Stefan Węgrzynowski

Parallel and nonparallel s-structures on Euclidean spaces

Czechoslovak Mathematical Journal, Vol. 33 (1983), No. 1, 76–91,92–94

Persistent URL: http://dml.cz/dmlcz/101857

Czechoslovak Mathematical Journal, 33 (108) 1983, Praha

PARALLEL AND NON-PARALLEL s-STRUCTURES ON EUCLIDEAN SPACES

STEFAN WĘGRZYNOWSKI, Szczecin

(Received February 20, 1981)

INTRODUCTION

Let E^n be an Euclidean space. An isometry of E^n with an isolated fixed point $x \in E^n$ is called a *symmetry of E^n at x*. A family $\{s_x, x \in E^n\}$ of symmetries on E^n is called *an s-structure on E^n*.

Let E^n be an Euclidean space with orthogonal coordinate system. Then the symmetry at every point $x = (x^1, \ldots, x^n) \in E^n$ may be written as

$$s_x(y) = x + A(x)(y - x), \quad y \in E^n, \tag{1}$$

where $A(x)$ is an orthogonal $(n \times n)$-matrix, and $I - A(x)$ is a nonsingular matrix.

Following [1], an s-structure $\{s_x\}$ on E^n is said to be *regular* if it satisfies the rule

$$s_x \circ s_y = s_u \circ s_x, \quad u = s_x(y) \tag{2}$$

for every two points $x, y \in E^n$.
i.e.

$$A(x) . A(y) = A(x + A(x)(y - x)) . A(x). \tag{3}$$

It follows by the regularity condition of an s-structure, that $A(x)$ is an analytic function of x, [2].

An s-structure $\{s_x\}$ on E^n is said to be *parallel* if $A(x) = \text{const}$. From (3) it is easy to observe that each parallel s-structure on E^n is regular.

Let E^n be an Euclidean space and s_0 an orthogonal transformation at the origin without fixed vectors. For each $x \in E^n$, let t_x denote the translation such that $t_x(0) = x$. Then the family

$$s_x = t_x \circ s_0 \circ t_x^{-1}, \quad x \in E^n \tag{4}$$

is a parallel regular s-structure. It is obvious that these families are the only parallel s-structures on E^n.

An interesting question is whether there are any non-parallel regular s-structures on Euclidean spaces.

O. Kowalski has proved that the spaces E^2, E^3, E^4 admit only parallel regular s-structures, and he has found a class of non-parallel regular s-structures in E^5. Those results have been obtained only as a random by product of the very complicated classification of all generalized symmetric Riemannian spaces of dimension $n \leqq 5$ [4].

The purpose of this paper is to reprove the above results for E^3, E^4 and to give the complete classification of all non-parallel regular s-structures in E^5, using a different, direct method. The new method exploits some basic facts on generalized symmetric spaces but it is much simpler than the general method of classification given in [4].

Therefore, the first unknown case E_6 seems to be also accessible by the new method.

1. REGULAR s-STRUCTURES AND THE GROUP OF TRANSVECTIONS OF $(E^n, \{s_x\})$

Let $(E^n, \{s_x\})$ be a regular s-structure on E^n. Following [2], an automorphism of $(E^n, \{s_x\})$ onto itself is a diffeomorphism $\phi : E^n \to E^n$ such that

$$\phi(s_x(y)) = s_{\phi(x)}\, \phi(y) \tag{4}$$

for every $x, y \in E^n$.

Obviously, all symmetries s_x of $(E^n, \{s_x\})$ are automorphisms.

Let $I(E^n)$ denote the group of all isometries of E^n, $T(E^n)$ the group of translations, and $\operatorname{Aut}(E^n, \{s_x\})$ the group of all automorphisms of the s-structure $(E^n, \{s_x\})$.

Let K denote the group of transvections of $(E^n, \{s_x\})$ i.e. the group generated by automorphisms of the form $s_x \circ s_y^{-1}$, $x, y \in E^n$. The group K is a connected normal Lie subgroup of $\operatorname{Aut}(E^n, \{s_x\})$ acting transitively on E^n. Also, K is a subgroup of $I(E^n)$ (see [5]).

If K_0 denote an isotropy subgroup at $o \in E^n$, then $E^n \approx K/K_0$.

Let $\mathfrak{K}$ be the Lie algebra of K and $\mathfrak{K}_0$ the Lie algebra of K_0. It is known [6] that there exists a subspace $\mathfrak{M} \subset \mathfrak{K}$ such that the following reductive decomposition holds:

$$\mathfrak{K} = \mathfrak{K}_0 \oplus \mathfrak{M}\,, \tag{5}$$

$$[\mathfrak{K}_0, \mathfrak{M}] \subset \mathfrak{M}\,. \tag{6}$$

The Lie algebra $i(E^n)$ of the isometry group $I(E^n)$ has the basis $\{X_i, X_{ij}\}$, where X_i, X_{ij} denote such vectors that the corresponding fundamental vector fields on E^n are

$$X_i^* = \frac{\partial}{\partial x^i}\,, \quad X_{ij}^* = x^j \frac{\partial}{\partial x^i} - x^i \frac{\partial}{\partial x^j}\,. \tag{7}$$

In the sequel we shall identify X_i, X_{ij} with X_i^*, X_{ij}^*, respectively.

Lemma 1. *The vector space $\mathfrak{M}$ defined by (5) is generated by vectors $A_i = X_i + a_i^{kl} X_{kl}$, $i, k, l = 1, 2, \ldots, n$, $k < l$.*

Proof. Let $I^+(E^n)$ denote the identity component of $I(E^n)$. Then $E^n \approx I^+(E^n)/SO(n)$, where as usual, $SO(n)$ denotes the subgroup of rotations of E^n at the origin. Let $\pi : I^+(E^n) \to E^n$ be the projection map i.e.

$$\pi(g) = g(o), \quad g \in I^+(E^n).$$

The Lie algebra $so(n)$ of $SO(n)$ is generated by vectors X_{ij}, $i < j$, $i, j = 1, 2, \ldots, n$.

Let $e \in I^+(E^n)$ denote the identity element and $\pi_{*e}: T_e(I^+(E^n)) \to T_0(E^n)$ the tangent map of the projection π at e. By the conditions (7) we have

$$\pi_{*e}(X_i) = (X_i^*)_0 = \left(\frac{\partial}{\partial x^i}\right)_0, \tag{8}$$

$$\pi_{*e}(X_{ij}) = (X_{ij}^*)_0 = 0.$$

Hence $\pi_{*e}(so(n)) = (0)$.

