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INTEGRAL EQUIVALENCE OF TWO SYSTEMS OF DIFFERENTIAL EQUATIONS

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Let two systems of differential equations

(a) $$x' = F(t, x)$$

and

(b) $$y' = G(t, y)$$

be given. Suppose that F and G are such that they guarantee the existence of solutions of (a) and (b), respectively, on the infinite interval $\langle 0, \infty)$.

Definition 1. Let $\psi(t)$ be a positive continuous function on an interval $\langle t_0, \infty)$ and let $p > 0$. We shall say that two systems (a) and (b) are *(ψ, p)-integral equivalent on $\langle t_0, \infty)$* iff for each solution $x(t)$ of (a) there exists a solution $y(t)$ of (b) such that

(c) $$\psi^{-1}(t)\,|x(t) - y(t)| \in L_p(t_0, \infty)$$

and conversely, for each solution $y(t)$ of (b) there exists a solution $x(t)$ of (a) such that (c) holds.

By restricted (ψ, p)-integral equivalence between (a) and (b) we shall mean that the relation (c) is satisfied for some subsets of solutions of (a) and (b), e.g. for the bounded solutions.

We will say that a function $z(t)$ is ψ-bounded on the interval $\langle t_0, \infty)$ iff

$$\sup_{t \geqq t_0} |\psi^{-1}(t)\, z(t)| < \infty .$$

Next we will consider special systems

(1) $$x' = A(t)\, x + f(t, x)$$

and

(2) $$y' = A(t)\, y ,$$

where $A(t)$ is an $n \times n$ matrix-function defined on $\langle t_0, \infty)$ whose elements are integrable on compact subsets of $\langle t_0, \infty)$; x and y are n-dimensional vectors and $f(t, x)$

is an n-dimensional vector-function defined on $\langle t_0, \infty) \times E_n$. $|\cdot|$ denotes any convenient matrix (vector) norm.

In order to obtain first information about conditions which guarantee (ψ, p)-integral equivalence between (1) and (2) let us consider at first a simpler case $f(t, x) = \varphi(t)$, i.e., we will consider systems

$$(1') \qquad x' = A(t)\, x + \varphi(t)$$

and

$$(2') \qquad y' = A(t)\, y\,.$$

Using the property that each solution of (1') can be represented in the form

$$x(t) = y(t) + x_0(t)\,,$$

where $x_0(t)$ is any given solution of (1') and $y(t)$ a suitable solution of (2'), we immediately obtain the following theorem:

Theorem 1. *The systems (1') and (2') are (ψ, p)-integral equivalent iff there exists (at least one) solution $x_0(t)$ of (1') such that*

$$\psi^{-1}(t)\, x_0(t) \in L_p(t_0, \infty)\,.$$

Our next problem, therefore, is to find sufficient conditions for the existence of a solution $x_0(t)$ of (1') which has the property that

$$\psi^{-1}(t)\, x_0(t) \in L_p(t_0, \infty)\,.$$

Suppose that $A(t) = A$ is a constant matrix and has the Jordan canonical form. Let $\mu_1 < \mu_2 < \ldots < \mu_s = \lambda$ be the distinct real parts of the eigenvalues $\lambda_i(A)$ of A and let m_i be the maximum order of those blocks in A which correspond to the eigenvalues with the real part μ_i. Denote $m_s = m$. Let μ be a real number. Let $l = m_j$ if $\mu_j = \mu$ and let $l = 1$ if no μ_j equals μ.

Suppose that $A = \operatorname{diag}(A_1, A_2)$, where A_1 and A_2 are square matrices such that $\operatorname{Re} \lambda_i(A_1) < \mu$, $\operatorname{Re} \lambda_i(A_2) \geqq \mu$ for all i. Then

$$Y(t) = \operatorname{diag}(e^{tA_1}, e^{tA_2})$$

is the fundamental matrix of (2) with $Y(0) = I$ (I-identity matrix).

Let

$$Y_1(t) = \operatorname{diag}(e^{tA_1}, 0)\,, \quad Y_2(t) = \operatorname{diag}(0, e^{tA_2})\,.$$

Then

$$(3) \qquad Y(t) = Y_1(t) + Y_2(t)\,, \quad Y(t)\, Y^{-1}(s) = Y_1(t)\, Y_1^{-1}(s) + Y_2(t)\, Y_2^{-1}(s)\,,$$
$$Y_i(t)\, Y_i^{-1}(s) = Y_i(t - s)\,, \quad i = 1, 2\,,$$

and there exist two constants $c_1 > 0$ and $c_2 > 0$ such that

$$(4) \qquad |Y_1(t)| \leqq c_1\, e^{(\mu - \delta).t}\, \chi_{m^*}(t)\,,$$
$$|Y_2(t)| = |Y_2(-t)| \leqq c_2\, e^{-\mu t}\, \chi_l(t) \quad \text{for} \quad t \geqq 0\,,$$

where $-\delta = \max_i [\operatorname{Re} \lambda_i(A_1) - \mu] < 0$, $m^* = m_i$ if $\mu_i - \mu \approx -\delta$ and

$$\chi_k(t) = \begin{cases} t^{k-1}, & t \geqq 1 \\ 1, & 0 \leqq t \leqq 1 . \end{cases}$$

We shall need the following results in our considerations:

Lemma 1. *Let σ be a positive constant and let $g(t) \geqq 0$, $g(t) \in L_1(0, \infty)$. Then*

$$\int_0^t e^{-\sigma(t-s)} g(s)\, ds \in L_p(0, \infty), \quad p \geqq 1 .$$

Proof. Observe first that our hypotheses imply that

$$\lim_{t \to \infty} \int_0^t e^{-\sigma(t-s)} g(s)\, ds = 0 \quad \text{(see Brauer [1])} .$$

