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ON CLASSES OF GRAPHS DETERMINED BY FORBIDDEN SUBGRAPHS

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1. INTRODUCTION

One of the most frequent ways of defining a class of graphs is by means of forbidden subgraphs. (For a survey see [2]). Let $\mathscr{G}$ be a set of graphs. We say that $\mathscr{G}$ is determined by a set $\mathscr{H}$ of forbidden subgraphs if $\mathscr{G} = \{G = (V, E) \mid |V| = n$ and G does not contain any $H \in \mathscr{H}$ as an induced subgraph$\}$. We can measure the complexity of a class $\mathscr{G}$ by minimum number k with the property: $\mathscr{G}$ is determined by a set $\mathscr{H}$ of forbidden graphs with at most k vertices. It appears that, for n large, it is not possible to divide all graphs with n vertices into two classes of small complexity. We give a quantitative expression of this fact in § 4.

In § 3 we study the following related question. What is the minimum number $\varphi_n(k)$ of graphs with k vertices so that every graph with n vertices contains at least one of them as an induced subgraph? (A set of graphs with this property is called *n-universal*.) This problem generalizes in a way the Ramsey numbers as $\varphi_n(k) = 2$ if n is so large that any graph with n vertices contains either a clique or an independent set of cardinality k.

2. BASIC NOTIONS

Let G be a graph, we shall denote by $V(G)$ and $E(G)$ the vertex and edge set, respectively.

We say that H is an induced subgraph of G if $V(H)$ is a subset of $V(G)$ and $E(H)$ is equal to the set $E(G)$ restricted to $V(H)$ (i.e. $E(H) = E(G) \cap [V(H)]^2$). Note that all subgraphs considered in this paper are induced.

By the symbol Gra^n we denote the set of all graphs with n vertices without loops. We define $\mathrm{Gra} = \bigcup_{n=1}^{\infty} \mathrm{Gra}^n$. Let $\mathscr{H}$ be a system of graphs. We define $\mathrm{Forb}\,\mathscr{H}$ as the class of all graphs not containing a subgraph isomorphic to H for any $H \in \mathscr{H}$. Put $\mathrm{Forb}^n = \mathrm{Gra}^n \cap \mathrm{Forb}\,\mathscr{H}$. Let $\mathscr{G}$ be a given set of graphs. It is easy to see that $\mathscr{H}$ with

Forb $\mathscr{H} = \mathscr{G}$ need not exist. On the other hand if $\mathscr{G} \subset \mathrm{Gra}^n$ then obviously for $\mathscr{H} = \mathrm{Gra}^n - \mathscr{G}$ we have $\mathscr{G} = \mathrm{Forb}^n \mathscr{H}$: thus the following question arises. What is the minimal k such that $\mathscr{H} \subset \mathrm{Gra}^k$ and $\mathscr{G} = \mathrm{Forb}^n \mathscr{H}$? The set $\mathscr{G}$ has in some sense a "simple structure" if the k with the above property is small — in this case we can recognize for a given graph $G \in \mathrm{Gra}^n$ whether $G \in \mathscr{G}$ in short time. Let $G \in \mathrm{Forb}^n$. Then obviously every graph from $\mathscr{U} = \mathrm{Gra}^n - \mathscr{G}$ contains a subgraph isomorphic to some $H \in \mathscr{H}$. In this case we say that $\mathscr{H}$ is n-universal for $\mathscr{U}$. This fact we denote by $\mathscr{U} = \mathrm{Univ}^n \mathscr{H}$. If $\mathscr{U} = \mathrm{Gra}^n$ we say that $\mathscr{H}$ is n-universal.

We shall conclude this section with one definition which will be often used in our paper: Let G_1, G_2 be two graphs and H be an induced subgraph of both G_1 and G_2. We say that a graph F is an amalgamation of G_1 and G_2 if $|V(F)| = |V(G_1)| + |V(G_2)| - |V(H)|$ and F contains (as induced subgraphs) copies of G_1 and G_2 the intersection of which is isomorphic to H.