The inclusion $K \subset I^+(E^n)$ implies $\mathfrak{K} \subset i(E^n)$ and $\mathfrak{K}_0 \subset so(n)$. Now K acts transitively on E^n i.e.

$$\forall x \in E^n \ \exists g \in K, \quad g(0) = x.$$

This means $\pi : K \to E^n$ is onto and therefore $\pi_{*e} : \mathfrak{K} \to T_0(E^n)$ maps also $\mathfrak{K}$ onto $T_0(E^n)$. But the equality $\pi_{*e}(\mathfrak{K}) = (0)$ implies $\pi_{*e}(\mathfrak{M}) = T_0(E^n)$. In other words, $\pi_{*e}|\mathfrak{M}$ is a vector space isomorphism. Now $T_0(E^n) = ((\partial/\partial x^i)_0)$, so for every $i = 1, 2, \ldots, n$ there exists $\tilde{X}_i \in \mathfrak{M}$ such that $\pi_{*e}(\tilde{X}_i) = (\partial/\partial x^i)_0$. Hence $\tilde{X}_i = a_i^j X_j + b_i^{kl} X_{kl}$ because $\mathfrak{M}$ is a subspace of $i(E^n)$. Finally $\pi_{*e}(\tilde{X}_i) = a_i^j(\partial/\partial x^j)_0 = (\partial/\partial x^i)_0$ implies $a_i^j = \delta_i^j$ and this completes the proof.

Theorem 1. *If the group of translations on the Euclidean space E^n is a subgroup in the group of all automorphisms of the regular s-structure $(E^n, \{s_x\})$, then the s-structure is parallel.*

Proof. Let t_x denote a translation of E^n such that $t_x(0) = x$. If for each $x \in E^n$, $t_x \in \operatorname{Aut}(E^n, \{s_x\})$ then we have by (4):

$$t_x(s_0(y)) = s_x(t_x(y)), \quad y \in E^n. \tag{9}$$

Therefore $s_x = t_x \circ s_0 \circ t_x^{-1}$, which means that the s-structure $(E^n, \{s_x\})$ is parallel.

Theorem 2. *If a regular s-structure on the Euclidean space E^n is parallel, then the group K of transvections of $(E^n, \{s_x\})$ coincides with the translation group $T(E^n)$.*

Proof. Let s_x, $x \in E^n$ be a family of parallel symmetries i.e.

$$s_x = t_x \circ s_0 \circ t_x^{-1}, \quad x \in E^n,$$

where s_0 is an orthogonal transformation at the origin without fixed vectors, and let t_x denote the translation with $t_x(0) = x$. It is easy to show that K is the group generated by all automorphisms of the form $s_0 \circ s_z^{-1}$, $z \in E^n$. Namely we have

$$s_x \circ s_y^{-1} = s_x \circ s_0^{-1} \circ s_0 \circ s_y^{-1} = (s_0 \circ s_x^{-1})^{-1} \circ (s_0 \circ s_y^{-1}) .$$

Hence in case that the s-structure $(E^n, \{s_x\})$ is parallel we have:

$$s_0 \circ s_x^{-1} = s_0 \circ (t_x \circ s_0 \circ t_x^{-1})^{-1} = s_0 \circ (t_x \circ s_0^{-1} \circ t_x^{-1}) = (s_0 \circ t_x \circ s_0^{-1}) \circ t_x^{-1} .$$

It is known that the translation group is normal in the group of all isometries on E^n, thus $(s_0 \circ t_x \circ s_0^{-1})$ is a translation, too. Hence the group K of transvections is generated by translations, and since K acts transitively on E^n it must be the whole $T(E^n)$.

2. REGULAR s-STRUCTURES ON E^3 AND E^4

Let $(E^n, \{s_x\})$ be a regular s-structure on E^n. In order to show that this structure is parallel it is sufficient to prove that the translation group $T(E^n)$ is contained in the group of transvections K.

In other words, it is sufficient to show that the subspace $\mathfrak{M}$ defined in (5) is generated by the vectors X_i.

It follows from Lemma 1 that $\mathfrak{M}$ is generated by vectors of the form:

$$(10) \qquad A_i = X_i + a_i^{kl} X_{kl} , \quad i, k, l = 1, 2, \ldots, n , \quad k < l .$$

Further we show that for every regular s-structure $\{s_x\}$ on E^3 and E^4 we get in (10) $a_i^{kl} = 0$, i.e., that all regular s-structures on E^3 and E^4 are parallel.

Let us consider a symmetry s_0 at the origin $o \in E^n$. By means of the formula

$$(11) \qquad g \to s_0 \circ g \circ s_0^{-1} , \quad g \in I(E^n) ,$$

this symmetry defines an automorphism of the group $I(E^n)$ [2], [5], which induces an automorphism of the group K of transvections. This latter automorphism defines an automorphism σ of the Lie algebra $\mathfrak{K}$, with the following properties:

$$(12) \qquad \sigma(\mathfrak{M}) = \mathfrak{M} ,$$

$$(13) \qquad \mathfrak{K}_0 = \mathfrak{K}^\sigma \subset (so(n))^\sigma$$

where $\mathfrak{K}^\sigma$ denotes the subalgebra of the fixed points of σ on $\mathfrak{K}$.

We will also make use of the following properties of the Lie algebra of the transvection group K [2]:

$$(14) \qquad \mathfrak{K} = \mathfrak{M} + [\mathfrak{M}, \mathfrak{M}] ,$$

$$(15) \qquad \mathfrak{K}_0 = \text{proj } [\mathfrak{M}, \mathfrak{M}]/so(n) .$$

Here (15) means the projection of $[\mathfrak{M}, \mathfrak{M}]$ into $so(n)$ with respect to the decomposition $i(E^n) = t(E^n) \oplus so(n)$

Theorem 3. *The 3-dimensional Euclidean space E^3 admits only parallel regular s-structures.*

Proof. Let $\{s_x\}$ be a regular s-structure on E^3. Let us consider a symmetry s_0 at the origin $o \in E^3$. In some orthogonal coordinate system it can be written in the form:

$$x^{1'} = x^1 \cos\alpha - x^2 \sin\alpha\,, \quad x^{2'} = x^1 \sin\alpha + x^2 \cos\alpha\,, \quad x^{3'} = -x^3\,, \tag{16}$$

where $\alpha \in (0, 2\pi)$, $\alpha = \text{const}$. Now by (7), (11) and (16) we have:

$$\begin{aligned} \sigma(X_1) &= X_1 \cos\alpha - X_2 \sin\alpha\,,\\ \sigma(X_2) &= X_1 \sin\alpha + X_2 \cos\alpha\,,\\ \sigma(X_3) &= -X_3\,,\\ \sigma(X_{12}) &= X_{12}\,,\\ \sigma(X_{23}) &= -X_{23}\cos\alpha - X_{13}\sin\alpha\,,\\ \sigma(X_{13}) &= X_{23}\sin\alpha - X_{13}\cos\alpha\,. \end{aligned} \tag{17}$$

In the case of E^3, condition (10) implies that the subspace $\mathfrak{M}$ is generated by the vectors:

$$\begin{aligned} A_1 &= X_1 + a_1 X_{12} + a_2 X_{23} + a_3 X_{13}\,,\\ A_2 &= X_2 + b_1 X_{12} + b_2 X_{23} + b_3 X_{13}\,,\\ A_3 &= X_3 + c_1 X_{12} + c_2 X_{23} + c_3 X_{13}\,. \end{aligned} \tag{18}$$

where a_i, b_i, c_i are real numbers. Using the condition (12) we obtain:

$$a_i = b_i = c_i = 0 \quad \text{for} \quad \alpha \neq \tfrac{1}{2}\pi + k\pi\,,$$

and

$$a_1 = b_1 = 0\,, \quad b_2 = a_3\,, \quad b_3 = -a_2$$

$$c_1 = c_2 = c_3 = 0 \quad \text{for} \quad \alpha = \tfrac{1}{2}\pi + k\pi\,.$$

