\theta = g$ for any $g \in G$ since $G \in \mathscr{F} \to g$ in H implies $\mathscr{F} \to g$ in G. Because G is large in H and θ is one-one on G, it follows that θ is one-one on H. □

The last several results of this section show α-convergence to be the coarsest reasonable l-convergence structure.

Pioposition 1.15. *α-convergence is the coarsest convex Hausdorff order closed l-convergence structure on any l-group G.*

Proof. Suppose $\mathscr{F} \Rightarrow 1$, where $\Rightarrow$ is any convex Hasudorff order closed l-convergence structure, and let $\mathscr{K}$ be $\mathrm{ocl}((\mathscr{F} \cap \dot{1})^{\sim})$. Conside1 $1 < g \in G$. Since $\mathscr{K} \Rightarrow 1$ by assumption, there is some $F \in \mathscr{F}$ such that $g \notin \mathrm{ocl}((F \cup \{1\})^{\sim})$. It follows that $F \wedge g \leqq y < g$ for some $y \geqq 1$. By the dual argument and Lemma 1.2, $\mathscr{K} \to 1$. Then $\mathscr{F} \supseteq \mathscr{K}$ yields $\mathscr{F} \to 1$. □

If P is a prime subgroup then a P *interval* is any set of the form $\{g \in G \mid Pc < < Pg < Pd\}$, denoted (Pc, Pd). If Γ is a set of primes then $\mathscr{C}(\Gamma)$ denotes $\{Y \subseteq \subseteq G \mid Y \supseteq \cap A,\ A \subseteq \Gamma,\ A \text{ finite}\}$ and $\mathscr{B}(\Gamma)$ denotes $\{Y \subseteq G \mid Y \supseteq \cap\{(P_i a_i^{-1}, P_i a_i) \mid P_i \in \Gamma,\ a_i \in G^+ \setminus P_i,\ 1 \leqq i \leqq n\}\}$. If Γ is a normal set of primes then both $\mathscr{B}(\Gamma)$ and $\mathscr{C}(\Gamma)$ are neighbourhood filters of the identity for unique convex l-topologies on G [2].

Half of the next important result was first proven by Papangelou [7] in the abelian case. Ellis [5] proved the converse and extended both results to substantially wider classes of l-groups. In full generality, the result is due to Madell [5].

Theorem 1.16. *α-convergence is topological if and only if G is completely distributive. In this case $\mathscr{F} \to 1$ if and only if $\mathscr{F} \supseteq \mathscr{B}(\Gamma)$, when Γ is the set of order closed primes of G.*

Proposition 1.17. *α-convergence is the coarsest Hausdorff l-convergence structure on G if and only if G is completely distributive.*

Proof. Suppose $\mathscr{F} \Rightarrow 1$, where $\Rightarrow$ is a Hausdorff l-convergence structure on the completely distributive l-group G. By Corollary 1.7 of [1] we may assume $\Rightarrow$ convex, which implies $\mathscr{K} = ((\mathscr{F} \vee 1) \cap \dot{1})^{\sim} \Rightarrow 1$. Consider an order closed prime P and element $a \in G^+ \setminus P$. By Lemma 3.1 of [4] there is some $x \in G$ with $1 < x \leqq Pa \cap G^+$. Since $\cap\mathscr{K} = \{1\}$, there must exist $F_1 \in \mathscr{F}$ such that $x \notin ((F_1 \vee 1) \cup \{1\})^{\sim}$. It follows that $Pf < Pa$ for all $f \in F_1$, for if not then $1 < x \leqq (f \vee 1) \wedge a \in Pa \cap G^+$ would imply $x \in ((F_1 \vee 1) \cup \{1\})^{\sim}$.

Likewise there is $F_2 \in \mathscr{F}$ such that $Pf \geqq Pa^{-1}$ for all $f \in F_2$. This shows $F_1 \cap \cap F_2 \subseteq (Pa^{-1}, Pa) \in \mathscr{F}$, meaning $\mathscr{F} \supseteq \mathscr{B}(\Gamma)$, where Γ is the set of order closed primes of G. By the previous Theorem, $\mathscr{F} \to 1$.