Let now $p \geqq 1$. Then for $T > 0$ we get

$$\int_0^T \left[\int_0^t e^{-\sigma(t-s)} g(s) ds \right]^p dt = \left[-\frac{1}{\sigma p} e^{-\sigma p t} \left(\int_0^t e^{\sigma s} g(s)\, ds \right)^p \right]_0^T +$$

$$+ \frac{1}{\sigma} \int_0^T e^{-\sigma p t} e^{\sigma t} g(t) \left[\int_0^t e^{\sigma s} g(s)\, ds \right]^{p-1} dt \leqq \frac{1}{\sigma} \int_0^T g(t) \left[\int_0^t g(s)\, ds \right]^{p-1} dt \leqq$$

$$\leqq \frac{1}{\sigma p} \left(\int_0^\infty g(s)\, ds \right)^p < \infty .$$

Lemma 2. *Let $g(t) \geqq 0$ be continuous on $0 \leqq t < \infty$ and such that*

$$\int_0^\infty s\, g(s)\, ds < \infty .$$

Then

$$\int_t^\infty g(s)\, ds \in L_p(0, \infty), \quad p \geqq 1 .$$

Proof. If $\int_0^\infty s\, g(s)\, ds < \infty$ then $\int_0^\infty g(s)\, ds < \infty$ and $\int_t^\infty g(s)\, ds$ is nonincreasing. Then for $p > 1$, using the second mean-value theorem we get

$$\int_0^\infty \left(\int_t^\infty g(s)\, ds \right)^p dt = \left(\int_0^\infty g(s)\, ds \right)^{p-1} \int_0^\xi \int_t^\infty g(s)\, ds\, dt \leqq$$

$$\leqq \left(\int_0^\infty g(s)\, ds \right)^{p-1} \int_0^\infty s\, g(s)\, ds < \infty$$

for some $\xi \in (0, \infty)$. For $p = 1$ we have

$$\int_0^\infty \int_t^\infty g(s)\, ds\, dt = \int_0^\infty \int_0^s g(s)\, dt\, ds = \int_0^\infty s\, g(s)\, ds < \infty .$$

Lemma 3. *Let $\psi(t)$ and $\varphi(t)$ be positive functions for $t \geqq 0$, $Y(t)$ a nonsingular matrix and P a projection. Further, suppose that*

$$\left[\int_0^t |\psi^{-1}(t)\, Y(t)\, P\, Y^{-1}(s)\, \varphi(s)|^p \,\mathrm{d}s\right]^{1/p} \leqq K \tag{5}$$

for $t \geqq 0$, $K > 0$, $p > 0$ and

$$\int_0^\infty \exp\left(-K^{-p}\int_0^t \varphi^p(s)\, \psi^{-p}(s)\,\mathrm{d}s\right)\mathrm{d}t < \infty\,. \tag{6}$$

Then

$$\lim \psi^{-1}(t)\,|Y(t)\,P| = 0 \quad \textit{as} \quad t \to \infty \tag{7}$$

and

$$|\psi^{-1}(t)\, Y(t)\, P| \in L_p(0, \infty)\,. \tag{8}$$

Proof. We follow first the proof due to T. G. Hallam [2]: Let

$$h(t) = \varphi^p(t)\,|Y(t)\,P|^{-p}\,.$$

We consider the identity

$$Y(t)\,P \int_0^t h(s)\,\mathrm{d}s = \int_0^t |\varphi^{-1}(s)\, Y(s)\, P|^{-p}\, Y(t)\, P\, Y^{-1}(s)\, \varphi(s)\, \varphi^{-1}(s)\, Y(s)\, P\,\mathrm{d}s\,. \tag{9}$$

Using the Hölder inequality we get

$$|Y(t)\,P| \leqq \left(\int_0^t h(s)\,\mathrm{d}s\right)^{-1/p}\left(\int_0^t |Y(t)\, P\, Y^{-1}(s)\, \varphi(s)|^p\,\mathrm{d}s\right)^{1/p} \tag{10}$$

and with respect to (5) we have

$$|\psi^{-1}(t)\, Y(t)\, P| \leqq K\left(\int_0^t h(s)\,\mathrm{d}s\right)^{-1/p}. \tag{11}$$

Denote

$$\mu(t) = \int_0^t h(s)\,\mathrm{d}s$$

Then (11) yields the inequality

$$\mu^{1/p}(t)\,|Y(t)\,P\,\varphi^{-1}(t)| \leqq K\,\varphi^{-1}\,\psi(t)$$

and since

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = h(t) = |Y(t)\,P|^{-p}\,\varphi^p(t)\,,$$

we get

$$\mu^{1/p}(t)\left(\frac{\mathrm{d}\mu}{\mathrm{d}t}\right)^{-1/p} \leqq K\,\varphi^{-1}(t)\,\psi(t)$$

or

$$\frac{d\mu}{dt} \geqq K^{-p} \varphi^p(t) \psi^{-p}(t) \mu(t).$$

Integrating from t_1 ($t_1 > 0$) to t we obtain the inequality (using Gronwall lemma)

$$\mu(t) \geqq \mu(t_1) \exp\left[K^{-p} \int_{t_1}^{t} \varphi^p(s) \psi^{-p}(s) ds\right]. \tag{12}$$

Note that (6) implies

$$\int_0^\infty \varphi^p(s) \psi^{-p}(s) ds = \infty .$$

Thus $\lim \mu(t) = \infty$ as $t \to \infty$ and then (11) yields (7) and

$$\int_{t_1}^{T} [\psi^{-1}(t) |Y(t) P|]^p dt \leqq K^p \int_{t_1}^{T} \mu^{-1}(t) dt .$$

In virtue of (12) we have

$$\int_{t_1}^{T} [\psi^{-1}(t) |Y(t) P|]^p dt \leqq K^p \mu^{-1}(t_1) \int_{t_1}^{T} \exp\left[-K^{-p} \int_{t_1}^{t} \varphi^p(s) \psi^{-p}(s) ds\right] dt,$$

which by (6) gives (8).