Hence in the case $\alpha = \frac{1}{2}\pi + k\pi$, the subspace $\mathfrak{M}$ has the following basis:

$$\begin{aligned} A_1 &= X_1 + a_2 X_{23} + a_3 X_{13}\,,\\ A_2 &= X_2 + a_3 X_{23} - a_2 X_{13}\,,\\ A_3 &= X_3\,. \end{aligned}$$

Since $\alpha \neq k\pi$, we get $(so(3))^\sigma = (X_{12})$, and clue to (13), $\mathfrak{K}_0 \subset (X_{12})$. Then we have the following two cases:

1) $\mathfrak{K}_0 = (0)$; then $\mathfrak{K} = \mathfrak{M}$ and consequently $[\mathfrak{M}, \mathfrak{M}] \subset \mathfrak{M}$. Hence $[A_1, A_2] = -(a_2^2 + a_3^2) X_{12} \in \mathfrak{M}$, so $a_2 = a_3 = 0$.

2) $\mathfrak{K}_0 \neq (0)$; then we have $\mathfrak{K}_0 = (X_{12})$ and by (6), $[X_{12}, A_1] \in \mathfrak{M}$. But the form of the bracket

$$[X_{12}, A_1] = X_2 - a_3 X_{23} + a_2 X_{13} \quad \text{implies} \quad [X_{12}, A_1] = A_2$$

and finally $a_2 = a_3 = 0$.

It follows that, in both cases,

$$\mathfrak{M} = (X_1, X_2, X_3)\,.$$

Hence we have proved that the translation group is contained in the group of transvections. It follows by theorem 1 that E^3 admits only parallel regular s-structures.

Theorem 4. *The Euclidean space E^4 admits only parallel regular s-structures.*

Proof. Let $\{s_x\}$ be a regular s-structure on E^4. Then there exists an orthogonal coordinate system in E^4 in which the symmetry s_0 can be written in the form

$$(19) \qquad s_0: \quad \begin{aligned} x^{1\prime} &= x^1 \cos\alpha - x^2 \sin\alpha\,, \\ x^{2\prime} &= x^1 \sin\alpha + x^2 \cos\alpha\,, \\ x^{3\prime} &= x^3 \cos\beta - x^4 \sin\beta\,, \\ x^{4\prime} &= x^3 \sin\beta + x^4 \cos\beta\,, \quad \alpha \neq 2k\pi\,, \quad \beta \neq 2k\pi\,. \end{aligned}$$

Let us introduce the complex coordinates in E^4:

$$(20) \qquad z = x^1 + \mathrm{i}x^2\,, \quad w = x^3 + \mathrm{i}x^4\,.$$

Then the symmetry (19) can be written in the form

$$(21) \qquad z' = z\mathrm{e}^{\mathrm{i}\alpha}\,, \quad w' = w\mathrm{e}^{\mathrm{i}\beta}\,.$$

By (20) we obtain

$$(22) \qquad \begin{aligned} \frac{\partial}{\partial z} &= \frac{1}{2}\left(\frac{\partial}{\partial x^1} - \mathrm{i}\frac{\partial}{\partial x^2}\right), \\ \frac{\partial}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial}{\partial x^1} + \mathrm{i}\frac{\partial}{\partial x^2}\right), \\ \frac{\partial}{\partial w} &= \frac{1}{2}\left(\frac{\partial}{\partial x^3} - \mathrm{i}\frac{\partial}{\partial x^4}\right), \\ \frac{\partial}{\partial \bar{w}} &= \frac{1}{2}\left(\frac{\partial}{\partial x^3} + \mathrm{i}\frac{\partial}{\partial x^4}\right). \end{aligned}$$

In the above notation, the basis of the algebra $i(E^4)^c$ is the following

$$Z = \frac{\partial}{\partial z}\,, \quad \bar{Z} = \frac{\partial}{\partial \bar{z}}\,, \quad W = \frac{\partial}{\partial w}\,, \quad \bar{W} = \frac{\partial}{\partial \bar{w}}\,,$$

$$A_1 = \bar{w}\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial w}, \quad \bar{A}_1 = w\frac{\partial}{\partial \bar{z}} - z\frac{\partial}{\partial \bar{w}},$$

$$A_2 = w\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{w}}, \quad \bar{A}_2 = \bar{w}\frac{\partial}{\partial \bar{z}} - z\frac{\partial}{\partial w},$$

$$A_3 = \mathrm{i}\left(\bar{z}\frac{\partial}{\partial \bar{z}} - z\frac{\partial}{\partial z}\right), \quad A_4 = \mathrm{i}\left(\bar{w}\frac{\partial}{\partial \bar{w}} - w\frac{\partial}{\partial w}\right).$$

The vectors $\{A_i, \bar{A}_i\}$ form a basis of the algebra $(so(4))^c$. Now the symmetry s_0 induces an automorphism σ of the algebra $\mathfrak{K}^c$. By (20)–(23) we then have:

(24)
$$\begin{aligned} \sigma(Z) &= \mathrm{e}^{-\mathrm{i}\alpha} Z, & \sigma(\bar{Z}) &= \mathrm{e}^{\mathrm{i}\alpha}\bar{Z}, \\ \sigma(W) &= \mathrm{e}^{-\mathrm{i}\beta} W, & \sigma(\bar{W}) &= \mathrm{e}^{\mathrm{i}\beta}\bar{W}, \\ \sigma(A_1) &= \mathrm{e}^{-\mathrm{i}(\alpha+\beta)} A_1, & \sigma(\bar{A}_1) &= \mathrm{e}^{\mathrm{i}(\alpha+\beta)}\bar{A}_1, \\ \sigma(A_2) &= \mathrm{e}^{-\mathrm{i}(\alpha-\beta)} A_2, & \sigma(\bar{A}_2) &= \mathrm{e}^{\mathrm{i}(\alpha-\beta)}\bar{A}_2, \\ \sigma(A_3) &= A_3, & \sigma(A_4) &= A_4. \end{aligned}$$

According to (10) the subspace $\mathfrak{M}^c$ is generated by the vectors:

(25)
$$\begin{aligned} A &= Z + \sum_{k=1}^{4} a_k A_k + \sum_{k=1}^{2} b_k \bar{A}_k, \\ B &= \bar{A}, \\ C &= W + \sum_{k=1}^{4} c_k A_k + \sum_{k=1}^{2} d_k \bar{A}_k, \\ D &= \bar{C}, \end{aligned}$$

where a_i, b_i, c_i, d_i are complex numbers. Making use of the condition $\sigma(\mathfrak{M}) = \mathfrak{M}$ we obtain

(26)
$$\begin{aligned} &a_1 = a_4 = a_3 = a_4 = 0, \\ &c_1 = c_3 = c_4 = 0, \\ &d_2 = 0 \end{aligned}$$

and also we obtain the following implications:

