Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 32 (1982), No. 2, 307–326

Persistent URL: http://dml.cz/dmlcz/101803

Czechoslovak Mathematical Journal, 32 (107) 1982, Praha

PARTIAL MONOUNARY ALGEBRAS WITH COMMON CONGRUENCE RELATIONS

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(Received June 23, 1981)

Partial monounary algebras have been investigated by W. Bartol [1]–[3] and by O. Kopeček [5]–[9] (in the papers [1]–[3], [6] and [8] the authors used the term "machine" instead of "partial monounary algebra").

Let (A, f) be a partial monounary algebra and let D_f be the set of all $z \in A$ such that $f(z)$ does not exist. By the symbol F we denote the set of all partial mappings of A into A. Let $\operatorname{Con}(A, f)$ be the system of all congruence relations on (A, f) and let $E(A)$ be the system of all equivalence relations on A. We denote

$$R(f) = \{g \in F: \operatorname{Con}(A, f) = \operatorname{Con}(A, g)\}\ .$$

In this paper the following result will be established:

(A) *Let* (A, f) *be a partial monounary algebra such that* $\operatorname{Con}(A, f) \neq E(A)$ *and* $f^{-1}(D_f) \neq \emptyset$. *Then we have*

$$\operatorname{card} R(f) \leqq 4\ ,$$

and this estimate is the best possible.

For each $i \in \{1, 2, 3, 4\}$ all partial monounary algebras with $\operatorname{Con}(A, f) \neq E(A)$, $f^{-1}(D_f) \neq \emptyset$ and $\operatorname{card} R(f) = i$ will be explicitly described. The formula for $\operatorname{card} R(f)$ in the case when $\operatorname{Con}(A, f) = E(A)$ is given in Corollary 1.6 below.

If $f^{-1}(D_f) = \emptyset$, then the question on the cardinality of $R(f)$ can be reduced to an analogous question concerning (complete) monounary algebras (this investigation will be performed elsewhere).

While the results on homomorphisms of partial monounary algebras obtained in [5], [6] are analogous to the results on homomorphisms of (complete) monounary algebras (M. Novotný [11]–[13]; cf. also M. Novotný and O. Kopeček [10]), when investigating congruence relations the situation is different: for a (complete) monounary algebra (A, f) with $\operatorname{Con}(A, f) \neq E(A)$ the cardinality of $R(f)$ can be infinite.

1. THE CASE $\operatorname{Con}(A,f) = E(A)$

Let (A,f) be a partial monounary algebra. An equivalence Θ on A will be called a congruence, if the following implication is valid (cf. also [4], p. 177):

$$(\forall x, y \in A - D_f)\,(x\Theta y \Rightarrow f(x)\,\Theta\, f(y)).$$

By the symbol N we denote the set of all positive integers. Further, put $D'_f = \bigcup_{z\in D_f} \bigcup_{n\in N} f^{-n}(z)$, $B_f = A - (D_f \cup D'_f)$. For $a, b \in A$ let $\Theta^f(a, b)$ be the smallest congruence Θ on (A,f) such that $a\Theta b$. Notice that $\Theta^f(a, b)$ exists for each $a, b \in A$. If no misunderstanding can occur, we shall also write $\Theta(a, b)$, D, D' and B instead of $\Theta^f(a, b)$, D_f, D'_f and B_f. Further, we shall not distinguish between a congruence Θ and a partition of the set A corresponding to Θ. Let $\Theta = \{T_\lambda : \lambda \in I\}$ and let $I' \subseteq I$ be such that card $T_\lambda = 1$ for each $\lambda \in I - I'$. Then we shall write $\Theta = [T_\lambda : \lambda \in I']$. Now let $S \subseteq \operatorname{Con}(A,f)$ be the system of all congruences Θ such that $\Theta = [T]$ and card $T = 2$. Then we put

$$P = \{z \in A \colon \Theta(z, x) \in S \text{ for each } x \in A,\ x \neq z\}.$$

We start with formulating four lemmas, the proofs of which are obvious; these lemmas will be frequently used without specific reference.

1.1. Lemma. $D \subseteq P$.

1.2. Lemma. *Let $x, y \in D'$. Then the partition corresponding to $\Theta(x, y)$ has only finitely many nontrivial classes. Further, $\Theta(x, y) \in S$ if and only if $x \neq y$ and $f(x) = f(y)$.*

1.3. Lemma. *Let $x \in D'$, $y \in B$. Then the partition corresponding to $\Theta(x, y)$ has only finitely many nontrivial classes and $\Theta(x, y) \notin S$.*

1.4. Lemma. *Let $x, y \in B$, $x \neq y$. Then $\Theta(x, y) \in S$ if and only if* (a) *$f(x) = x$, $f(y) = y$, or* (b) *$f(x) = f(y)$, or* (c) *$f(x) = y$, $f(y) = x$.*

1.5. Theorem. *The following conditions are equivalent*:

(1) $\operatorname{Con}(A,f) = E(A)$.

(2) $P = A$.

(3) *Some of the conditions* (a)–(d) *is satisfied*:

(a) *There are $A_1, A_2 \subseteq A$ such that $A = A_1 \cup A_2$, $A_1 = D$, $f(z) = z$ for each $z \in A_2$.*

(b) *There are $A_1, A_2 \subseteq A$, $a \in A$ such that $A = A_1 \cup A_2 \cup \{a\}$, $A_1 = D, f(z) = a$ for each $z \in A_2 \cup \{a\}$.*

(c) *There are $A_1 \subseteq A$, $a, b \in A$, $a \neq b$, such that $A = A_1 \cup \{a, b\}$, $A_1 = D$, $f(a) = b$, $f(b) = a$.*

(d) *There are $A_1, A_2 \subseteq A$, $a \in A$ such that $A = A_1 \cup A_2 \cup \{a\}$, $A_1 \cup \{a\} = D$, $f(z) = a$ for each $z \in A_2$.*

Proof. The assertion (1) $\Leftrightarrow$ (2) is obvious, since in both the cases we have $\Theta(x, y) = [\{x, y\}]$ for each $x, y \in A$, $x \neq y$.

It is easy to see that (3) $\Rightarrow$ (2).