Now we are able to prove some theorems concerning the (ψ, p)-integral equivalence of the systems (1), (2).

Theorem 2. *Let A be a constant matrix. Let $\varphi(t)$ be continuous on $\langle 0, \infty)$, and let*

$$\int_0^\infty t^l |\varphi(t)| dt < \infty . \tag{13}$$

Then the systems (1′) and (2′) are $(1, p)$-integral equivalent, $p \geqq 1$.

Proof. For a solution $x(t)$ of (1′) we have $x(t) = y(t) + x_0(t)$, where $x_0(t)$ is a particular solution of (1′) and $y(t)$ a suitable solution of (2′). To prove our theorem it suffices to prove the existence of such solution $\mu_0(t)$ of (1′) that

$$\mu_0(t) \in L_p(0, \infty).$$

Let $Y(t)$ be the fundamental matrix of (2′), $\mu = 0$ and $Y_1(t)$, $Y_2(t)$ the corresponding matrices mentioned above. Let x_0 be such that

$$x_0 + \int_0^\infty Y_2^{-1}(s) \varphi(s) ds = 0 .$$

Then the solution $x(t)$, $x(0) = x_0$, of (1′) satisfies

$$x(t) = Y_1(t) x_0 + \int_0^t Y_1(t - s) \varphi(s) ds - \int_t^\infty Y_2(t - s) \varphi(s) ds \tag{14}$$

To prove that $x(t)$ belongs to $L_p(0, \infty)$ it is sufficient to prove that each of the three terms on the right hand side in (14) belongs to $L_p(0, \infty)$.

Owing to (5), we have

$$\int_0^\infty [|Y_1(s)|\,|x_0|]^p \,\mathrm{d}s \leqq |x_0|^p C_1^p \int_0^\infty \exp[-p\delta s]\, \chi_{m^*}^p(s)\,\mathrm{d}s < +\infty .$$

Then

$$\int_0^\infty \left|\int_0^t Y_1(t-s)\,\varphi(s)\,\mathrm{d}s\right|^p \mathrm{d}t \leqq C_1^p \int_0^\infty \left[\int_0^t \mathrm{e}^{-\delta(t-s)}\, \chi_{m^*}(t-s)\,|\varphi(s)|\,\mathrm{d}s\right]^p \mathrm{d}t .$$

But, using the Minkowski inequality, we have

$$\left\{\int_0^\infty \left[\int_0^t \mathrm{e}^{-\delta(t-s)}\,\chi_{m^*}(t-s)\,|\varphi(s)|\,\mathrm{d}s\right]^p \mathrm{d}t\right\}^{1/p} \leqq$$

$$\leqq \left\{\int_0^\infty \left[\int_0^{t/2} \mathrm{e}^{-\delta(t-s)}\,\chi_{m^*}(t-s)\,|\varphi(s)|\,\mathrm{d}s\right]^p \mathrm{d}t\right\}^{1/p} +$$

$$+ \left\{\int_0^\infty \left[\int_{t/2}^t \mathrm{e}^{-\delta(t-s)}\,\chi_{m^*}(t-s)\,|\varphi(s)|\,\mathrm{d}s\right]^p \mathrm{d}t\right\}^{1/p} = I_1 + I_2 .$$

For $t \geqq 2$, $0 \leqq s \leqq \frac{1}{2}t$ we have $t \geqq t - s \geqq \frac{1}{2}t \geqq 1$, therefore

$$I_1^p = \int_0^\infty \left[\int_0^{t/2} \mathrm{e}^{-\delta(t-s)/2}\,\mathrm{e}^{-\delta(t-s)/2}(t-s)^{m^*-1}\,|\varphi(s)|\,\mathrm{d}s\right]^p \mathrm{d}t \leqq$$

$$\leqq B_1 \int_0^\infty \left[\int_0^{t/2} \mathrm{e}^{-\delta(t-s)/2}|\varphi(s)|\,\mathrm{d}s\right]^p \mathrm{d}t$$

where we have used the inequality $\mathrm{e}^{-\delta u/2}u^{m^*-1} \leqq B_1$ for $u \geqq 0$ and $\delta > 0$. Now, from Lemma 1 we have $I_1^p < \infty$.

Using the same fact that $\mathrm{e}^{-\delta(t-s)}\,\chi_{m^*}(t-s) \leqq B_2$ for $t - s \geqq 0$ we get

$$I_2^p = \int_0^\infty \left[\int_{t/2}^t \mathrm{e}^{-\delta(t-s)}\,\chi_{m^*}(t-s)\,|\varphi(s)|\,\mathrm{d}s\right]^p \mathrm{d}t \leqq$$

$$\leqq B_2^p \int_0^\infty \left[\int_{t/2}^t |\varphi(s)|\,\mathrm{d}s\right]^p \mathrm{d}t \leqq B_2 \int_0^\infty \left[\int_{t/2}^\infty |\varphi(s)|\,\mathrm{d}s\right]^p \mathrm{d}t < \infty ,$$

following Lemma 2. Thus the second term on the right-hand side in (14) belongs to $L_p(0, \infty)$.