(27)
$$\begin{aligned} 2\alpha + \beta &\neq 2k\pi \Rightarrow b_1 = 0, \\ 2\alpha - \beta &\neq 2l\pi \Rightarrow b_2 = 0, \\ \alpha + 2\beta &\neq 2n\pi \Rightarrow d_1 = 0, \\ \alpha - 2\beta &\neq 2m\pi \Rightarrow c_2 = 0, \end{aligned}$$

where k, l, m, n are integers. If all of the conditions (27) are satisfied, then we have

$$\mathfrak{M}^c = (Z, \bar{Z}, W, \bar{W}) .$$

Let us restrict ourselves for a while to the conditions (26); then for the subspace $\mathfrak{M}^c$ we obtain the following generators:

$$(28) \qquad \begin{aligned} A &= Z + b_1\bar{A}_1 + b_2\bar{A}_2 , \\ B &= \bar{A} , \\ C &= W + c_2A_2 + d_1\bar{A} , \\ D &= \bar{C} . \end{aligned}$$

Now, we apply the conditions (13) and (15) in the complexified form i.e.

$$(29) \qquad \mathfrak{K}_0^c = [\text{proj}\,[\mathfrak{M}^c, \mathfrak{M}^c]/(so(4))^c] \subset (so^c(4))^\sigma .$$

By (23) and (28) we have:

$$(30) \qquad \begin{aligned} [A, B] &= -\mathrm{i}(b_1\bar{b}_1 + b_2\bar{b}_2)\,A_3 - \mathrm{i}(b_1\bar{b}_1 - b_2\bar{b}_2)\,A_4 , \\ [C, D] &= -\mathrm{i}(d_1\bar{d}_1 + c_2\bar{c}_2)\,A_3 - \mathrm{i}(d_1\bar{d}_1 - c_2\bar{c}_2)\,A_4 , \\ [A, C] &= -b_1B - d_1D + (b_1\bar{b}_1 + d_1\bar{d}_1)\,A_1 + b_1\bar{b}_2A_2 + d_1\bar{c}_2\bar{A}_2 - \\ &\quad - \mathrm{i}b_2c_2A_3 + \mathrm{i}b_2c_2A_4 , \\ [A, D] &= - b_2B - \bar{c}_2C + \bar{b}_1b_2A_1 + d_1\bar{c}_2A_1 + (b_2\bar{b}_2 + c_2\bar{c}_2)\,A_2 - \\ &\quad - \mathrm{i}b_1d_1A_3 - \mathrm{i}b_1\bar{d}_1A_4 , \\ [B, D] &= \overline{[A, C]} , \\ [B, C] &= \overline{[A, D]} . \end{aligned}$$

Now it is easy to show that at least two of the conditions (27) must be satisfied.

Hence the following cases may still occur:

1)
$$\begin{aligned} 2\alpha + \beta &= 2k\pi \quad \text{implies} & \alpha + 2\beta &\neq 2n\pi \Rightarrow d_1 = 0 , \\ 2\alpha - \beta &= 2l\pi & \alpha - 2\beta &\neq 2m\pi \Rightarrow c_2 = 0 , \\ & & \alpha + \beta &\neq 2s\pi \Rightarrow \sigma(A_1) \neq A_1 , \\ & & \alpha - \beta &\neq 2p\pi \Rightarrow \sigma(A_2) \neq A_2 . \end{aligned}$$

Then by (29) and (30) we obtain $b_1 = b_2 = 0$.

2)
$$\begin{aligned} 2\alpha + \beta &= 2k\pi \quad \text{implies} & 2\alpha - \beta &\neq 2l\pi \Rightarrow b_2 = 0 , \\ \alpha + 2\beta &= 2n\pi & \alpha - 2\beta &\neq 2m\pi \Rightarrow c_2 = 0 , \\ & & \alpha + \beta &\neq 2s\pi \Rightarrow \sigma(A_1) \neq A_1 . \end{aligned}$$

Then by (29) and (30) we obtain $b_1 = d_1 = 0$.

3) $$\begin{aligned} 2\alpha - \beta &= 2l\pi \quad \text{implies} & 2\alpha + \beta &\neq 2k\pi \Rightarrow b_1 = 0\,, \\ \alpha - 2\beta &= 2m\pi & \alpha + 2\beta &\neq 2n\pi \Rightarrow d_1 = 0\,, \\ & & \alpha - \beta &\neq 2p\pi \Rightarrow \sigma(A_2) \neq A_2\,. \end{aligned}$$

Then by (29) and (30) we obtain $b_2 = c_2 = 0$.

4) $$\begin{aligned} 2\alpha + \beta &= 2k\pi \quad \text{implies} & 2\alpha - \beta &\neq 2l\pi \Rightarrow b_2 = 0\,, \\ \alpha - 2\beta &= 2m\pi & \alpha + 2\beta &\neq 2n\pi \Rightarrow d_1 = 0\,, \\ & & \alpha + \beta &\neq 2s\pi \Rightarrow \sigma(A_1) \neq A_1\,, \\ & & \alpha - \beta &\neq 2p\pi \Rightarrow \sigma(A_2) \neq A_2\,. \end{aligned}$$

Then by (29) and (30) we obtain $b_1 = c_2 = 0$. The remaining cases

5) $$\alpha + 2\beta = 2n\pi\,,$$ $$\alpha - 2\beta = 2m\pi,$$

6) $$2\alpha - \beta = 2l\pi,$$ $$\alpha + 2\beta = 2n\pi,$$

are similar to 1) and 4), respectively.

The cases where only one equality holds can be treated analogously to the cases where two equalities hold. (Using conditions (29) and (30) we get again

$$b_1 = b_2 = c_2 = d_1 = 0\,.)$$

Consequently, we obtain in each case

$$\mathfrak{M}^c = (Z, \bar{Z}, W, \bar{W})\,.$$

Hence we have proved that the translation group is contained in the group of transvections.

By theorem 1 it follows that E^4 admits only parallel regular s-structures.

3. REGULAR s-STRUCTURES ON E^5

Theorem 5. *The Euclidean space E^5 admits non-parallel s-structures. Each non-parallel regular s-structure $\{s_x\}$ on E^5 can be described in the following way: there is a system of orthogonal coordinates $(x^1, \ldots, x^5)$ in E^5 such that, with respect to the complex coordinates $z = x^1 + ix^2$, $w = x^3 + ix^4$ and the real coordinates $t = x^5$, the transvection group K is given by*

$$K: \quad z' = z \,.\, e^{i\varrho t_0} + z_0\,, \quad w' = w \,.\, e^{-i\varrho t_0} + w_0\,, \quad t' = t + t_0$$

and each symmetry s_x, $x = (z_0, w_0, t_0) \in E^5$ *has the form*:

$$\begin{Vmatrix} z' \\ w' \\ t' \end{Vmatrix} = \begin{Vmatrix} z_0 \\ w_0 \\ t_0 \end{Vmatrix} + \begin{Vmatrix} 0 & e^{i(\alpha+2\varrho t_0)} & 0 \\ e^{-i(\alpha+2\varrho t_0)} & 0 & 0 \\ 0 & 0 & -1 \end{Vmatrix} . \begin{Vmatrix} z - z_0 \\ w - w_0 \\ t - t_0 \end{Vmatrix} .$$