Suppose that (2) is satisfied, i.e., $\Theta(x, y) = [\{x, y\}]$ for each $x, y \in A$, $x \neq y$. Let $B \neq \emptyset$. Then 1.3 implies that $D' = \emptyset$. According to 1.4 we obtain that either $f(x) = x$ for each $x \in B$, or there exists $a \in B$ with $f(x) = a$ for each $x \in B$, or B is a cycle with two elements. Hence (a), (b) or (c) is valid. Now assume that $B = \emptyset$. From 1.2 it follows that $f(x) = f(y)$ for each $x, y \in D'$, $x \neq y$, thus $f^{-2}(D) = \emptyset$ and hence (d) holds.

1.6. Corollary. *Assume that* Con $(A, f) = E(A)$. *If A is infinite, then* card $R(f) = 2^{\text{card}\, A}$. *If* card $A = n < \aleph_0$, *then*

$$\text{card}\, R(f) = 2^n(n + 1) + \frac{n(n - 5)}{2}.$$

Proof. Suppose that Con $(A, f) = E(A)$ and let card $A = n$. From 1.5 it follows that for an operation belonging to $R(f)$ some of the conditions (a)–(d) is satisfied, hence we have the following number of possibilities: 2^n in the case (a) (a subset $A_1 \subseteq A$ can be chosen in 2^n ways); $n \,.\, 2^{n-1}$ in the case (b) (the choice of an element $a \in A$, the choice of $A_1 \subseteq A - \{a\}$); $\frac{1}{2}n(n - 1)$ in the case (c) (the choice of $\{a, b\}$); $n \,.\, 2^{n-1}$ in the case (d) (the choice of $a \in A$ and $A_2 \subseteq A - \{a\}$). But these possibilities are not all independent. If $A_2 = \emptyset$ in the case (b), then this possibility is considered also in the case (a) (for each $a \in A$). Similarly, if $A_2 = \emptyset$ in the case (d), this possibility is included in the case (a) as well, for each $a \in A$. All the other possibilities are independent, hence we obtain

$$\text{card}\, R(f) = 2^n + n \,.\, 2^{n-1} + \frac{n(n - 1)}{2} + n\, 2^{n-1} - 2n = 2^n\,(n + 1) + \frac{n(n - 5)}{2}.$$

If $n \geqq \aleph_0$, then card $R(f) = 2^n$.

2. AUXILIARY RESULTS

In what follows we suppose that Con $(A, f) \neq E(A)$.

2.1. Lemma. $P = D$.

Proof. From 1.1 it follows that we have to prove only the relation $A - D \subseteq A - P$. Since $P \neq A$, according to 1.5 we obtain that the condition (3) in 1.5 fails to hold, hence none of the conditions (a)–(d) in 1.5 is satisfied. First consider the case $B = \emptyset$. Then either (a) there exist distinct elements $a, b, z \in A$ with $f(b) = a$, $f(a) = z \in D$, or (b) there exist distinct elements $a, b, u, z \in A$ with $f(a) = z \in D$, $f(b) = u \in D$. Suppose that (a) is valid. Then $\Theta(b, a) = [\{b, a, z\}] \notin S$, hence $b \notin P$. Let $x \in A - D$, $x \neq b$. Then $f(x)\; \Theta(x, b)\; a$. If $f(x) \neq a$, then $x \notin P$. If $f(x) =$

$= a$, then $\Theta(x, a) = [\{x, a, z\}]$, hence $x \notin P$ as well. If (b) holds, then $\Theta(a, b) = [\{a, b\}, \{z, u\}] \notin S$, thus $b \notin P$. For each $x \in A - D$, $x \neq b$, we obtain $f(x) \,.\, \Theta(x, b)\, u$. If $f(x) \neq u$, then $x \notin P$. If $f(x) = u$, then $\Theta(a, x) = [\{a, x\}, \{z, u\}]$, and $x \notin P$.

Further, suppose that $B \neq \emptyset$. From 1.5 it follows that some of the following cases occurs: (a) $D' \neq \emptyset$; (b) $D' = \emptyset$ and there exist distinct elements $a, b, c \in B$ with $f(a) = b$, $f(b) = c$; (c) $D' = \emptyset$ and there exist distinct elements $a, b, c \in B$ with $f(a) = f(b) = b, f(c) = c$; (d) $D' = \emptyset$ and there exist distinct elements $a, b, a', b' \in B$ with $f(a) = b$, $f(b) = a$, $f(a') = b'$, $f(b') = a'$; (e) $D' = \emptyset$ and there exist distinct elements $a, b, c \in B$ with $f(a) = b$, $f(b) = a$, $f(c) = c$. In the case (a) there exist distinct elements $a, b, z \in A$ with $f(b) = z \in D$, $a \in B$. Then $\Theta(a, b) \notin S$, hence $b \notin P$. Let $x \in A - D$, $x \neq b$. We get $f(x)\, \Theta(x, b)\, z$. If $f(x) \neq z$, then $x \notin P$. If $f(x) = z$, then $z\, \Theta(x, a) f(a)$, thus $\Theta(x, a) \notin S$ and $x \notin P$. Let (b) be valid. Since $\Theta(a, b) \neq [\{a, b\}]$, we have $a \notin P$. Let $x \in A - D$, $x \neq a$. Then $f(x)\, \Theta(x, a)\, b$. If $f(x) \neq b$, then $x \notin P$. If $f(x) = b$, then $b\, \Theta(x, b)\, c$, hence $x \notin P$. Now suppose that (c) holds. Since $\Theta(a, c) = [\{a, b, c\}]$, we have $a \notin P$. Let $x \in A - D$, $x \neq a$. If $f(x) \neq b$, then $f(x)\, \Theta(x, a)\, b$, hence $x \notin P$; if $f(x) = b$, then $\Theta(x, c) \notin S$ and $x \notin P$. In the case (d) or (e) we obtain respectively $\Theta(a, a') \notin S$ or $\Theta(a, c) \notin S$, thus $a \notin P$. Let $x \in A - D$, $x \neq a$. If $f(x) \neq b$, then $f(x)\, \Theta(x. a)\, b$, hence $x \notin P$. If $f(x) = b$, then $f(x)\, \Theta(x, b)\, a$, and therefore $x \notin P$.

Remark. In the following Lemmas 2.2–2.16 let us assume that distinct symbols $x, y, z, \ldots$ denote distinct elements. Moreover, we shall not prove the implication (1) $\Rightarrow$ (2) in 2.2–2.16; it can be easily verified. The figure corresponding to Lemma 2.2 is denoted as Fig. 2.2, and similarly for other lemmas in this section.