For the third term on the right-hand side in (14) we have

$$\left\{\int_0^\infty \left|\int_t^\infty Y_2(t-s)\,\varphi(s)\,\mathrm{d}s\right|^p \mathrm{d}t\right\}^{1/p} \leqq C_2\left\{\int_0^\infty \left[\int_t^\infty \chi_l(s-t)\,|\varphi(s)|\,\mathrm{d}s\right]^p \mathrm{d}t\right\}^{1/p} \leqq$$

$$\leqq C_2\left\{\int_0^\infty \left[\int_t^{t+1} |\varphi(s)|\,\mathrm{d}s\right]^p \mathrm{d}t\right\}^{1/p} + C_2\left\{\int_0^\infty \left[\int_{t+1}^\infty s^{l-1}|\varphi(s)|\,\mathrm{d}s\right]^p \mathrm{d}t\right\}^{1/p} \leqq$$

$$\leqq C_2\left\{\int_0^\infty\left[\int_t^\infty |\varphi(s)|\,\mathrm{d}s\right]^p \mathrm{d}t\right\}^{1/p} + C_2\left\{\int_0^\infty\left[\int_{t+1}^\infty s^{l-1}|\varphi(s)|\,\mathrm{d}s\right]^p \mathrm{d}t\right\}^{1/p} < \infty$$

according to Lemma 2.

Theorem 2 and its proof give us some ideas how to establish the (ψ, p)-integral equivalence between (1) and (2). There are three things to be used: the formula of variation of constants, the decomposition of the fundamental matrix $Y(t)$ of (2) into two matrices $Y_1(t)$, $Y_2(t)$ which have similar properties as (3), (4), the estimation and the growth of $f(t, x)$. The last task will be easier if we know an apriori estimate for the solutions $x(t)$ of (1).

Theorem 3. *Let $Y(t)$ be a fundamental matrix of (2), $\psi(t)$ and $\varphi(t)$ positive continuous functions for $t \geqq 0$.*

Suppose that:

a) *there exist supplementary projectors P_1, P_2, a constant $K > 0$ and $2 \leqq \leqq p < \infty$ such that*

$$\left[\int_0^t |\psi^{-1}(t)\, Y(t)\, P_1\, Y^{-1}(s)\, \varphi(s)|^p\, \mathrm{d}s + \int_t^\infty |\psi^{-1}(t)\, Y(t)\, P_2\, Y^{-1}(s)\, \varphi(s)|^p\, \mathrm{d}s\right]^{1/p} \leqq K$$

for all $t \geqq 0$,

b) *there exists g: $\langle 0, \infty) \times \langle 0, \infty) \to \langle 0, \infty)$ such that* (i) *$g(t, u)$ is monotone nondecreasing in u for each fixed $t \in \langle 0, \infty)$ and integrable on compact subsets of $\langle 0, \infty)$ for fixed $u \in \langle 0, \infty)$,* (ii) *$\int_0^\infty s\, g^{p'}(s, c)\, \mathrm{d}s < \infty$ for any constant $c \geqq 0$, where $1/p + 1/p' = 1$,* (iii) *let $f(t, x)$ be continuous on $\langle 0, \infty) \times R^n$ and such that for each $x \in R^n$*

$$|f(t, x)| \leqq \varphi(t)\, g(t, \psi^{-1}(t)\, |x|)$$

a.e. on $< 0, \infty)$,

c) $$\int_0^\infty \exp\left\{-K^{-p}\int_0^t \varphi^p(s)\, \psi^{-p}(s)\, \mathrm{d}s\right\} \mathrm{d}t < \infty\,,$$

d) $$\int_0^\infty |P_1\, Y^{-1}(s)\, \varphi(s)|\, g(s, c)\, \mathrm{d}s < \infty\,.$$

Then the sets of ψ-bounded solutions of (1) and of (2) are (ψ, p)-integral equivalent.

Proof. Let $y(t)$ be a ψ-bounded solution of (2) on $\langle t_0, \infty)$, $t_0 \geqq 0$. Then there is $\varrho > 0$ such that $y \in B_{\psi,\varrho}$, where

$$B_{\psi,\varrho} = \{z : z \text{ is continuous on } \langle t_0, \infty) \text{ and } \sup_{t \geqq t_0} |\psi^{-1}(t)\, z(t)| \leqq \varrho\}\,.$$

Define for $x \in B_{\psi,2\varrho}$ the operator

$$Tx(t) = y(t) + \int_{t_0}^t Y(t)\, P_1\, Y^{-1}(s) f(s, x(s))\, \mathrm{d}s - \int_t^\infty Y(t)\, P_2\, Y^{-1}(s) f(s, x(s))\, \mathrm{d}s\,.$$

The existence of

$$\int_t^\infty Y(t)\, P_2\, Y^{-1}(s) f(s, x(s))\, ds$$

is guaranteed by a) and b).