3. n-UNIVERSAL GRAPHS

Denote by $\varphi_n(k) = \min \{|\mathscr{H}|; \mathscr{H} \subset \mathrm{Gra}^k \text{ and } \mathscr{H} \text{ is } n\text{-universal}\}$. In this section we shall give some bounds for the behavior of the function $\varphi_n(k)$. The problem of determination of values of $\varphi_n(k)$ includes the problem of determination of Ramsey numbers as the following holds:

3.1. Proposition.

α) $n^{(k)} = 1$ *for* $k = 1$

β) $n^{(k)} = 2$ *for* $2 \leqq k \leqq r(n)$

γ) $n^{(k)} > 2$ *for* $k > r(n)$,

where $r(n)$ *is the maximal* k *such that every graph with* n *vertices contains either the complete graph with* k *vertices* K_k *or a discrete graph with* k *vertices* $\emptyset_k$ *as an induced subgraph.*

For the proof it is sufficient to realize that if $\mathscr{H}$ is n-universal then both K_k and $\emptyset_k$ are contained in $\mathscr{H}$.

The bounds for the number $r(n)$ are given by the following

3.2. Proposition. (See [1], § 12.)

$$\tfrac{1}{2} \log_2 n < r(n) < 2 \log_2 n .$$

Let us note that the slight improvements of the above bounds are known (see [5], [4]). As we are able to give rough bounds for the quantities studied in our paper only, the restrictions given by Proposition 3.2. are sufficiently exact for our purposes.

3.3. Theorem.

A) $\dfrac{2^{\binom{k}{2}}}{k!} \cdot \dfrac{1}{\binom{n}{k}} \leqq \varphi_n(k)$ *for every* n *and* $k \leqq n$.

Moreover, if $k \geqq r(n)$, *then*

B) $\varphi_n(k) < \dfrac{2^{2k}}{2n}$ *for* $\frac{1}{2}\log_2 n < k \leqq \log_2 n$,

C) $\varphi_n(k) < 2^{\binom{k}{2}} \binom{n}{2k}^{-k/2}$ *for* $\log_2 n < k < n/2$; $k \geqq 4$,

D) $\varphi_n(k) \leqq 2 \cdot 2^{\binom{k-1}{2}} \left(k - \left\lceil \dfrac{n-1}{2} \right\rceil\right)$ *for* $k \geqq n/2$,

where $\lceil x \rceil$ *denotes the upper integer part of the number* x.

Proof. First we prove the inequality A). Without loss of generality suppose that $\mathrm{Gra}^n = \{G; V(G) = \{1, 2, \ldots, n\}\}$. Let $\varphi_n(k) = p$; hence there exists $\mathscr{H} \subset \mathrm{Gra}^k$ such that $\mathscr{H} = \{H_1, H_2, \ldots, H_p\}$ is n-universal.

For an arbitrary $H \in \mathrm{Gra}^k$ we have

$$\left|\{G \in \mathrm{Gra}^n;\ H \text{ is isomorphic to a subgraph of } G\}\right| \leqq k! \cdot \binom{n}{k} \cdot 2^{\binom{n}{2} - \binom{k}{2}} .$$

Thus,

$$2^{\binom{n}{2}} = \left|\{G \in \mathrm{Gra}^n \mid \exists i : H_i \text{ isomorphic to a subgraph of } G\}\right| \leqq p \cdot k! \cdot \binom{n}{k} \frac{2^{\binom{n}{2}}}{2^{\binom{k}{2}}}$$

and hence $\varphi_n(k) \geqq 2^{\binom{k}{2}} / \left(k! \cdot \binom{n}{k}\right)$.

Before proving the inequalities B), C), D) choose in every $G \in \mathrm{Gra}^n$ a fixed sequence of vertices $x_1^G, x_2^G, \ldots, x_{t+1}^G$, where $t = [\log_2 n]$, and a sequence of independent sets $X = X_1^G \supset X_2^G \supset \ldots \supset X_{t+1}^G$ such that the following holds.

i) $x_i^G \in X_i^G - X_{i+1}^G$, $x_{t+1} \in X_{t+1}$, for every $i = 1, 2, \ldots, t$,
ii) $E_i^G \subset E(G)$ or $E_i^G \cap E(G) = \emptyset$ for $i = 1, 2, \ldots, t$ and $E_i^G = \{(x_i^G, y), y \in X_{i+1}^G\}$.

Now we prove the inequality B). Define the set of sequences $\mathscr{P} \subset \{0, 1\}^{k-1}$ by

$$p = (p_1, p_2, \ldots, p_{k-1}) \in \mathscr{P} \text{ iff either } p_i = 0 \text{ for every } i = 1, \ldots, t - k + 2$$
$$\text{or } p_i = 1 \text{ for every } i = 1, \ldots, t - k + 2 .$$

As $t = (2(t - k + 2) - 1) + ((k - 1) - (t - k + 2))$, for every $s = (s_1, s_2, \ldots, s_t) \in \{0, 1\}^t$ we can choose $i_1 < i_2 < \ldots < i_{k-1}$ such that $p = (s_{i_1}, s_{i_2}, \ldots, s_{i_{k-1}}) \in \mathscr{P}$.