Here $\varrho > 0$ *and* $\alpha \in (0, 2\pi)$, $\alpha \neq \pi$, *are real parameters.*

Proof. Put

$$E^5(x^1, x^2, x^3, x^4, x^5) = C^2(z, w) \times R^1(t), \qquad z = x^1 + ix^2, \quad w = x^3 + ix^4, \quad t = x^5, \tag{31}$$

and let $\{s_x\}$ be a regular s-structure on E^5. There exists an orthogonal coordinate system in E^5 in which the symmetry s_0 at the origin $o \in E^5$ has the form

$$z' = ze^{i\alpha}, \quad w' = we^{i\beta}, \quad t' = -t, \quad \alpha \neq 2k\pi, \quad \beta \neq 2k\pi. \tag{32}$$

In these notations, the algebra $i(E^5)^c$ has the following basis:

$$Z = \frac{\partial}{\partial z}, \qquad \bar{Z} = \frac{\partial}{\partial \bar{z}}, \tag{33}$$

$$W = \frac{\partial}{\partial w}, \qquad \bar{W} = \frac{\partial}{\partial \bar{w}},$$

$$T = \frac{\partial}{\partial t},$$

$$A_1 = \bar{w}\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial w}, \qquad \bar{A}_1 = w\frac{\partial}{\partial \bar{z}} - z\frac{\partial}{\partial \bar{w}},$$

$$A_2 = w\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{w}}, \qquad \bar{A}_2 = \bar{w}\frac{\partial}{\partial \bar{z}} - z\frac{\partial}{\partial w},$$

$$A_3 = 2t\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial t}, \qquad \bar{A}_3 = 2t\frac{\partial}{\partial \bar{z}} - z\frac{\partial}{\partial t},$$

$$A_4 = 2t\frac{\partial}{\partial w} - \bar{w}\frac{\partial}{\partial t}, \qquad \bar{A}_4 = 2t\frac{\partial}{\partial \bar{w}} - w\frac{\partial}{\partial t},$$

$$A_5 = i\left(\bar{z}\frac{\partial}{\partial \bar{z}} - z\frac{\partial}{\partial z}\right), \quad A_6 = i\left(\bar{w}\frac{\partial}{\partial \bar{w}} - w\frac{\partial}{\partial w}\right).$$

Again, the symmetry s_0 induces an automorphism σ of $\mathfrak{K}^c$, and from (31), (32) and (33) we have:

$$\sigma(Z) = e^{-i\alpha}Z, \qquad \sigma(W) = e^{-i\beta}W, \quad \sigma(T) = -T, \tag{34}$$

$$\sigma(A_1) = e^{-i(\alpha+\beta)}A_1\,, \quad \sigma(A_2) = e^{-i(\alpha-\beta)}A_2\,,$$
$$\sigma(A_3) = -e^{i\alpha}A_3\,, \quad \sigma(A_4) = -e^{-i\beta}A_4\,,$$
$$\sigma(A_5) = A_5\,, \quad \sigma(A_6) = A_6\,.$$

According to (10), the subspace $\mathfrak{M}^c$ is generated by:

(35)
$$A = Z + \sum_{k=1}^{6} p_k A_k + \sum_{k=1}^{4} q_k \bar{A}_k\,,$$
$$B = \bar{A}\,,$$
$$C = W + \sum_{k=1}^{6} r_k A_k + \sum_{k=1}^{4} s_k \bar{A}_k\,,$$
$$D = \bar{C}\,,$$
$$E = T + \sum_{k=1}^{6} a_k A_k + \sum_{k=1}^{4} \bar{a}_k \bar{A}_k\,,$$

where p_i, q_i, r_i, s_i, a_i are complex numbers. Using conditions $\sigma(\mathfrak{M}) = \mathfrak{M}$ again we obtain

(36)
$$p_1 = p_2 = p_3 = p_5 = p_6 = 0\,,$$
$$r_1 = r_4 = r_5 = r_6 = s_2 = 0\,,$$
$$a_3 = a_4 = 0\,,$$

and the following implications:

(37)
(1) $2\alpha + \beta \neq 2k\pi \Rightarrow q_1 = 0\,,$

(2) $2\alpha - \beta \neq 2l\pi \Rightarrow q_2 = 0\,,$

(3) $\alpha + 2\beta \neq 2n\pi \Rightarrow s_1 = 0\,,$

(4) $\alpha - 2\beta \neq 2m\pi \Rightarrow r_2 = 0\,,$

(5) $\alpha \neq \frac{1}{2}\pi + p\pi \Rightarrow q_3 = 0\,,$

(6) $\beta \neq \frac{1}{2}\pi + q\pi \Rightarrow s_4 = 0\,,$

(7) $\alpha + \beta \neq (2s + 1)\pi \Rightarrow q_4 = s_3 = a_1 = 0\,,$

(8) $\alpha - \beta \neq (2r + 1)\pi \Rightarrow p_4 = r_3 = a_2 = 0\,,$

where k, l, m, n, p, r, s are integers. If all of the conditions (37) are satisfied, then

$$\mathfrak{M}^c = (Z, \bar{Z}, W, \bar{W}, T)\,.$$

It is easy to see that one of the conditions (7) or (8) must be always satisfied. Hence it suffices to consider the following cases:

$$\text{I} \quad \alpha - \beta = (2r + 1)\pi, \quad \alpha + \beta \neq (2s + 1)\pi,$$

$$\text{II} \quad \alpha - \beta \neq (2r + 1)\pi, \quad \alpha + \beta = (2s + 1)\pi,$$

$$\text{III} \quad \alpha - \beta \neq (2r + 1)\pi, \quad \alpha + \beta \neq (2s + 1)\pi.$$

Ad I. Here $2\alpha - \beta \neq 2l\pi$ and $\alpha - 2\beta \neq 2m\pi$, and the following three possibilities occur:

$$1) \quad 2\alpha + \beta \neq 2k\pi \quad \text{and} \quad \alpha + 2\beta \neq 2n\pi,$$

$$2) \quad 2\alpha + \beta = 2k\pi \quad \text{and} \quad \alpha + 2\beta \neq 2n\pi,$$

$$3) \quad 2\alpha + \beta \neq 2k\pi \quad \text{and} \quad \alpha + 2\beta = 2n\pi.$$

Ad I, 1) In this case, according to (35)–(37), we obtain for the subspace $\mathfrak{M}^c$ the following basis:

$$(38) \qquad A = Z + pA_4 + q\bar{A}_3, \quad B = \bar{A},$$
$$C = W + rA_3 + s\bar{A}_4, \quad D = \bar{C},$$
$$E = T + aA_2 + \bar{a}\bar{A}_2.$$

p, q, r, s, a — are arbitrary complex numbers.