Then

$$|\psi^{-1}(t)\, Tx(t)| \leqq$$

$$\leqq \psi^{-1}(t)\, |y(t)| + \int_{t_0}^t |\psi^{-1}(t) Y(t)\, P_1 Y^{-1}(s)|\, |f(s, x(s))|\, ds + \int_t^\infty |\psi^{-1}(t) Y(t) P_2 Y^{-1}(s)|\, .$$

$$. |f(s, x(s))|\, ds \leqq \varrho + \int_{t_0}^t |\psi^{-1}(t)\, Y(t)\, P_1\, Y^{-1}(s)|\, \varphi(s)\, g(s, \psi^{-1}(s)\, |x(s)|)\, ds +$$

$$+ \int_t^\infty |\psi^{-1}(t)\, Y(t)\, P_2\, Y^{-1}(s)|\, \varphi(s)\, g(s, \psi^{-1}(s)\, |x(s)|)\, ds\, .$$

Using the Hölder inequality, a), b) we get

$$|\psi^{-1}(t)\, Tx(t)| \leqq$$

$$\leqq \varrho + \left\{\int_{t_0}^t |\psi^{-1}(t)\, Y(t)\, P_1\, Y^{-1}(s)\, \varphi(s)|^p\, ds\right\}^{1/p} \left\{\int_{t_0}^t g^{p'}(s, 2\varrho)\, ds\right\}^{1/p'} +$$

$$+ \left\{\int_t^\infty |\psi^{-1}(t)\, Y(t)\, P_2\, Y^{-1}(s)\, \varphi(s)|^p\, ds\right\}^{1/p} \left\{\int_t^\infty g^{p'}(s, 2\varrho)\, ds\right\}^{1/p'} \leqq$$

$$\leqq \varrho + K \left\{\int_{t_0}^\infty g^{p'}(s, 2\varrho)\, ds\right\}^{1/p'} .$$

If we choose t_0 such that

$$K \left\{\int_{t_0}^\infty g^{p'}(s, 2\varrho)\, ds\right\}^{1/p'} \leqq \varrho\, ,$$

we have that T maps $B_{\psi,2\varrho}$ into itself.

Now we are going to prove the continuity of T on $B_{\psi,2\varrho}$. Let $x_n(t), x(t) \in B_{\psi,2\varrho}$, $x_n(t)$ converge to $x(t)$ uniformly on compact intervals of $\langle t_0, \infty)$. For $Tx_n(t) - Tx(t)$ we have

$$|Tx_n(t) - Tx(t)| \leqq$$

$$\leqq \int_{t_0}^t |Y(t)\, P_1\, Y^{-1}(s)|\, |f(s, x_n(s)) - f(s, x(s))|\, ds +$$

$$+ \int_t^\infty |Y(t)\, P_2\, Y^{-1}(s)|\, |f(s, x_n(s)) - f(s, x(s))|\, ds \leqq$$

$$\leqq \psi(t) \int_{t_0}^t |\psi^{-1}(t)\, Y(t)\, P_1\, Y^{-1}(s)\, \varphi(s)|\, \varphi^{-1}(s)\, |f(s, x_n(s)) - f(s, x(s))|\, ds +$$

$$+ \psi(t) \int_t^\infty \left|\psi^{-1}(t)\, Y(t)\, P_2\, Y^{-1}(s)\, \varphi(s)\right| \varphi^{-1}(s) \left|f(s, x_n(s)) - f(s, x(s))\right| \mathrm{d}s \leqq$$

$$\leqq \psi(t) \left\{\int_{t_0}^t \left|\psi^{-1}(t)\, Y(t)\, P_1\, Y^{-1}(s)\, \varphi(s)\right|^p \mathrm{d}s\right\}^{1/p} .$$

$$\cdot \left\{\int_{t_0}^t \varphi^{-p'}(s) \left|f(s, x_n(s)) - f(s, x(s))\right|^{p'} \mathrm{d}s\right\}^{1/p'} +$$

$$+ \psi(t) \left\{\int_t^\infty \left|\psi^{-1}(t)\, Y(t)\, P_2\, Y^{-1}(s)\, \varphi(s)\right|^p \mathrm{d}s\right\}^{1/p} .$$

$$\cdot \left\{\int_t^\infty \varphi^{-p'}(s) \left|f(s, x_n(s)) - f(s, x(s))\right|^{p'} \mathrm{d}s\right\}^{1/p'} \leqq$$

$$\leqq \psi(t)\, K \left\{\int_{t_0}^t \varphi^{-p'}(s) \left|f(s, x_n(s)) - f(s, x(s))\right|^{p'} \mathrm{d}s\right\}^{1/p'} +$$

$$+ \psi(t)\, K \left\{\int_t^\infty \varphi^{-p'}(s) \left|f(s, x_n(s)) - f(s, x(s))\right|^{p'} \mathrm{d}s\right\}^{1/p'} \leqq$$

$$\leqq 2K\, \psi(t) \left\{\int_{t_0}^\infty \varphi^{-p'}(s) \left|f(s, x_n(s)) - f(s, x(s))\right|^{p'} \mathrm{d}s\right\}^{1/p'} \leqq$$

$$\leqq 2K\, \psi(t) \left\{\int_{t_0}^{t_1} \varphi^{-p'}(s) \left|f(s, x_n(s)) - f(s, x(s))\right|^{p'} \mathrm{d}s + \right.$$

$$\left. + \int_{t_1}^\infty \varphi^{-p'}(s) \left|f(s, x_n(s)) - f(s, x(s))\right|^{p'} \mathrm{d}s\right\}^{1/p'} .$$

On $\langle t_0, t_1 \rangle$, $x_n(s)$ converges to $x(s)$ uniformly. Then the continuity of $f(t, x)$ implies that to $\varepsilon > 0$ there is $n_0(t_1)$ such that for $n \geqq n_0(t_1)$ we have

$$\varphi^{-1}(s) \left|f(s, x_n(s)) - f(s, x(s))\right| < \frac{\varepsilon}{4K(t_1 - t_0)^{1/p'}} \quad \text{for} \quad s \in \langle t_0, t_1 \rangle .$$

Applying this and b) (iii), we have for $n \geqq n_0$

$$\left|Tx_n(t) - Tx(t)\right| \leqq 2K\, \psi(t) \left[\frac{\varepsilon^{p'}}{4^{p'} K^{p'}} + 2 \int_{t_1}^\infty \varphi^{-p'}(s)\, \varphi^{p'}(s)\, g^{p'}(s, 2\varrho)\, \mathrm{d}s\right]^{1/p'} .$$

Choose t_1 such that

$$\int_{t_1}^\infty g^{p'}(t, 2\varrho)\, \mathrm{d}t < \frac{1}{2} \frac{\varepsilon^{p'}}{4^{p'} K^{p'}} .$$

Then we get

$$\left|Tx_n(t) - Tx(t)\right| < \varepsilon\, \psi(t) .$$

This shows that T is continuous on $B_{\psi, 2\varrho}$.