For every sequence $p \in \mathscr{P}$ we define the graph H_p with the vertex set $\{v_1, v_2, \ldots, v_k\}$ such that for $i < j$

$$\{v_i, v_j\} \in E(H_p) \quad \text{iff} \quad p_i = 1\,.$$

Put $\mathscr{H} = \{H_p;\ p \in \mathscr{P}\}$. For a given graph $G \in \mathrm{Gra}^n$ we define a 0,1-sequence $s = (s_1, s_2, \ldots, s_t)$ by

$$s_i = \begin{cases} 1 & \text{for} \quad E_i^G \subset E(G) \\ 0 & \text{for} \quad E_i^G \cap E(G) = \emptyset\,. \end{cases}$$

Choose $p \in \mathscr{P}$ such that p is a subsequence of S. Clearly H_p is an induced subgraph of G. Hence

$$|\mathscr{H}| = 2\,.\,2^{(k-1)-(t-k+2)} = \frac{2^{2k}}{4\,.\,2^t} < \frac{2^{2k}}{n}\,.$$

C) Let t_0 be the largest positive integer such that $n \geqq k\,.\,2^{t_0}$. Define the set $\mathscr{H}$ as follows:

$$H = (V, E) \in \mathscr{H} \quad \text{iff} \quad V = \{v_1, v_2, \ldots, v_{t_0}, v_{t_0+1}, \ldots, v_k\} \quad \text{and}$$

$$\text{for every} \quad i = 1, \ldots, t_0 \quad \text{and} \quad E_i = \{\{v_i, v_j\};\ i < j \leqq k\}$$

$$\text{either} \quad E_i \cap E = \emptyset \quad \text{or} \quad E_i \subset E\,.$$

$\mathscr{H}$ is universal for Gra^n as every subgraph induced on vertices $x_1^G, x_2^G, \ldots, x_{t_0}^G, y_{t_0+1}, \ldots, y_k$ where $\{y_{t_0+1}, \ldots, y_k\} \subset X_{t_0}$ is isomorphic to some $H \in \mathscr{H}$.

Estimate the cardinality of

$$|\mathscr{H}| \leqq 2^{t_0}\, 2^{\binom{k-t_0}{2}} = \frac{2^{\binom{k}{2}}}{2^{t_0(k-(t_0+3)/2)}} < \frac{2^{\binom{k}{2}}}{\left(\dfrac{n}{2k}\right)^{k/2}} \quad \text{for} \quad k \geqq 4$$

as $t_0 + 3 \leqq \log_2 (8n/k)$ and for $k \geqq 4$ also $\log_2 (8n/k) \leqq \log_2 n + 1$.

D) Define $\mathscr{H}$ as follows:

$$H = (V, E) \in \mathscr{H} \quad \text{iff} \quad V = \{v_1, v_2, \ldots, v_k\} \quad \text{and there exists } d\,,$$

$$k - 1 \geqq d \geqq \lceil (n-1)/2 \rceil \quad \text{such that for} \quad E_1 = \{\{v_1, v_j\},\ 2 \leqq i \leqq d\}$$

$$\text{either} \quad E_1 \cap E = \emptyset \quad \text{or} \quad E_1 \subset E\,.$$

As in the previous case it is easy to verify that $\mathscr{H}$ is universal and

$$|\mathscr{H}| \leqq 2\,.\,2^{\binom{k-1}{2}}\,.\,(k - \lceil (n-1)/2 \rceil)\,.$$

A pair $\mathscr{G}_1, \mathscr{G}_2$ of nonempty sets of graphs is called a cut if $\mathscr{G}_1 \cup \mathscr{G}_1 = \mathrm{Gra}^n$ for some n, and moreover $\mathscr{G}_1 \cap \mathscr{G}_2 = \emptyset$. In this section we study the following question. Let k, l be such that there exist $\mathscr{H}_1 \subset \mathrm{Gra}^k$, $\mathscr{H}_2 \subset \mathrm{Gra}^l$ such that the sets $\mathscr{G}_1 = \mathrm{Forb}^n \mathscr{H}_1$, $\mathscr{G}_2 = \mathrm{Forb}^n \mathscr{H}_2$ form a cut. What is the relation among n, k and l? For $n \geqq 2$ obviously both $\mathscr{H}_1$ and $\mathscr{H}_2$ are nonempty and thus also $k \geqq 2$ and $l \geqq 2$. Choose an $H_1 \in \mathscr{H}_1$ and $H_2 \in \mathscr{H}_2$ and consider the disjoint sum $H_1 + H_2$. The cardinality of the vertex set of the graph $H_1 + H_2$ is at least $n + 1$. In the opposite case the graph $H_1 + H_2$ would be a subgraph of a graph F with n vertices and hence $F \notin \mathrm{Forb}^n \mathscr{H}_1 \cup \mathrm{Forb}^n \mathscr{H}_2$. Thus we have proved that $k + l > n$.