If the same figure is related also to some lemma of § 3, then we denote it also by the number of the corresponding lemma from § 3.

In the figures we use the following denotation:

$\bullet\ a$ elements with the property $f(a) = a$;

$\square\ b$ elements of D;

$\circ\ d$

$\uparrow$

$\circ\ c$ a pair of elements $c, d \in A$ with $f(c) = d$ (the possibilities $d \in D$ or $f(d) = d$ being not excluded).

2.2. Lemma. *Let $x, y, u, z \in A$. The following conditions are equivalent:*

(1) $f(u) = z \in D$, $f(x) = y \notin D$, $f(y) \notin \{x, y, z\}$.

(2) $z \in P$, $\Theta(x, y) = [x\, \Theta(x, y)] \notin S$, $\{z\} \notin \Theta(u, y)$, $\Theta(u, x) = [\{u, x\}, \{z, y\}]$.

Proof. Let (2) be valid. According to 2.1 we have $z \in D$. Since $\Theta(x, y) \notin S$, $\Theta(u, x) \notin S$, we obtain $y \notin D$, $x \notin D$, $u \notin D$ and hence $f(y)$, $f(x)$ and $f(u)$ exist. Further, we have $f(u)\ \Theta(u, x)\ f(x)$. If $f(u) = f(x)$ or $\{f(u), f(x)\} = \{u, x\}$, then $\Theta(u, x) \in S$, which is a contradiction. Therefore $\{f(u), f(x)\} = \{z, y\}$. Now let $f(u) = y$, $f(x) = z$. Then $\Theta(u, y) = [\{f^i(u) \colon i \in N \cup \{0\}\}]$, and the fact that $\{z\} \notin \Theta(u, y)$ implies that $z = f^i(u)$ for some $i \in N$. Obviously, $i \geqq 2$, since $z \notin \{u, y\}$. Let $i > 2$, i.e., $f(y) \neq z$. We obtain $\Theta(x, y) = [\{x, y\}, \{z, f(y)\}] \neq [x\, \Theta(x, y)]$, which is a contradiction. Thus $i = 2, f(y) = z$. But then $\Theta(x, y) \in S$,

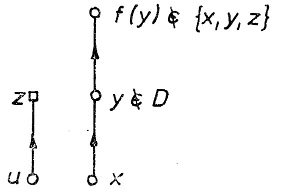


Fig. 2.2, 3.4

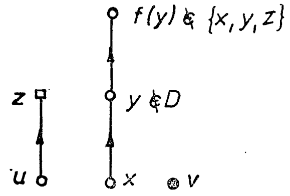


Fig. 2.2.1

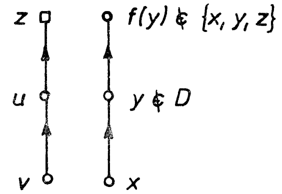


Fig. 2.2.2

a contradiction. Therefore $f(u) = z$, $f(x) = y$. If $f(y) \in \{x, y\}$, then $\Theta(x, y) \in S$; if $f(y) = z$, then $\{z\} \in \Theta(u, y) \in S$. Hence the condition (1) is satisfied.

2.2.1. Lemma. *Let $x, y, u, z, v \in A$. The following conditions are equivalent*:

(1) $f(u) = z \in D$, $f(x) = y \notin D$, $f(y) \notin \{x, y, z\}$, $f(v) = v$.

(2) $z \in P$, $\Theta(x, y) = [x\, \Theta(x, y)] \notin S$, $\{z\} \notin \Theta(u, y)$, $\Theta(u, x) = [\{u, x\}, \{z, y\}]$, $\Theta(v, u) = [\{v, u, z\}]$, $\Theta(v, x) = [v\, \Theta(v, x)]$.

Proof. Let (2) be valid. According to 2.2 we obtain $f(u) = z \in D$, $f(x) = y \notin D$, $f(y) \notin \{x, y, z\}$. Since $\Theta(v, u) \notin S$, we have $v \notin D$ and then $f(v)\, \Theta(v, u)\, z$, hence $f(v) \in \{v, u, z\}$. If $f(v) = z$, then $\Theta(v, x) = [\{v, x\}, \{z, y\}] \neq [v\, \Theta(v, x)]$, which is a contradiction. If $f(v) = u$, then $\Theta(v, x) = [\{v, x\}, \{u, y\}, \{z, f(y)\}] \neq [v\, \Theta(v, x)]$, a contradiction. Therefore $f(v) = v$.

2.2.2. Lemma. *Let $x, y, u, z, v \in A$. The following conditions are equivalent*:

(1) $f(u) = z \in D$, $f(x) = y \notin D$, $f(y) \notin \{x, y, z\}$, $f(v) = u$.

(2) $z \in P$, $\Theta(x, y) = [x\, \Theta(x, y)] \notin S$, $\{z\} \notin \Theta(u, y)$, $\Theta(u, x) = [\{u, x\}, \{z, y\}]$, $\Theta(v, u) = [\{v, u, z\}]$, $\{u, y\} \in \Theta(v, x)$.

Proof. Let (2) be valid. According to 2.2 we obtain $f(u) = z \in D$, $f(x) = y \notin D$,

$f(y) \notin \{x, y, z\}$. Further, $v \notin D$ and $f(v)\, \Theta(v, u)\, z$, hence $f(v) \in \{v, u, z\}$. If $f(v) = z$, then $\Theta(v, x) = [\{v, x\}, \{z, y\}]$, and if $f(v) = v$, then $\Theta(v, x) = [v\, \Theta(v, x)]$; in both the cases we obtained a contradiction. Therefore $f(v) = u$.

2.3. Lemma. *Let $u, z, x, y \in A$. The following conditions are equivalent*:

(1) $f(u) = z \in D$, $f(x) = y$, $f(y) = x$.

(2) $z \in P$, $\Theta(u, x) = [\{u, x\}, \{z, y\}]$, $\Theta(u, y) = [\{u, y\}, \{z, x\}]$, $\Theta(x, y) \in S$.