Now we make use of conditions (13) and (15) in the complex form again e.g.

$$(39) \qquad \mathfrak{K}_0^c = [\text{proj}\,[\mathfrak{M}^c, \mathfrak{M}^c]/(so(5))^c] \subset ((so(5))^c)^\sigma.$$

By (33) and (38) we have:

$$(40) \qquad [A, B] = -2p\bar{q}A_1 + 2\bar{p}q\bar{A}_1 - 2q\bar{q}\,iA_5 + 2p\bar{p}\,iA_6,$$
$$[C, D] = 2r\bar{s}A_1 - 2s\bar{r}\bar{A}_1 + 2r\bar{r}\,iA_5 - 2s\bar{s}\,iA_6,$$
$$[A, C] = -2prA_1 + 2qs\bar{A}_1 + 2qr\,iA_5 + 2ps\,iA_6,$$
$$[A, D] = (p - \bar{r})E - a(p - \bar{r})A_2 + [2(q\bar{s} - p\bar{r}) - \bar{a}(p - \bar{r})]\bar{A}_2,$$
$$[A, E] = -2qB - (\bar{a} + 2p)C + [2q\bar{q} + r(\bar{a} + 2p) + ap]A_3 +$$
$$+ [2q\bar{p} + s(\bar{a} + 2p) - aq]\bar{A}_4,$$
$$[C, E] = (a - 2r)A - 2sD + [-q(a - 2r) + 2\bar{r}s + \bar{a}s]\bar{A}_3 +$$
$$+ [-p(a - 2r) + 2s\bar{s} - \bar{a}r]A_4,$$
$$[B, D] = \overline{[A, C]}, \quad [B, C] = \overline{[A, D]}, \quad [B, E] = \overline{[A, E]}, \quad [D, E] = \overline{[C, E]}.$$

Obviously, in our case (I, 1)

$$\alpha \neq \pi, \quad \beta \neq \pi \quad \text{and} \quad \alpha - \beta \neq 2k\pi.$$

hence by (34)

$$\sigma(A_2) \neq A_2\,, \quad \sigma(A_3) \neq A_3\,, \quad \sigma(A_4) \neq A_4\,.$$

Moreover, two following possibilities occur:

(i) $$\alpha + \beta \neq 2k\pi \Rightarrow \sigma(A_1) \neq A_1\,,$$

(ii) $$\alpha + \beta = 2k\pi \Rightarrow \sigma(A_1) = A_1\,,$$

By (39) and (40) we get:

(41)
$$\begin{aligned}
&a(p - \bar{r}) = 0\,,\\
&2(q\bar{s} - p\bar{r}) - \bar{a}(p - \bar{r}) = 0\,,\\
&2q\bar{q} + r(\bar{a} + 2p) + ap = 0\,,\\
&2\bar{p}q + s(\bar{a} + 2p) - aq = 0\,,\\
&q(a - 2r) - 2\bar{r}s - \bar{a}s = 0\,,\\
&p(a - 2r) - 2s\bar{s} + \bar{a}r = 0\,, \quad \text{if} \quad \alpha + \beta = 2k\pi\,.
\end{aligned}$$

If $\alpha + \beta \neq 2k\pi$, then we have the following additional equalities:

(42) $$p\bar{q} = 0\,, \quad r\bar{s} = 0\,, \quad pr = 0\,, \quad qs = 0\,.$$

First equation in (41) gives

$$a = 0 \quad \text{or} \quad p = \bar{r}\,.$$

First of all, let us consider the possibility $a = 0$.

In the case (I, 1, i), the conditions (41) and (42) give rise to $q = 0$, $s = 0$ and $pr = 0$, i.e., either $p = 0$, or $r = 0$.

1° Let $p = 0$ then $\mathfrak{M}^c$ is generated by:

(43)
$$\begin{aligned}
&A = Z\,, \quad B = \bar{Z}\,,\\
&C = W + rA_3\,, \quad D = \bar{W} + \bar{r}\bar{A}_3\,,\\
&E = T\,.
\end{aligned}$$

Then by (40) we have:

$$\begin{aligned}
&[A, B] = 0\,, \qquad [C, D] = 2r\bar{r}\,\mathrm{i}A_5\,, \quad [A, C] = 0\,,\\
&[A, D] = -\bar{r}E\,, \quad [A, E] = 0\,, \qquad [C, E] = -2rA\,.
\end{aligned}$$

The conditions (6), (13) and (15) in the complex form give the inclusion:

(44) $$[\mathrm{proj}\,[\mathfrak{M}^c, \mathfrak{M}^c]/(so(n))^c, \mathfrak{M}^c] \subset \mathfrak{M}^c\,.$$

Applying (44) in our case (I, 1, i, 1°) we have:

$$[[C, D], C] = -2r^2\bar{r}A_3 \in \mathfrak{M}^c\,.$$

Hence $r = 0$, too, and $\mathfrak{M}^c = (Z, \bar{Z}, W, \bar{W}, T)$.

2° For $r = 0$, the subspace $\mathfrak{M}^c$ is generated by:

$$(45) \qquad A = Z + pA_4\,, \quad B = \bar{Z} + \bar{p}\bar{A}_4\,, \quad C = W\,, \quad D = \bar{W}\,, \quad E = T\,.$$

Then we get by (40):

$$[A, B] = 2p\bar{p}\, iA_6\,, \quad [C, D] = 0\,, \quad [A, C] = 0\,,$$
$$[A, D] = pE\,, \quad [A, E] = -2pC\,, \quad [C, E] = 0\,.$$

Condition (44) implies

$$[[A, B], A] = 2p^2\bar{p}A_4 \in \mathfrak{M}^c\,.$$

Hence $p = 0$, too, and $\mathfrak{M}^c = (Z, \bar{Z}, W, \bar{W}, T)$.

In the second case (I, 1, ii), conditions (41) give:

$$\operatorname{rank} \begin{Vmatrix} p & q & -s & -p \\ s & r & r & q \end{Vmatrix} = 1\,.$$

Suppose $p \neq 0$, then we have $q = -\bar{\varrho}p$, $r = \varrho\bar{\varrho}\bar{p}$, $s = \bar{\varrho}\bar{p}$ where ϱ – is complex number.

Then for the subspace $\mathfrak{M}^c$ we obtain the following basis:

$$A = Z + pA_4 - \bar{\varrho}p\bar{A}_3\,, \qquad B = \bar{A}\,,$$
$$C = W - \varrho\bar{\varrho}\bar{p}A_3 + \bar{\varrho}\bar{p}\bar{A}_4\,, \qquad D = \bar{C}\,,$$
$$E = T\,.$$

By (40) we obtain

$$[A, B] = 2p\bar{p}(\varrho A_1 - \bar{\varrho}\bar{A}_1 - \varrho\bar{\varrho}\, iA_5 + iA_6)\,,$$
$$[C, D] = -\varrho\bar{\varrho}[A, B]\,,$$
$$[A, C] = \bar{\varrho}[A, B]\,,$$
$$[A, D] = p(1 + \varrho\bar{\varrho})\, T\,,$$
$$[A, E] = 2p(\bar{\varrho}B - C)\,,$$
$$[C, E] = 2\varrho\bar{\varrho}(\varrho A - D)\,.$$

and using (44) we have:

$$[A, [A, B]] = 2p\bar{p}(-\varrho\bar{\varrho}A + \bar{\varrho}D + \bar{p}(1 - \varrho\bar{\varrho})\, A_4 - 2p\bar{\varrho}(1 + \varrho\bar{\varrho})\, \bar{A}_3 \in \mathfrak{M}^c$$

which finally gives $p = 0$, a contradiction. Hence we have by (41) $s = q = 0$, $r =$ an arbitrary complex number, and this possibility has been already discussed previously (cf. (43)).