Now suppose that α-convergence is the coarsest Hausdorff l-convergence structure on G, and let Δ be the set of all primes of G. $\mathscr{C}(\Delta)$ is the neighbourhood filter of 1 of a Hausdorff l-topology [2] whose convergence we may denote $\Rightarrow$. Then $\mathscr{C}(\Delta) \Rightarrow 1$ implies $\mathscr{C}(\Delta) \to 1$ and $\mathrm{ocl}(\mathscr{C}(\Delta)) \to 1$. Therefore $1 = \cap \mathrm{ocl}(\mathscr{C}(\Delta)) = \cap \Gamma$, the distributive radical of G. That is, G is completely distributive.

We close this section with a question. Are the completely distributive l-groups the only ones which admit a coarsest Hausdorff l-convergence structure?

2. THE α-COMPLETION

G is *α-complete* if $G^{\alpha} = G$. H is an *α-completion of* G if G is large in H, H is α-complete, and if $G \leqq K < H$ implies K is not α-complete. In this section we prove that every l-group G has an α-completion which is unique up to l-isomorphism over G. The α-completion of G can be obtained by iterating the construction of the previous section to obtain a chain of l-groups $G \leqq G^{\alpha} \leqq G^{\alpha\alpha} \leqq \ldots$, taking unions at limit stages. That the members of this chain eventually cease to grow larger is proven by showing that each is bounded in cardinality by $|2^G|$. The α-completion of G is denoted $G^{i\alpha}$, where the $i\alpha$ is meant to stand for "iterated α". This approach begs the fundamental open question of whether G^{α} is α-complete.

The following notion of extension provides the means to prove the cardinality bound on $G^{i\alpha}$. Define $G \leqq H$ to mean that for all $h_1 < h_2$ in H there exists $g_1 < g_2$ in G such that $(h_i \vee g_1) \wedge g_2 = g_i$, $i = 1, 2$. Though not relevant here, one can show that $G \leqq H$ if and only if H is an essential extension of G in the category of distributive lattices (that is, every lattice homomorphism on H which is one-one on G must be one-one on H). See also [3] for a related use of this concept.

Proposition 2.1. $G \leqq G^{\alpha}$.

Proof. Consider $h_1 < h_2$ in G^{α}; let $\mathscr{F}_1$ and $\mathscr{F}_2$ be filters on G such that $h_i = [\mathscr{F}_i]$. Since $\mathscr{F}_2\mathscr{F}_1^{-1} \to h_2h_1^{-1} > 1$, there exist sets $F_i \in \mathscr{F}_i$ with $\wedge(F_2F_1^{-1} \vee 1) \neq 1$, say $F_2F_1^{-1} \vee 1 \geqq a$ for some $1 < a \in G$. Because $\mathscr{F}_2\mathscr{F}_2^{-1} \to 1$, there is some $K \in \mathscr{F}_2$ such that $K \subseteq F_2$ and $KK^{-1} \wedge a \leqq b < a$ for some $b \geqq 1$. Fix $x \in K$. Observe that for $k \in K$, $xk^{-1} \wedge a \leqq b$ implies $kx^{-1} \vee a^{-1} \geqq b^{-1}$, meaning $K \vee a^{-1}x \geqq$ $\geqq b^{-1}x > a^{-1}x$. Secondly, note that for $f \in F_1$, $xf^{-1} \vee 1 \geqq a$ implies $xf^{-1} \vee b \geqq$ $\geqq a$ or $fx^{-1} \wedge b^{-1} \leqq a^{-1}$, meaning $F_1 \wedge b^{-1}x \leqq a^{-1}x < b^{-1}x$. If we let $g_1 =$ $= a^{-1}x$ and $g_2 = b^{-1}x$, we have $\dot{g}_i = (\mathscr{F}_i \vee g_1) \wedge g_2 \to (h_i \vee g_1) \wedge g_2$, or or $(h_i \vee g_1) \wedge g_2 = g_i$, $i = 1, 2$. □

Proposition 2.2. *Suppose $G \leqq H \leqq K$. Then $G \leqq H \leqq K$ if and only if $G \leqq K$.*

Proposition 2.3. *If $\mathscr{C}$ is a collection of l-groups totally ordered by $\leqq$ then $C \leqq \cup\mathscr{C}$ for any $C \in \mathscr{C}$.*

Proposition 2.4. $G \leqq H$ implies $|H| \leqq |2^G|$.