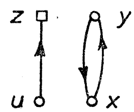


Fig. 2.3, 3.3

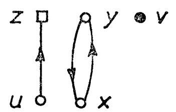


Fig. 2.3.1

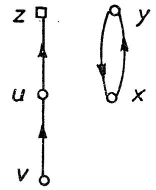


Fig. 2.3.2

Proof. If (2) is valid, then acccording to 2.1 we have $z \in D$, $u \notin D$, $x \notin D$, $y \notin D$. Further, $f(u)\, \Theta(u, x) f(x)$ and similarly as in the proof of 2.2 the relation $\{f(u), f(x)\} = \{z, y\}$ holds, since $\Theta(u, x) \in S$ in the remaining cases. Now let $f(u) = y$, $f(x) = z$. Then $z\, \Theta(x, y) f(y)$, and the fact that $\Theta(x, y) \in S$ yields that $f(y) = z$. This implies $\Theta(u, y) = [\{u, y, z\}]$, a contradiction. Hence $f(u) = z$, $f(x) = y$. Then $z\, \Theta(u, y) f(y)$, thus either $f(y) = z$ or $f(y) = x$ (in virtue of the relation $\Theta(u, y) = [\{u, y\}, \{z, x\}]$). In the case $f(y) = z$ we get $\Theta(u, y) \in S$, which is a contradiction, and therefore $f(y) = x$.

2.3.1. Lemma. *Let $x, y, z, u, v \in A$. The following conditions are equivalent*:

(1) $f(u) = z \in D$, $f(x) = y$, $f(y) = x$, $f(v) = v$.

(2) $z \in P$, $\Theta(u, x) = [\{u, x\}, \{z, y\}]$, $\Theta(u, y) = [\{u, y\}, \{z, x\}]$, $\Theta(x, y) \in S$, $\Theta(v, x) = \Theta(v, y) = [\{v, x, y\}]$.

Proof. Let (2) be valid. According to 2.3 we have $f(u) = z \in D$, $f(x) = y$, $f(y) = x$. Since $\Theta(v, x) \notin S$, we obtain $v \notin D$ and $f(v)\, \Theta(v, x)\, y$. This implies $f(v) \in \{v, x, y\}$. If $f(v) \in \{x, y\}$, then $\Theta(v, x) \neq \Theta(v, y)$, which is a contradiction. Thus $f(v) = v$.

2.3.2. Lemma. *Let $x, y, z, u, v \in A$. The following conditions are equivalent*:

(1) $f(u) = z \in D$, $f(x) = y$, $f(y) = x$, $f(v) = u$.

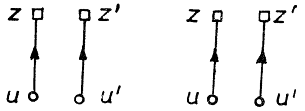
(2) $z \in P$, $\Theta(u, x) = [\{u, x\}, \{z, y\}]$, $\Theta(u, y) = [\{u, y\}, \{z, x\}]$, $\Theta(x, y) \in S$, $\Theta(v, u) = [\{v, u, z\}]$, $\{u\} \notin \Theta(v, x)$.

Proof. Let (2) be valid. From 2.3 it follows that $f(u) = z \in D$, $f(x) = y$, $f(y) = x$. Since $\Theta(v, x) \notin S$, we obtain $v \notin D$, $f(v)\, \Theta(v, u)\, z$, and this implies $f(v) \in \{v, z, u\}$. If $f(v) = v$, then $\Theta(v, x) = [\{v, x, y\}]$; if $f(v) = z$, then $\Theta(v, x) = [\{v, x\}, \{z, y\}]$. This is a contradiction, hence $f(v) = u$.

2.4. Lemma. *Let* $u, z, u', z' \in A$. *The following conditions are equivalent*:

(1) (a) $f(u) = z \in D, f(u') = z' \in D$, *or* (b) $f(u) = z' \in D, f(u') = z \in D$.
(2) $\{z, z'\} \subseteq P$, $\Theta(u, u') = [\{u, u'\}, \{z, z'\}]$.

Proof. If (2) is valid, then 2.1 implies that $\{z, z'\} \subseteq D$, $u \notin D$, $u' \notin D$. Further, we have $f(u)\ \Theta(u, u')\ f(u')$. If $\{f(u), f(u')\} = \{u, u'\}$ or $f(u) = f(u')$, then $\Theta(u, u') \in S$. Hence $\{f(u), f(u')\} = \{z, z'\}, f(u) \neq f(u')$, and thus the condition (1) is satisfied.



(a) (b)

Fig. 2.4, 3.5

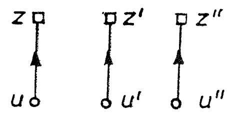


Fig. 2.5, 3.1

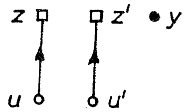


Fig. 2.6, 3.2

2.5. Lemma. *Let* $u, u', u'', z, z', z'' \in A$. *The following conditions are equivalent*:

(1) $f(u) = z$, $f(u') = z'$, $f(u'') = z''$, $\{z, z', z''\} \subseteq D$.
(2) $\{z, z', z''\} \subseteq P$, $\Theta(u, u') = [\{u, u'\}, \{z, z'\}]$, $\Theta(u, u'') = [\{u, u''\}, \{z, z''\}]$.

Proof. If (2) is valid, then from 2.4 we have either (a) $f(u) = z \in D$, $f(u') = z' \in D$, or (b) $f(u) = z' \in D, f(u') = z \in D$. Similarly, for the elements u, z, u'', z'' it follows from 2.4 that either (a') $f(u) = z \in D$, $f(u'') = z'' \in D$, or (b') $f(u) = z'' \in D$, $f(u'') = z \in D$. It is obvious that (a) and (a') hold, hence the condition (1) is satisfied.

2.6. Lemma. *Let* $u, z, u', z', y \in A$. *The following conditions are equivalent*:

(1) $f(u) = z \in D$, $f(u') = z' \in D$, $f(y) = y$.
(2) $\{z, z'\} \subseteq P$, $\Theta(u, u') = [\{u, u'\}, \{z, z'\}]$, $\Theta(y, u) = [\{y, u, z\}]$, $\Theta(y, u') = [\{y, u', z'\}]$.

Proof. Let (2) be valid. Since $\Theta(y, u) \notin S$, we have $y \notin D$. Further, from 2.4 it follows that either (a) $f(u) = z \in D$, $f(u') = z' \in D$, or (b) $f(u) = z' \in D$, $f(u') = z \in D$. In the case (b) we obtain $z'\ \Theta(u, y)\ f(y)$, hence $f(y) = z'$. Then $\Theta(y, u') = [\{y, u'\}, \{z, z'\}]$, which is a contradiction. In the case (a) we have $z\ \Theta(u, y)\ f(y)$, thus $f(y) \in \{y, z, u\}$. Further, $z'\ \Theta(u', y)\ f(y)$, which implies $f(y) \in \{y, u', z'\}$. Therefore we obtain that $f(y) = y$.