The functions in $TB_{\psi,2\varrho}$ are evidently uniformly bounded for each $t \geqq t_0$ because $TB_{\psi,2\varrho} \subset B_{\psi,2\varrho}$. Because $z = Tx$ is a solution of the equation

$$z' = A(t)\, z + f(t, x(t)),$$

the derivatives of the functions in $TB_{\psi,2\varrho}$ are uniformly bounded on every compact interval. Thus the functions in $TB_{\psi,2\varrho}$ are equicontinuous on every compact subinterval of $\langle t_0, \infty)$.

Then Schauder's fixed point theorem yields the existence of a fixed point $x(t)$ of T in $B_{\psi,2\varrho}$. A direct verification shows that this fixed point $x(t)$ is a ψ-bounded solution of (1).

Conversely, let $x(t)$ be a ψ-bounded solution of (1). Define

$$y(t) = x(t) - \int_{t_0}^{t} Y(t)\, P_1\, Y^{-1}(s) f(s, x(s))\, ds + \int_{t}^{\infty} Y(t)\, P_2\, Y^{-1}(s) f(s, x(s))\, ds.$$

It is easy to prove that $y(t)$ is a ψ-bounded solution of (2).

Now we have to prove that

$$\psi^{-1}(t)\, |x(t) - y(t)| \in L_p(t_0, \infty).$$

We have

$$\psi^{-1}(t)\, [x(t) - y(t)] =$$

$$= \int_{t_0}^{t} \psi^{-1}(t)\, Y(t)\, P_1\, Y^{-1}(s) f(s, x(s))\, ds - \int_{t}^{\infty} \psi^{-1}(t)\, Y(t)\, P_2\, Y^{-1}(s) f(s, x(s))\, ds.$$

It is sufficient to show that the terms on the right-hand side belong to $L_p(t_0, \infty)$. By the assumptions of the theorem and the Hölder inequality we get

$$\left|\int_{t_0}^{t} \psi^{-1}(t)\, Y(t)\, P_1\, Y^{-1}(s) f(s, x(s))\, ds\right| \leqq \int_{t_0}^{t} |\psi^{-1}(t) Y(t)\, P_1 Y^{-1}(s)|\, \varphi(s)\, g(s, 2\varrho)\, ds \leqq$$

$$\leqq |\psi^{-1}(t)\, Y(t)\, P_1| \int_{t_0}^{\infty} |P_1\, Y^{-1}(s)\, \varphi(s)\, g(s, 2\varrho)|\, ds.$$

Since (from Lemma 3)

$$|\psi^{-1}(t)\, Y(t)\, P_1| \in L_p(t_0, \infty)$$

and d) holds, it is evident that this first term belongs to $L_p(t_0, \infty)$. For the second term we have

$$\int_{t}^{\infty} |\psi^{-1}(t)\, Y(t)\, P_2\, Y^{-1}(s)|\, |f(s, x(s))|\, ds \leqq \int_{t}^{\infty} |\psi^{-1}(t) Y(t) P_2 Y^{-1}(s)|\, \varphi(s)\, g(s, 2\varrho)\, ds \leqq$$

$$\leqq \left(\int_{t}^{\infty} |\psi^{-1}(t) Y(t) P_2 Y^{-1}(s)\, \varphi(s)|^p\, ds\right)^{1/p} \left(\int_{t}^{\infty} g^{p'}(s, 2\varrho)\, ds\right)^{1/p'} \leqq$$

$$\leqq K \left(\int_{t}^{\infty} g^{p'}(s, 2\varrho)\, ds\right)^{1/p'}.$$

Thus from b) (ii) and Lemma 2 we get that also this term belongs to $L_p(t_0, \infty)$. The proof of the theorem is complete.

Corollary 3.1. *Let $p = 1$ (and thus $p' = \infty$). Assume that the assumptions of Theorem 3 are satisfied except* b) (ii) *which is substituted by the conditions*

$$\lim_{t\to\infty} \gamma_c(t) = 0 \quad \textit{for each} \quad c \geqq 0 \quad \textit{and} \quad \gamma_c(t) \in L_1(0, \infty),$$

where $\gamma_c(t) = \sup_{s\geqq t} g(s, c)$. Then the conclusion of Theorem 3 holds true.

Corollary 3.2. *Let $p = \infty$* (and $p' = 1$). *Let the condition* a) *of Theorem 3 be replaced by*

$$\sup_{\tau_0\leqq s\leqq t} |\psi^{-1}(t)\, Y(t)\, P_1\, Y^{-1}(s)\, \varphi(s)| + \sup_{t<s<\infty} |\psi^{-1}(t)\, Y(t)\, P_2\, Y^{-1}(s)\, \varphi(s)| \leqq K$$

and

$$|\psi^{-1}(t)\, Y(t)\, P_1| \in L_v(0, \infty), \quad v > 1$$

and let all the other assumptions of Theorem 3 hold.