If we replace in the above argument the disjoint sum $H_1 + H_2$ by a graph which is an amalgamation of graphs H_1 and H_2 in a vertex (one-point amalgamation) we prove the following.

4.1. Proposition.

$$k + l > n + 1\,.$$

In this section we find some refinements of the above statement. More precisely, for given n, k $(k < n)$ we define $\psi(k, n)$ as the minimum l such that there exists a cut $\mathscr{G}_1, \mathscr{G}_2$ with the above properties. We give some estimation for the function $\psi(k, n)$.

4.2. Theorem. *Let* $n \geqq 2$, $k \geqq 2$. *Then*

A) $\psi(n - k, n) \leqq 2k + 2$;

B) $\psi(n - k, n) > k + \frac{1}{2}\log_2 \xi$, *where* $\xi = \min(k, n - k)$,
if

$$n \geqq k \frac{k + (\log_2 k)/2}{k - (\log_2 k)^2}\,. \tag{1}$$

Proof. First we prove the inequality A). Put

$$\mathscr{G}_1 = \mathrm{Forb}\,\{\emptyset_{n-k}\}\,, \quad \mathscr{G}_2 = \mathrm{Forb}\,\{H \in \mathrm{Gra}^{2k+2}\,\beta(H) \geqq k + 1\}\,,$$

where $\beta(H) = \min\{|A|;\, A \subset V(H)$ and $e \cap A \neq \emptyset$ for every $e \in E(H)\}$. We prove now that $\mathscr{G}_1 \cup \mathscr{G}_2 = \mathrm{Gra}^n$. Let $G \in \mathscr{G}_1$, i.e. G contains $\emptyset_{n-k}$ and hence $\beta(G) \leqq k$. Thus $G \in \mathscr{G}_2$.

The proof of $\mathscr{G}_1 \cap \mathscr{G}_2 = \emptyset$ will follow from the following

4.3. Lemma. *Let* $\beta(G) = p$. *Then there exists a subgraph* H *of* G *such that* $|V(H)| \leqq 2p$ *and* $\beta(H) = p$.

Proof of lemma. Put $G = (V, E)$. Let $A \subset V$, $|A| = p$ be such that each edge of G contains a vertex of A. Define a relation $R \subset A \times E$ by

$$(x, e) \in R \quad \text{iff} \quad x \in e\,.$$

The existence of a matching $F = \{(x_1, e_1), \ldots, (x_p, e_p)\} \subset R$ of the cardinality p follows from the König-Hall Theorem [3]. The graph H induced on the set

$$\bigcup_{i=1}^{p} \{v_i, x_i\}$$

where $e_i = \{v_i, x_i\}$ has the required properties.

Let now $G \in \mathcal{G}_2$, i.e. if H is a subgraph of G which has $2k + 2$ vertices then $\beta(H) \leqq k$. According to Lemma 4.3, $\beta(G) \leqq k$ and hence G contains $\emptyset_{n-k}$ as a subgraph.

We prove the inequality B). Let n and k be given. Consider a cut $\mathcal{G}_1, \mathcal{G}_2$ with the minimum l such that

$$\mathcal{G}_1 = \text{Forb}\,\mathcal{H}_1\,, \qquad \mathcal{H}_1 \subset \text{Gra}^{n-k}\,,$$

$$\mathcal{G}_2 = \text{Forb}\,\mathcal{H}_2\,, \qquad \mathcal{H}_2 \subset \text{Gra}^{l}\,.$$

Moreover, let k be such that (1) holds. We shall consider three cases.

α) Suppose $K_n \in \mathcal{G}_1$, $\emptyset_n \in \mathcal{G}_2$ (the case $\emptyset_n \in \mathcal{G}_1$, $K_n \in \mathcal{G}_2$ is analogous as all the properties considered here are invariant with respect to complement).