2.7. Lemma. *Let $x, y, z \in A$. The following conditions are equivalent:*

(1) (a) $f(x) = y$, $f(y) = z \in D$, *or* (b) $f(y) = x$, $f(x) = z \in D$, *or* (c) $f(y) = z \in D, f(x) = x$, *or* (d) $f(x) = z \in D, f(y) = y$.

(2) $z \in P$, $\Theta(x, y) = [\{x, y, z\}]$.

Proof. Let (2) be valid. Then $z \in D$, $x \notin D$, $y \notin D$ according to 2.1. Hence we have $f(x)\,\Theta(x, y)\,f(y)$. If $f(x) = f(y)$, then $\Theta(x, y) \in S$, which is a contradiction, and therefore $f(x) \neq f(y)$, $\{f(x), f(y)\} \subseteq \{x, y, z\}$. If $\{f(x), f(y)\} = \{x, y\}$, then $\Theta(x, y) \in S$, a contradiction. The case $f(x) = y, f(y) = z$ is the case (a); if $f(x) = z$, $f(y) = x$, we have the case (b); if $f(x) = x, f(y) = z$, then (c) is valid, and if $f(x) = z, f(y) = y$, then (d) is valid.

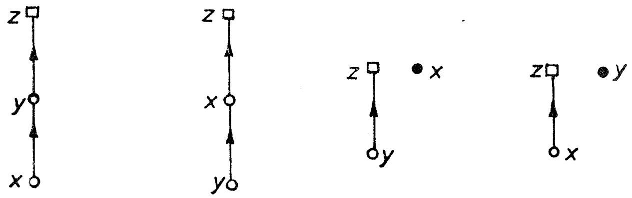


Fig. 2.7, 3.14

2.8. Lemma. *Let $x, x', y, z \in A$. The following conditions are equivalent:*

(1) (a) $f(x) = f(x') = y$, $f(y) = z \in D$, *or* (b) $f(x) = f(x') = z \in D$, $f(y) = y$, *or* (c) $f(y) = z \in D, f(x) = x, f(x') = x'$.

(2) $z \in P$, $\Theta(x, y) = [\{x, y, z\}]$, $\Theta(x, x') \in S$, $\Theta(x', y) = [\{x', y, z\}]$.

Proof. If (2) is valid, then $z \in D, x \notin D, x' \notin D, y \notin D$. According to 2.7 some of the following conditions is satisfied: (a') $f(x) = y$, $f(y) = z$; (b') $f(y) = x$, $f(x) = z$;

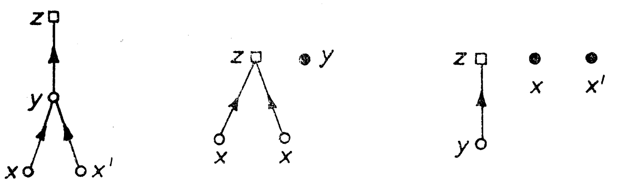


Fig. 2.8, 3.12

(c') $f(y) = z, f(x) = x$; (d') $f(x) = z, f(y) = y$. Let (a') hold. Then $f(x')\,\Theta(x', x)\,y$, hence $f(x') = y$, and it is the case (a). If (b') is valid, then $f(x')\,\Theta(x', x)\,z$, thus $f(x') =$

$= z$ and $\Theta(x', y) = [\{x', y\}, \{x, z\}]$, which is a contradiction. Let (c′) hold. Then $f(x')\,\Theta(x', x)\,x$, and this implies that either $f(x') = x$ or $f(x') = x'$ holds. If $f(x') = x$, then $\Theta(x', y) = [\{x', y\}, \{z, x\}]$, a contradiction. Thus $f(x') = x'$, and we have the condition (c). If (d′) is satisfied, then $f(x')\,\Theta(x', x)\,z$, hence $f(x') = z$, and this is the case (b).

2.9. Lemma. *Let $x, x', y, z, u \in A$. The following conditions are equivalent:*

(1) (a) $f(x) = f(x') = y$, $f(y) = z \in D$, $f(u) = u$, *or* (b) $f(x) = x$, $f(x') = x'$, $f(u) = y$, $f(y) = z \in D$.

(2) $z \in P$, $\Theta(x, y) = [\{x, y, z\}]$, $\Theta(x, x') \in S$, $\Theta(x', y) = [\{x', y, z\}]$, $\Theta(u, y) = [\{u, y, z\}]$, $\Theta(u, x) \notin S$.

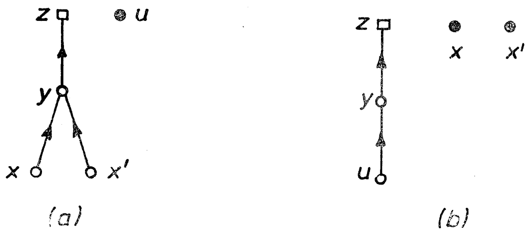


Fig. 2.9, 3.10

Proof. Let (2) be valid. Then $z \in D$, $u \notin D$ in virtue of 2.1. Further, 2.8 implies that some of the following three conditions is satisfied: (a′) $f(x) = f(x') = y, f(y) = z$; (b′) $f(x) = f(x') = z$, $f(y) = y$; (c′) $f(y) = z$, $f(x) = x$, $f(x') = x'$. Suppose that (a′) holds. Then $f(u)\,\Theta(u, y)\,z$, hence $f(u) \in \{u, y, z\}$. If $f(u) = z$, then $\Theta(u, y) \in S$, which is a contradiction. If $f(u) = y$, then $\Theta(u, x) \in S$, a contradiction. Thus $f(u) = u$, and it is the case (a). Now let (b′) be valid. Then $f(u)\,\Theta(u, y)\,y$, $f(u) \in \{u, y, z\}$. If $f(u) \in \{u, y\}$ we obtain $\Theta(u, y) \in S$; if $f(u) = z$, then $\Theta(u, x) \in S$. Hence (b′) does not hold. In the case (c′) we have $f(u) \in \{u, y, z\}$. If $f(u) = u$, then $\Theta(u, x) \in S$; if $f(u) = z$, then $\Theta(u, y) \in S$. This is a contradiction, and therefore $f(u) = y$; thus the condition (b) is satisfied.