Then the sets of ψ-bounded solutions of (1) *and of* (2) *are (ψ, v)-integral equivalent.*

Theorem 4. *Let $\psi(t)$, $\alpha(t)$ and $\beta(t)$ be positive continuous functions for $t \geqq \tau_0 \geqq 0$ with*

$$\lim_{t\to\infty} \psi^{-1}(t) = 0$$

and $\beta(t)$ bounded on $\langle\tau_0, \infty)$.

Let $Y(t)$ be a fundamental matrix of (2).

Let $w : \langle\tau_0, \infty) \times I \to I$, $I = \langle 0, \infty)$ be such that

a) *$w(t, r)$ is monotone nondecreasing in r for each fixed $t \in \langle\tau_0, \infty)$, $w(t, c\,\psi(t))$ is integrable on compact subsets of $\langle\tau_0, \infty)$ for each $c \geqq 0$,*

b) *$\int_{\tau_0}^{\infty} s\, \alpha(s)\, w(s, \psi(s))\, ds < \infty$ for each $c \geqq 0$,*

c) *$\int_{\tau_0}^{t} \beta(t - s)\, \alpha(s)\, w(s, c\,\psi(s))\, ds \in L_v(\tau_0, \infty)$ for each $c \geqq 0$; $v > 1$.*

Let there exist two supplementary projectors P_1, P_2 and a constant $c > 0$ such that

d)
$$|Y(t)\, P_1\, Y^{-1}(s)\, \alpha^{-1}(s)| \leqq c\, \beta(t - s) \quad \textit{for} \quad t \geqq s \geqq \tau_0,$$
$$|Y(t)\, P_2\, Y^{-1}(s)\, \alpha^{-1}(s)| \leqq c \qquad \textit{for} \quad \tau_0 \leqq t \leqq s < \infty.$$

Then the sets of ψ-bounded solutions of (1) *and of* (2) *are* $(1, v)$*-integral equivalent.*

Proof. Let $t_0 \geqq \tau_0$ be fixed and let $y(t)$ be a ψ-bounded solution of (2) with

$$|y(t)|_\psi = \sup_{t\geqq\tau_0} |\psi^{-1}(t)\, y(t)| \leqq \delta \quad \text{for} \quad \delta > 0.$$

Let

$$\varphi(t) = \alpha^{-1}(t), \quad g(t, r) = \alpha(t)\, w(t, \psi(t)\, r) \quad \text{for} \quad t \geqq \tau_0.$$

Then for $t \geqq \tau_0$, $g(t, r)$ is monotone nondecreasing in r and for each fixed $r \in I$, $g(t, r)$ is integrable on compact subsets of $\langle \tau_0, \infty)$ by condition a).

From b) we get

$$\int_{\tau_0}^{\infty} g(s, c)\, ds < \infty \quad \text{for} \quad c \geqq 0 .$$

Moreover, condition d) and the assumptions on $\psi(t)$ and $\beta(t)$ imply that there exists $K > 0$ such that

$$\sup_{\tau_0 \leqq s \leqq t} |\psi^{-1}(t)\, Y(t)\, P_1\, Y^{-1}(s)\, \varphi(s)| + \sup_{t \leqq s < \infty} |\psi^{-1}(t)\, Y(t)\, P_2\, Y^{-1}(s)\, \varphi(s)| < K .$$

Then the same reasoning as in the proof of Theorem 3 with $p = \infty$ gives the existence of a solution $x(t)$ of (1) such that

$$|x(t)|_{\psi} = \sup_{t \geqq \tau_0} |\psi^{-1}(t)\, x(t)| \leqq 2\delta$$

and

$$x(t) = y(t) + \int_{t_0}^{t} Y(t)\, P_1\, Y^{-1}(s) f(x, x(s))\, ds - \int_{t}^{\infty} Y(t) P_2 Y^{-1}(s) f(s, x(s))\, ds. \tag{15}$$

Conversely, given a solution $x(t)$ of (1) we have that

$$y(t) = x(t) - \int_{t_0}^{t} Y(t)\, P_1\, Y^{-1}(s) f(s, x(s))\, ds + \int_{t}^{\infty} Y(t)\, P_2\, Y^{-1}(s) f(s, x(s))\, ds$$

is a solution of (2).

Now, we have to prove that

$$|x(t) - y(t)| \in L_v(t_0, \infty)$$

if (15) holds. We have

$$|x(t) - y(t)| \leqq \int_{t_0}^{t} |Y(t)\, P_1\, Y^{-1}(s)|\, |f(s, x(s))|\, ds + \int_{t}^{\infty} |Y(t) P_2 Y^{-1}(s)|\, |f(s, x(s))| ds \leqq$$

$$\leqq \int_{t_0}^{t} |Y(t)\, P_1\, Y^{-1}(s)\, \alpha^{-1}(s)|\, \alpha(s)\, w(s, 2\delta\, \psi(s))\, ds +$$

$$+ \int_{t}^{\infty} |Y(t)\, P_2\, Y^{-1}(s)\, \alpha^{-1}(s)|\, \alpha(s)\, w(s, 2\delta\, \psi(s))\, ds \leqq$$

$$\leqq c \left\{ \int_{t_0}^{t} \beta(t - s)\, \alpha(s)\, w(s, 2\delta\, \psi(s))\, ds + \int_{t}^{\infty} \alpha(s)\, w(s, 2\delta\, \psi(s))\, ds \right\} .$$

It is sufficient to prove that each of the two terms on the righthand side belongs to $L_v(t_0, \infty)$.

For the first term the statement is true by assumption c) and for the second term it follows from b) and Lemma 2.