Proof. With each $h \in H$ associate the set of pairs (a, b) in the Cartesian product $G \times G$ such that $h \vee a \geqq b$. The definition of $\leqq$ assures that this association is one-one. □

Theorem 2.5. *Every l-group G has an α-completion $G^{i\alpha}$ which is unique up to α-isomorphism over G. G and $G^{i\alpha}$ satisfy the same positive universal formulas and hence generate the same variety of l-groups.*

Proof. Define $G_0 = G$, $G_{\beta+1} = (G_\beta)^\alpha$, and $G_\gamma = \cup\{G_\delta \mid \delta < \gamma\}$ for limit ordinals γ. By Propositions 2.1, 2.2, and 2.3, $G \leqq G_\beta$ for all ordinals β. By Proposition 2.4, there is an ordinal δ such that G_δ is α-complete. The Theorem then follows from Proposition 2.22 of [1]. □

Theorem 2.6. *H is l-isomorphic to $G^{i\alpha}$ over G if and only if G is a large l-subgroup of H, H is α-complete, and every l-monomorphism ψ from G onto a large l-subgroup of the α-complete l-group M can be uniquely extended to an l-monomorphism $\psi^\wedge : H \to M$.*

Proof. Proposition 2.23 of [1]. □

The coarseness of the α-convergence structure (Proposition 1.13) implies that G^α is the largest Cauchy completion that can be obtained from G by convex Hasudorff order closed l-Cauchy structures.

Proposition 2.7. *Let $\mathscr{D}$ be any l-Cauchy structure which induces a convex Hausdorff order closed l-convergence structure ⇒ on G. Then there is an l-isomorphism from $G^{\mathscr{D}}$ into G^α over G.*

Proof. By Proposition 1.15 the identity map from $(G, \Rightarrow)$ to $(G, \rightarrow)$ is continuous, hence Proposition 2.11 of [1] furnishes the required l-monomorphism. □

Corollary 2.8. *If G is α-complete then G is Cauchy complete with respect to any Hasudorff order closed l-Cauchy structure on G. In particular, G is order Cauchy and polar Cauchy complete.*

Corollary 2.9. *G^α contains a copy of the Dedekind MacNeill completion $G^\wedge$ of G. $G^{i\alpha}$ also contains a copy of the polar Cauchy completion G^{ip} of G, and hence of the lateral completion G^L of G. Therefore an α-complete l-group is both laterally and Dedekind MacNeill complete.*

Proof. In section 4 of [1] it is shown that $G^\wedge$ is the completion of G with respect to the order Cauchy structure, which by Proposition 2.7 is l-isomorphic to an l-subgroup of G^α over G. G^p and G^{ip} are the subjects of section 5 of [1]; a similar argument shows $G^p \leqq G^\alpha$ and $G^{ip} \leqq G^{i\alpha}$. That G^{ip} is laterally complete is Corollary 5.23 of [1]. □

Proposition 2.7 raises an interesting unsettled question. Suppose $\Rightarrow$ is a convex Hausdorff l-convergence which is both order closed and strongly normal on G. Suppose in addition that $G \leqq G^{\mathscr{D}}$, where $\mathscr{D}$ is the l-Cauchy structure generated from $\Rightarrow$ by declaring $\mathscr{F} \in \mathscr{D}$ whenever $\mathscr{F}\mathscr{F}^{-1}, \mathscr{F}^{-1}\mathscr{F} \Rightarrow 1$. Must $\Rightarrow$ be finer than α-convergence?

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