We prove that

$$\psi(n - k, n) > k + \tfrac{1}{2}\log_2 k\,. \tag{2}$$

Suppose that (2) does not hold, i.e.

$$l \leqq k + \tfrac{1}{2}\log_2 k\,. \tag{3}$$

From (1) and (3) we get that

$$k(n - k - 1) \geqq n(l - k - 1)^2 + k(l - k - 1)\,. \tag{4}$$

By Proposition 4.1 we hawe $l - k - 1 > 0$ and hence

$$\frac{n - k - 1}{l - k - 1} \geqq \frac{n(l - k - 1)}{k} + 1\,. \tag{5}$$

We show that we can choose positive integers a, b such that

$$a(l - k - 1) \leqq n - k - 1 \tag{6}$$

$$b(l - k - 1) < l - 1\,. \tag{7}$$

Now (5) implies the existence of a positive integer a such that

$$\frac{n - k - 1}{l - k - 1} \geqq a \geqq \frac{n(l - k - 1)}{k}\,, \tag{8}$$

which clearly implies the inequality (6). Put $b = \lceil n/c \rceil$, from (8) it follows that

$$b \leqq \left\lceil \frac{k}{l - k - 1} \right\rceil < \frac{k}{l - k - 1} + 1 = \frac{l - 1}{l - k - 1}\,.$$

Consider a partition of an n-point set $X = \bigcup_{i=1}^{b} X_i$ such that $|X_i| = a$ for every $i \leqq \leqq [n/a]$ and define a complete b-partite graph F with the vertex set X such that $x \in X_i$ and $x' \in X_j$ are joined by an edge if $i \neq j$. From (6) and (7) it follows that every $n - k$ and l-subset of $X = V(F)$ contains K_{l-k} and $\emptyset_{l-k}$, respectively.

If $F \in \mathscr{G}_1$ then $F \notin \text{Forb}\, \mathscr{H}_2$ and hence there exists a subgraph H of F such that $H \in \mathscr{H}_2$ and thus H does not contain $\emptyset_{l-k}$ as a subgraph. From the assumption $\emptyset_n \in \mathscr{G}_2 = \text{Forb}\, \mathscr{H}_2$ it follows that $\emptyset_n \notin \text{Forb}\, \mathscr{H}_1$ and hence $\emptyset_{n-k} \in \mathscr{H}_1$. The amalgamation of H and $\emptyset_{n-k}$ in $\emptyset_{l-k}$ is a graph which contains graphs from both $\mathscr{H}_1$ and $\mathscr{H}_2$ which contradicts $\text{Forb}\, \mathscr{H}_1 \cup \text{Forb}\, \mathscr{H}_2 = \text{Gra}^n$.

Analogously if $F \in \mathscr{G}_2$ then there exists an $H \in \mathscr{H}_1$ such that K_{l-k} is a subgraph of H. From $K_n \in \mathscr{G}_1$ it follows that $K_l \in \mathscr{H}_2$ and hence there exists a graph with n vertices containing both K_l and G as subgraphs.

2) Suppose $K_n, \emptyset_n \in \mathscr{G}_1$ and thus $K_l, \emptyset_l \in \mathscr{H}_2$. As $|V(H)| = n - k$ for $H \in \mathscr{H}_1$, H contains either $K_{r(n-k)}$ or $\emptyset_{r(n-k)}$. Suppose that $l \leqq k + r(n - k)$. Fix an $H \in \mathscr{H}_1$ and consider the amalgamation of H and either K_l or $\emptyset_l$ in $K_{r(n-k)}$ or $\emptyset_{r(n-k)}$, respectively. Thus we obtain a graph F with $n - k + l - r(n - k) \leqq n$ vertices, which contains either K_l or $\emptyset_l$ and hence $F \notin \mathscr{G}_2$. As H is a subgraph of F we also have $F \in \mathscr{G}_2$ — a contradiction. Thus we proved $l > k + r(n - k)$ and as $r(m) > > \frac{1}{2} \log_2 m$ for every m we also have $l > k + \frac{1}{2} \log_2 (n - k)$.

3) Suppose that $K_n, \emptyset_n \in \mathscr{G}_2$ and hence $K_{n-k}, \emptyset_{n-k} \in \mathscr{H}_1$. Analogously to 2) the assumption $l \leqq k + r(k)$ leads to the existence of a graph of order n which is not an element of $\mathscr{G}_1$ and $\mathscr{G}_2$, respectively. Thus $l > k + \frac{1}{2} \log_2 k$.

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