Lemma 4. (Theorem 5, M. Švec [3].) *Let* $|f(t, x)| \leqq w(t, |x|)$ *a.e. on* I *for each*

$x \in R^n$, where $w(t, u)$ is a nonnegative continuous function in (t, u) on $I \times I$, nondecreasing in u for each fixed $t \in I$.

Suppose further that

$$\int_0^\infty \mathrm{e}^{-\lambda t} w(t, c\mathrm{e}^{\lambda t} \chi_m(t))\,\mathrm{d}t < \infty \quad \textit{for each} \quad c \geqq 0$$

and

$$\lim_{t_0 \to \infty} \frac{1}{c} \int_{t_0}^\infty \mathrm{e}^{-\lambda t} w(t, c\mathrm{e}^{\lambda t} \chi_m(t))\,\mathrm{d}t = 0$$

uniformly with respect to $c \in \langle 1, \infty)$.

Then every solution $x(t) = x(t; t_0, x_0)$ of (1) is defined on $\langle t_0, \infty)$ and the estimate

$$|x(t)| \leqq D\mathrm{e}^{\lambda t} \chi_m(t), \quad t \geqq t_0 \geqq 0$$

holds for some constant $D > 0$.

Corollary 4.1. *Let l, m and λ be defined as above. Suppose that there exists a nonnegative function w on $I \times I$ such that*

a) *$w(t, r)$ is monotone nondecreasing in r for each $t \in I$ and $w(t, c\mathrm{e}^{\lambda t} \chi_m(t))$ is integrable on compact subsets of I for each c.*

b) *$f(t, x) \leqq w(t, |x|)$ a.e. on I for each $x \in R^n$,*

c) *$\int_0^\infty t^l w(t, c\mathrm{e}^{\lambda t} \chi_m(t))\,\mathrm{d}t < \infty$ for each $c \geqq 0$ if $\lambda \geqq 0$,*
   $\int_0^\infty \mathrm{e}^{-\lambda t} w(t, c\mathrm{e}^{\lambda t} \chi_m(t))\,\mathrm{d}t < \infty$ for each $c \geqq 0$ if $\lambda < 0$,

d) *$\lim_{t_0 \to \infty} 1/c \int_{t_0}^\infty \mathrm{e}^{-\lambda t} w(t, c\mathrm{e}^{\lambda t} \chi_m(t))\,\mathrm{d}t = 0$*
   uniformly with respect to $c \in \langle 1, \infty)$

e) *$\int_{t_0}^t \mathrm{e}^{-\delta(t-s)} \chi_{m*}(t - s)\, t^{l-1} w(t, c\mathrm{e}^{\lambda t} \chi_m(t))\,\mathrm{d}t \in L_v(t_0, \infty)$, $v \geqq 1$.*

Then the systems (1) and (2) are $(1, v)$-integral equivalent.

Proof. Since all the hypotheses of Lemma 4 are satisfied, solutions $x(t)$ of (1) and $y(t)$ of (2) satisfy the estimate

$$|x(t)| \leqq D_1 \mathrm{e}^{\lambda t} \chi_m(t), \quad t \geqq t_0$$

and

$$|y(t)| \leqq D_2 \mathrm{e}^{\lambda t} \chi_m(t), \quad t \geqq t_0 .$$

Let

$$\psi(t) = \mathrm{e}^{\lambda t} \chi_m(t) .$$

Then it follows that all the solutions of (1) as well as those of (2) are ψ-bounded. Therefore if $\lambda < 0$, all solutions of (1) converge to zero as $t \to \infty$ and the same is true for the solutions of (2). Then, using the Minkowski inequality, we get

$$\left[\int_{t_0}^{\infty} |x(t) - y(t)|^v \, dt\right]^{1/v} \leqq \left(\int_{t_0}^{\infty} |x(t)|^v \, dt\right)^{1/v} + \left(\int_{t_0}^{\infty} |y(t)|^v \, dt\right)^{1/v} \leqq$$

$$\leqq D_1 \left(\int_{t_0}^{\infty} e^{\lambda v t} \chi_{m+v}(t) \, dt\right)^{1/v} + D_2 \left(\int_{t_0}^{\infty} e^{\lambda v t} \chi_{m+v}(t) \, dt\right)^{1/v} < \infty .$$

We need to consider the case $\lambda \geqq 0$ only. We may proceed as in [3, p. 56]. We get

$$x(t) - y(t) = \int_{t_0}^{t} Y_1(t - s) f(s, x(s)) \, ds - \int_{t}^{\infty} Y_2(t - s) f(s, x(s)) \, ds$$

and using (4) and b),

$$|x(t) - y(t)| \leqq c_1 \int_{t_0}^{t} e^{-\delta(t - t_0)} \chi_{m^*}(t - s) \, w(s, D_2 e^{\lambda s} \chi_m(s)) \, ds +$$

$$+ c_2 \int_{t}^{\infty} \chi_l(s - t) \, w(s, D_1 e^{\lambda s} \chi_m(s)) \, ds \leqq$$

$$\leqq c_1 \int_{t_0}^{t} e^{-\delta(t - t_0)} \chi_{m^*}(t - s) \, w(s, D_1 e^{\lambda s} \chi_m(s) \, ds +$$

$$+ c_2 \int_{t}^{\infty} w(s, D_1 e^{\lambda s} \chi_m(s)) \, ds + c_2 \int_{t}^{\infty} s^{l-1} \, w(s, D_1 e^{\lambda s} \chi_m(s)) \, ds .$$

The first term belongs to $L_v(t_0, \infty)$ by e), the second and the third belong to the same space as a consequence of Lemma 2.

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