2.10. Lemma. *Let $x, x', y, y', z \in A$. The following conditions are equivalent:*

(1) (a) $f(x) = f(x') = z \in D$, $f(y) = y$, $f(y') = y'$, *or* (b) $f(y) = f(y') = z \in D$, $f(x) = x$, $f(x') = x'$.

(2) $z \in P$, $\Theta(x, y) = [\{x, y, z\}]$, $\Theta(x, x') \in S$, $\Theta(x', y) = [\{x', y, z\}]$, $\Theta(y, y') \in S$, $\Theta(x, y') = [\{x, y', z\}]$.

Proof. Let (2) be valid. Then $z \in D$, $y' \notin D$ in virtue of 2.1. From 2.8 it follows that either (a′) $f(x) = f(x') = y$, $f(y) = z$, or (b′) $f(x) = f(x') = z$, $f(y) = y$, or (c′) $f(y) = z, f(x) = x, f(x') = x$. Suppose that (a′) holds. Then $f(y')\,\Theta(y', y)\,z$, hence $f(y') = z$ and $\Theta(x, y') = [\{x, y'\}, \{y, z\}]$, a contradiction. Let (b′) hold.

Then $f(y')\,\Theta(y', y)\,y$ implies that $f(y') \in \{y, y'\}$. If $f(y') = y$, we have $\Theta(x, y') = [\{x, y'\}, \{z, y\}]$, which is a contradiction, therefore $f(y') = y'$ and (a) is valid. If (c') holds, then $f(y')\,\Theta(y', y)\,z$, thus $f(y') = z$, and we have the case (b).

Fig. 2.10, 3.11

2.11. Lemma. *Let $x, y, z, u \in A$. The following conditions are equivalent:*

(1) (a) $f(x) = y, f(y) = z \in D, f(u) = u$, *or* (b) $f(u) = y, f(y) = z \in D, f(x) = x$.

(2) $z \in P,\ \Theta(x, y) = [\{x, y, z\}],\ \Theta(u, x) = [\{u, x, y, z\}],\ \Theta(u, y) = [\{u, y, z\}]$.

Proof. Let (2) be valid. Then $z \in D$, $u \notin D$ with respect to 2.1. Further, 2.7 implies that some of the following conditions is satisfied: (a') $f(x) = y, f(y) = z$; (b') $f(y) = x$, $f(x) = z$; (c') $f(y) = z$, $f(x) = x$; (d') $f(x) = z$, $f(y) = y$. We always have

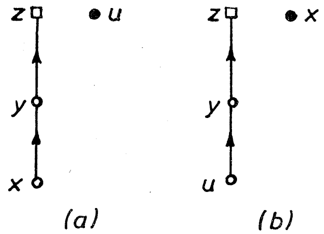


Fig. 2.11, 3.13

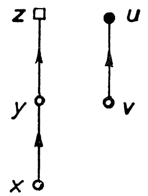


Fig. 2.12, 3.7

$f(x) \in \{x, y, z\}$, $f(u)\,\Theta(u, x)\,f(x)$, hence $f(u) \in \{u, x, y, z\}$. Suppose that (a') holds. If $f(u) = x$, then $\Theta(u, y) = [\{u, y\}, \{x, z\}]$; if $f(u) = y$, then $\Theta(u, x) \in S$; if $f(u) = z$, then $\Theta(u, x) = [\{u, x\}, \{z, y\}]$. Thus the only possibility is $f(u) = u$, and we have the case (a). If (b') is valid, then $f(u)\,\Theta(u, y)\,x$, hence $f(u) = x$, but then $\Theta(u, x) = [\{u, x, z\}]$, which is a contradiction. Suppose that (c') is valid. We obtain $f(u)\,\Theta(u, y)\,z$, hence $f(u) \in \{u, y, z\}$. If $f(u) = z$, then $\Theta(u, y) \in S$; if $f(u) = u$, then $\Theta(u, x) \in S$. In both the cases we have obtained a contradiction. Hence $f(u) = y$, and therefore (b) is valid. Now let (d') hold. If $f(u) \in \{u, y\}$, then $\Theta(u, y) \in S$, and if $f(u) \in \{x, y\}$, then $\Theta(u, y) \neq [\{u, y, z\}]$, a contradiction.

2.12. Lemma. *Let $x, y, z, u, v \in A$. The following conditions are equivalent:*

(1) $f(x) = y,\ f(y) = z \in D,\ f(v) = f(u) = u$.

(2) $z \in P$, $\Theta(x, y) = [\{x, y, z\}]$, $\Theta(u, x) = [\{u, x, y, z\}]$, $\Theta(u, y) = [\{u, y, z\}]$, $\Theta(v, u) \in S$, $\{u\} \notin \Theta(v, x)$.

Proof. Let (2) be valid. Then $z \in D$, $v \notin D$ according to 2.1. From 2.11 it follows that either (a) $f(x) = y$, $f(y) = z$, $f(u) = u$, or (b) $f(u) = y$, $f(y) = z$, $f(x) = x$. In the case (b) we obtain $f(v)\ \Theta(v, u)\ y$, thus $f(v) = y$. But then $\{u\} \in \Theta(v, x)$, and this is a contradiction. If (a) holds, then $f(v)\ \Theta(v, u)\ u$, hence $f(v) \in \{v, u\}$. If $f(v) = v$, we get $\{u\} \in \Theta(v, x)$, a contradiction. Therefore $f(v) = u$ and the condition (1) is satisfied.

2.13. Lemma. *Let $x, y, z, u \in A$. The following conditions are equivalent*:

(1) (a) $f(x) = y, f(y) = f(u) = z \in D$, *or* (b) $f(x) = z \in D, f(u) = f(y) = y$.

(2) $z \in P$, $\Theta(x, y) = [\{x, y, z\}]$, $\Theta(u, y) \in S$, $\Theta(u, x) = [\{u, x\}, \{y, z\}]$.