It remains the second possibility $p = \bar{r}$. In this case (independently of conditions (42)), (41) implies

$$p = q = s = r = 0\,, \quad a = \text{an arbitrary complex number}\,.$$

Hence, in this case we get the following basis for the subspace $\mathfrak{M}^c$:

$$(46) \qquad A = Z\,, \quad B = \bar{Z}\,, \quad C = W\,, \quad D = \bar{W}\,,$$
$$E = T + aA_2 + \bar{a}\bar{A}_2\,.$$

By (40) we have:

$$[A, B] = 0\,, \quad [A, C] = 0\,, \quad [A, D] = 0\,, \quad [A, E] = -\bar{a}W\,,$$
$$[B, C] = 0\,, \quad [B, D] = 0\,, \quad [B, E] = a\bar{W}\,, \quad [C, D] = 0\,,$$
$$[C, E] = aZ\,, \quad [D, E] = \bar{a}\bar{Z}\,.$$

We have obtained a 5-dimensional Lie algebra which is essentially distined from the Lie algebra of the group of translations. This implies together with Theorem 2 that the Euclidean space E^5 admits non-parallel regular s-structure.

Ad I, 2) In this case the following additional conditions must be fulfilled:

$$\alpha \neq \tfrac{1}{2}\pi + p\pi\,, \quad \beta \neq \tfrac{1}{2}\pi + q\pi\,, \quad \alpha + \beta \neq 2k\pi\,,$$
$$\alpha \neq \pi\,, \qquad \beta \neq \pi\,.$$

According to (35)–(37) we then obtain for the subspace $\mathfrak{M}^c$ the following basis:

$$A = Z + pA_4 + q\bar{A}_1\,, \quad B = \bar{A}\,,$$
$$C = W + rA_3\,, \quad D = \bar{C}\,,$$
$$E = T + aA_2 + \bar{a}\bar{A}_2\,.$$

From the condition (39) (similarly to the case (I, 1)) we get $p = q = r = 0$, $a =$ an arbitrary complex number. Hence we obtain again the algebra (46).

Ad I, 3) It reduces to case (I, 2).

Ad II. Here $2\alpha + \beta \neq 2k\pi$ and $\alpha + 2\beta \neq 2l\pi$, and another three possibilities can appear

1) $2\alpha - \beta \neq 2l\pi$ and $\alpha - 2\beta \neq 2m\pi$,
2) $2\alpha - \beta = 2l\pi$ and $\alpha - 2\beta \neq 2m\pi$,
3) $2\alpha - \beta = 2l\pi$ and $\alpha - 2\beta = 2m\pi$.

Ad II, 1) In this case we get for $\mathfrak{M}^c$ the following basis:

$$(47) \qquad A = Z + p\bar{A}_3 + q\bar{A}_4\,, \quad B = \bar{A}\,,$$
$$C = W + r\bar{A}_3 + s\bar{A}_4\,, \quad D = \bar{C}\,,$$
$$E = T + aA_1 + \bar{a}\bar{A}_1\,,$$

p, q, r, s, a — are arbitrary complex numbers. By (33) we have:

$$(48)\qquad \begin{aligned}[A, B] &= -2\bar{p}qA_2 + 2p\bar{q}\bar{A}_2 - 2p\bar{p}\,\mathrm{i}A_5 - 2q\bar{q}\,\mathrm{i}A_6\,,\\ [C, D] &= -2\bar{r}sA_2 + 2r\bar{s}\bar{A}_2 - 2r\bar{r}\,\mathrm{i}A_5 - 2s\bar{s}\,\mathrm{i}A_6\,,\\ [A, D] &= -2\bar{r}qA_2 + 2p\bar{s}\bar{A}_2 - 2p\bar{r}\,\mathrm{i}A_5 - 2q\bar{s}\,\mathrm{i}A_6\,,\\ [A, C] &= (q - r)E - a(q - r)A_1 + [2(ps - qr) - \bar{a}(q - r)]\,A_1\,,\\ [A, E] &= -2pB - (\bar{a} + 2q)D + [2p\bar{p} + (\bar{a} + 2q)\bar{r} + aq]\,A_3 + \\ &\quad + [2p\bar{q} + (\bar{a} + 2q)\bar{s} - ap]\,A_4\,,\\ [C, E] &= (\bar{a} - 2r)B - 2sD + [-p(\bar{a} - 2r) + 2s\bar{r} + as]\,A_3 + \\ &\quad + [-q(\bar{a} - 2r) + 2s\bar{s} - ar]\,A_4\,.\end{aligned}$$

Similarly as in the case (I, 1), we obtain by (39), (44) and (48):

$$(49)\qquad \begin{aligned} &p = q = r = s = 0\,,\\ &a = \text{an arbitrary complex number}\,.\end{aligned}$$

Hence, the basis of $\mathfrak{M}^c$ has the following form:

$$(50)\qquad \begin{aligned} &A = Z\,,\quad B = \bar{Z}\,,\quad C = W\,,\quad D = \bar{W}\,,\\ &E = T + aA_1 + \bar{a}\bar{A}_1\,.\end{aligned}$$

It is easy to check that this is a 5-dimensional Lie algebra, which is isomorphic to algebra (46).

The remaining cases (II, 2) and (II, 3) are completely analogous to (I, 2) and (I, 3). Each of them implies conditions (49).

Ad III. According to (35) and (37) the subspace $\mathfrak{M}^c$ is generated by:

$$(51)\qquad \begin{aligned} &A = Z + q_1\bar{A}_1 + q_2\bar{A}_2 + q_3\bar{A}_3\,,\quad B = \bar{A}\,,\\ &C = W + r_2A_2 + s_1\bar{A}_1 + s_4\bar{A}_4\,,\quad D = \bar{C}\\ &E = T.\end{aligned}$$

Then we have by (33) and (51):

$$(52)\qquad \begin{aligned}[A, B] &= (q_2\bar{q}_3 - q_3\bar{q}_1)\,A_4 + (q_1\bar{q}_3 - q_3\bar{q}_2)\,A_4 - \\ &\quad - (q_1\bar{q}_1 + q_2\bar{q}_4 + 2q_3\bar{q}_3)\,\mathrm{i}A_5 + (q_1\bar{q}_1 + q_2\bar{q}_2)\,\mathrm{i}A_6\,,\\ [C, D] &= (s_4\bar{s}_1 - r_2\bar{s}_4)\,A_3 + (s_4\bar{r}_2 - s_1\bar{s}_4)\,\bar{A}_3 + (r_2\bar{r}_2 - s_1\bar{s}_1)\,\mathrm{i}A_5 - \\ &\quad - (r_2\bar{r}_2 + s_1\bar{s}_1 + 2s_4\bar{s}_4)\,\mathrm{i}A_6\,,\end{aligned}$$

$$
\begin{aligned}
[A, C] &= - q_1 B - s_1 D + (q_1\bar{q}_1 + s_1\bar{s}_1) A_1 + 2q_3 s_4 \bar{A}_1 + q_1\bar{q}_2 A_2 + \\
&+ s_1\bar{r}_2\bar{A}_2 + q_1\bar{q}_3 A_3 - q_2 s_4 \bar{A}_3 + s_1\bar{s}_4 A_4 - q_3 r_2 \bar{A}_4 - \\
&- q_2 r_2 \, \mathrm{i}A_5 + q_2 r_2 \, \mathrm{i}A_6 , \\
[A, D] &= - q_2 B - \bar{r}_2 C + q_2\bar{q}_1 A_1 + s_1\bar{r}_2\bar{A}_1 + (q_2\bar{q}_2 + r_2\bar{r}_2) A_2 + \\
&+ 2q_3\bar{s}_4\bar{A}_2 + q_2\bar{q}_3 A_3 - q_1\bar{s}_4\bar{A}_3 - q_3\bar{s}_1 A_4 + s_4\bar{r}_2\bar{A}_4 - \\
&- q_1\bar{s}_1 \, \mathrm{i}A_5 - q_1\bar{s}_1 \, \mathrm{i}A_6 , \\
[A, E] &= - 2q_3 B + 2q_3\bar{q}_1 A_1 + 2q_3\bar{q}_2 A_2 + 2q_3\bar{q}_3 A_3 , \\
[C, E] &= - 2s_4 D + 2s_4\bar{r}_2\bar{A}_2 + 2s_4\bar{s}_1 A_1 + 2s_4\bar{s}_4 A_4 .
\end{aligned}
$$

It is easy to check by means of conditions (34), (39) and (52) that each of the six possibilities (1)–(6) in (37) implies

$$q_1 = q_2 = q_3 = r_2 = s_1 = s_4 = 0 .$$

Hence in the case III we have:

$$\mathfrak{M}^c = (Z, \bar{Z}, W, \bar{W}, T) .$$

Now we shall determine a Lie group of transformations of E^5, the Lie algebra of which is isomorphic to (46).

For this purpose we find first the 1-parameter group of transformations corresponding to the vector field E. By solving the system of diffeıential equations

$$\frac{\mathrm{d}z}{\mathrm{d}s} = aw ,$$

$$\frac{\mathrm{d}w}{\mathrm{d}s} = -\bar{a}z ,$$

$$\frac{\mathrm{d}t}{\mathrm{d}s} = 1 ,$$

we get:

$$
\begin{aligned}
z &= z_0 \cos(\sqrt{|a|}) s + \frac{\bar{a}}{\sqrt{|a|}} w_0 \sin(\sqrt{|a|}) s , \\
w &= w_0 \cos(\sqrt{|a|}) s - \frac{a}{\sqrt{|a|}} z_0 \sin(\sqrt{|a|}) s , \\
t &= t_0 + s .
\end{aligned}
$$

Hence the 1-parameter group of transformations has the form:

$$z' = z \cos(\sqrt{|a|}) s + \frac{\bar{a}}{\sqrt{|a|}} w \sin(\sqrt{|a|}) s ,$$

$$w' = -\frac{a}{\sqrt{|a|}} z (\sin \sqrt{|a|}) s + w \cos (\sqrt{|a|}) s ,$$

$$t' = t + s .$$

Therefore our 5-dimensional Lie group of transformations of $E^5 \approx C^2 \times R^1$ has the form:

$$z' = z \cos (\sqrt{|a|}) t_0 + \frac{\bar{a}}{\sqrt{|a|}} w \sin (\sqrt{|a|}) t_0 + z_0 ,$$

$$w' = -\frac{a}{\sqrt{|a|}} z \sin (\sqrt{|a|}) t_0 + w \cos (\sqrt{|a|}) t_0 + w_0 ,$$

$$t' = t + t_0 .$$

This is isomorphic to the matrix group

$$\left\| \begin{matrix} \cos (\sqrt{|a|}) t_0 & \frac{\bar{a}}{\sqrt{|a|}} \sin (\sqrt{|a|}) t_0 & 0 & z_0 \\ -\frac{a}{\sqrt{|a|}} \sin (\sqrt{|a|}) t_0 & \cos (\sqrt{|a|}) t_0 & 0 & w_0 \\ 0 & 0 & 1 & t_0 \\ 0 & 0 & 0 & 1 \end{matrix} \right\|$$

Introducing new coordinates (admissible in the space C^2).

$$z_1 = \frac{z + \mathrm{i}\tilde{w}}{\sqrt{2}} , \quad w_1 = \frac{z - \mathrm{i}\tilde{w}}{\sqrt{2}} ,$$

where

$$\tilde{w} = \frac{\bar{a}}{\sqrt{|a|}} w$$

one can write our transformation group in the form

$$z_1' = z_1 e^{-\mathrm{i}\lambda t_0} + z_1^0 ,$$

$$w_1' = w_1 e^{\mathrm{i}\lambda t_0} + w_1^0 ,$$

$$t' = t + t_0 ,$$

where $\lambda = \sqrt{|a|}$.

The symmetry s_0 looks out as follows:

$$z_1' = w_1 e^{\mathrm{i}\alpha} ,$$

$$w_1' = z_1 e^{\mathrm{i}\alpha} ,$$

$$t' = -t .$$

The family $\{g \circ s_0 \circ g^{-1} : g \in K\}$ is the regular non-parallel s-structure on $E^5 \approx$ $\approx C^2 \times R$.

And can be expressed in the form:

$$\begin{Vmatrix} z' \\ w' \\ t' \end{Vmatrix} = \begin{Vmatrix} z_0 \\ w_0 \\ t_0 \end{Vmatrix} + \begin{Vmatrix} 0 & e^{-i(\alpha+2\lambda t_0)} & 0 \\ e^{i(\alpha+2\lambda t_0)} & 0 & 0 \\ 0 & 0 & -1 \end{Vmatrix} \cdot \begin{Vmatrix} z - z_0 \\ w - w_0 \\ t - t_0 \end{Vmatrix}.$$

References

[1] *Graham, P. J., Ledger, A. J.:* s-regular manifolds. Differential geometry — in honour of Kentaro Yano, pp. 133—144. Tokyo 1972.

[2] *O. Kowalski:* Generalized symmetric spaces, Lecture Notes in Mathematics, 805, Springer-Verlag Berlin—Heidelberg—New York 1980.

[3] *O. Kowalski:* Riemannian manifolds with general symmetries. Math. Zeitschrift *136*, pp. 135—150 (1974).

[4] *O. Kowalski:* Classification of generalized symmetric Riemannian spaces of dimension $n \leqq 5$, Rozpravy ČSAV, Řada MPV, No 8, *85* (1975).

[5] *O. Kowalski:* Smooth and affine s-manifolds. Period. Math. Hung. *80* (1977), 205—208.

[6] *O. Kowalski:* Affine reductive spaces. Beiträge zur Algebra und Geometrie *8* (1979), 99—105.

Author's address: Department of Mathematics, Technical University, 70-310 Szczecin, Al. Piastów 17, Poland.