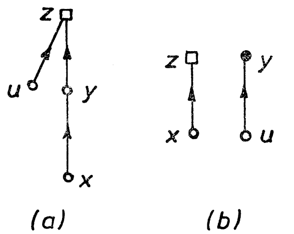


Fig. 2.13, 3.9

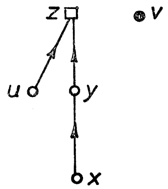


Fig. 2.14, 3.6(a)

Proof. If (2) is valid, then 2.1 implies that $z \in D$, $u \notin D$. Further, according to 2.7, one of the following conditions is satisfied: (a′) $f(x) = y, f(y) = z$; (b′) $f(y) = x$, $f(x) = z$; (c′) $f(y) = z$, $f(x) = x$; (d′) $f(x) = z$, $f(y) = y$. Let (a′) hold. Then $f(u)\ \Theta(u, y)\ z$, hence $f(u) = z$, and it is the case (a). If (b′) is valid, then $f(u)\ \Theta(u, y)\ x$, hence $f(u) = x$, and then $\Theta(u, x) = [\{u, x, z\}]$, which is a contradiction. If (c′) is valid, then $f(u)\ \Theta(u, y)\ z$, thus $f(u) = z$, but this implies a contradiction, namely $\Theta(u, x) = [\{u, x, z\}]$. Suppose that (d′) holds. We have $f(u)\ \Theta(u, y)\ y$, and hence $f(u) \in \{u, y\}$. If $f(u) = u$, then $\Theta(u, x) = [\{u, x, z\}]$, which is impossible, therefore $f(u) = y$ and the condition (b) is satisfied.

2.14. Lemma. *Let $x, y, z, u, v \in A$. The following conditions are equivalent*:

(1) $f(x) = y, f(y) = f(u) = z \in D, f(v) = v$.

(2) $z \in P$, $\Theta(x, y) = [\{x, y, z\}]$, $\Theta(u, y) \in S$, $\Theta(u, x) = [\{u, x\}, \{y, z\}]$, $\Theta(v, u) = [\{v, u, z\}]$, $\Theta(v, y) = [\{v, y, z\}]$.

Proof. Let (2) be valid. Then $z \in D$, $v \notin D$ with respect to 2.1. According to 2.13 we have either (a) $f(x) = y$, $f(y) = f(u) = z$, or (b) $f(x) = z$, $f(u) = f(y) = y$. Suppose that (b) holds. We obtain $f(v)\ \Theta(v, u)\ y$ and hence $f(v) = y$. Then $\Theta(v, y) \in S$, which is a contradiction. Let us consider the case (a). We have $f(v)\ \Theta(v, u)\ z$, thus $f(v) \in \{v, u, z\}$. If $f(v) = z$, then $\Theta(v, u) \in S$; if $f(v) = u$, then $\Theta(v, y) = [\{v, y\}, \{u, z\}]$. Hence we get a contradiction except for the possibility $f(v) = v$.

2.15. Lemma. *Let $x, y, z, u, v \in A$. The following conditions are equivalent*:

(1) $f(x) = y$, $f(y) = f(u) = z \in D$, $f(v) = u$.

(2) $z \in P$, $\Theta(x, y) = [\{x, y, z\}]$, $\Theta(u, y) \in S$, $\Theta(u, x) = [\{u, x\}, \{y, z\}]$, $\Theta(v, u) = [\{v, u, z\}]$, $\Theta(v, y) = [\{v, y\}, \{u, z\}]$.

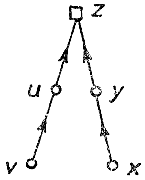


Fig. 2.15, 3.6(b)

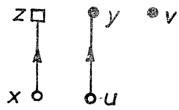


Fig. 2.16, 3.8

Proof. Let (2) be valid. Then $z \in D$, $v \notin D$ according to 2.1. From 2.13 it follows that either (a) $f(x) = y$, $f(y) = f(u) = z$, or (b) $f(x) = z$, $f(u) = f(y) = y$. If (b) holds, then we have $f(v)\ \Theta(v, u)\ y$, hence $f(v) = y$, but then $\Theta(v, y) \in S$, which is a contradiction. In the case (a) we obtain $f(v)\ \Theta(v, u)\ z$, thus $f(v) \in \{z, v, u\}$. If $f(v) = z$, then $\Theta(v, y) \in S$; if $f(v) = v$, then $\Theta(v, y) = [\{v, y, z\}]$. Hence we have a contradiction, and the remaining case is $f(v) = u$.

2.16. Lemma. *Let $x, y, z, u, v \in A$. The following conditions are equivalent*:

(1) $f(x) = z \in D$, $f(u) = f(y) = y$, $f(v) = v$.

(2) $z \in P$, $\Theta(x, y) = [\{x, y, z\}]$, $\Theta(u, y) \in S$, $\Theta(u, x) = [\{u, x\}, \{y, z\}]$, $\Theta(v, u) = [\{v, u, y\}]$, $\Theta(v, y) \in S$.

Proof. Let (2) be valid. Then $z \in D$, $v \notin D$ in virtue of 2.1. Further, 2.13 implies that either (a) $f(x) = y$, $f(y) = f(u) = z$, or (b) $f(x) = z$, $f(u) = f(y) = y$. In the case (a) we have $f(v)\ \Theta(v, u)\ z$, hence $f(v) = z$, and then $\Theta(v, u) \in S$, which is a contradiction. In the case (b) we obtain $f(v)\ \Theta(v, u)\ y$, thus $f(v) \in \{y, v, u\}$. If $f(v) = y$, then $\Theta(v, u) \in S$, a contradiction. If $f(v) = u$, then $\Theta(v, y) = [\{v, y, u\}]$, which is a contradiction as well. Therefore $f(v) = v$.

3. UPPER BOUND FOR card $R(f)$

We start with a lemma which has an auxiliary character.

We suppose that the system Con (A, f) is given and that Con $(A, f) \neq E(A)$.

3.0. Lemma. (i) *The set D is uniquely determined by* Con (A, f).

(ii) *Let $u, z \in A$, $f(u) = z \in D$. Further, let $a \in A - \{u, z\}$, $a \notin D$. The case* *when* $f(a) \notin \{a, u\} \cup D$ *can be characterized by* $\mathrm{Con}(A, f)$ *and in this case* $f(a)$ *is uniquely determined by* $\mathrm{Con}(A, f)$.

Proof. The assertion (i) follows from 2.1. Let the assumptions of (ii) be satisfied. We distinguish the following cases.

a) There exists $b \in A - \{u, z, a\}$ such that u, z, a, b fulfil the condition that we get from (2) of 2.3 if we write u, z, a, b instead of u, z, x, y. According to 2.3 we obtain $f(a) = b$, $f(b) = a$.

b) There exists $b \in A - \{u, z, a\}$ such that u, z, a, b fulfil the condition that we obtain from (2) of 2.2 writing u, z, a, b instead of u, z, x, y. From 2.2 it follows that $f(a) = b$ and $b \notin D$, $f(b) \notin \{a, b, z\}$.

c) Neither a) nor b) holds. Then the remaining cases are: α) $f(a) \in \{a, u\} \cup D$, β) $f^2(a) = z$, $f(a) \neq u$, γ) $f^2(a) = f(a) \neq a$. We have to characterize the cases β) and γ) by means of $\mathrm{Con}(A, f)$.

c1) First suppose that there exists $b \in A - \{a, u, z\}$ such that the condition obtained from (2) of 2.13 writting u, b, a, z instead of x, y, u, z is satisfied. Then 2.13 implies that either $f(u) = b$, $f(a) = f(b) = z$, or $f(u) = z$, $f(a) = f(b) = b$. Since we assume that $f(u) = z$, we have $f(a) = f(b) = b$, and this is the case γ).

c2) Suppose that the condition assumed to be valid in c1) does not hold and that there exists $b \in A - \{a, b, z\}$ such that a, b, u, z fulfil the condition which we get from (2) of 2.13 writing a, b, u, z instead of x, y, u, z. Then 2.13 implies that either $f(a) = b, f(u) = f(b) = z$, or $f(a) = z, f(u) = f(b) = b$. Because the relation $f(u) = z$ is valid, we have $f(a) = b \neq u$, $f(b) = z$. This is the case β).

Now we shall investigate the following question: Assume that the system $\mathrm{Con}(A, f) \neq E(A)$ is given. To what extend are we able to reconstruct the partial operation f?

First we notice that the conditions (2) in 2.2.–2.16 are expressed merely by the properties of the system $\mathrm{Con}(A, f)$, without using explicitly the partial operation f itself.

In the following lemmas distinct cases concerning $\mathrm{Con}(A, f)$ will be investigated; e.g., in Lemma 3.5 we suppose that no condition assumed in Lemmas 3.1–3.4 is valid. Again let us remark that the figure which is related to some of the following lemmas is denoted by the same number as the corresponding lemma.

3.1. Lemma. *Let there exist distinct elements* $u, z, u', z', u'', z'' \in A$ *fulfilling the condition* (2) *from* 2.5. *Then* f *is uniquely determined by* $\mathrm{Con}(A, f)$.

Proof. From 2.5 it follows that $f(u) = z \in D$, $f(u') = z' \in D$, $f(u'') = z'' \in D$. Let $a \in A - \{u, z, u', z', u'', z''\}$. With respect to 3.0 we obtain that the only case we have to investigate is the case $f(a) \in \{a, u\} \cup D$. If a is such that u, z, u', z', a fulfil the condition that we get from (2) of 2.6 (writing u, z, u', z', a instead of u, z, u', z', y), then $f(a) = a$. If there exists $b \in A - \{a, u, z, u', z'\}$ such that u, z, u', z', a, b fulfil the condition that we obtain from (2) of 2.5 (with u, z, u', z', a, b

instead of u, z, u', z', u'', z''), then $f(a) = b \in D$. Notice that here $b = z''$ can hold. From 3.0 (for u', z' instead of u, z) it follows that the case $f(a) = u$ can be characterized by $\operatorname{Con}(A, f)$. Hence $f(a)$ is uniquely determined by $\operatorname{Con}(A, f)$ for each $a \in A$.

Let the assumption of 3.1 be not satisfied.

3.2. Lemma. *Let there exist distinct elements $u, z, u', z', y \in A$ fulfilling the condition* (2) *from* 2.6. *Then f is uniquely determined by* $\operatorname{Con}(A, f)$.

Proof. From 2.6 it follows that $f(u) = z \in D$, $f(u') = z' \in D$, $f(y) = y$. Let $a \in A - \{u, z, u', z', y\}$. According to 3.0, the only case we have to investigate is the case $f(a) \in \{a, u\} \cup D$. If we consider 3.0 for the elements u', z' instead of u, z, then we see that $f(a)$ is uniquely determined by $\operatorname{Con}(A, f)$ for $f(a) \notin \{a, u'\} \cup D$. Since the assumption of 3.1 is not satisfied, we obtain $f(a) \notin D - \{z, z'\}$. Hence we only have to characterize the case $f(a) \in \{a, z, z'\}$ by $\operatorname{Con}(A, f)$.

First suppose that u, z, u', z', a fulfil the condition that we get from (2) of 2.6 (with u, z, u', z', a instead of u, z, u', z', y). Then $f(a) = a$. The case when $f(a) = z$ can be described by $\operatorname{Con}(A, f)$ according to 2.6 (with a, z, u', z', y instead of u, z, u', z', y), and the case $f(a) = z'$ is analogous.

Let the assumptions of 3.1 and 3.2 be not satisfied.

3.3. Lemma. *Let there exist distinct elements $u, z, x, y \in A$ fulfilling the condition* (2) *from* 2.3. *Then f is uniquely determined by* $\operatorname{Con}(A, f)$.

Proof. Let $a \in A - \{x, y, u, z\}$. According to 3.0, if $f(a) \notin \{a, u\} \cup D$, then $f(a)$ is uniquely determined by $\operatorname{Con}(A, f)$. From 2.3.1 (with u, z, x, y, a instead of u, z, x, y, v) it follows that the case $f(a) = a$ can be characterized by $\operatorname{Con}(A, f)$; by 2.3.2 (with u, z, x, y, a instead of u, z, x, y, v), the case $f(a) = u$ can be characterized by $\operatorname{Con}(A, f)$, and the case $f(a) \in D$ follows from 2.3 (there exists $b \in D$ such that a, b, x, y fulfil the condition that we obtain from (2) of 2.3 for a, b, x, y instead of u, z, x, y).

Let the assumptions of 3.1 – 3.3 be not satisfied.

3.4. Lemma. *Let there exist distinct elements $u, z, x, y \in A$ fulfilling the condition* (2) *from* 2.2. *Then f is uniquely determined by* $\operatorname{Con}(A, f)$.

Proof. Let $a \in A - \{x, y, u, z\}$. If $f(a) \notin \{a, u\} \cup D$, then $f(a)$ is uniquely determined by $\operatorname{Con}(A, f)$ according to 3.0. From 2.2.1 it follows that the case when $f(a) = a$ can be characterized by $\operatorname{Con}(A, f)$; the same is valid for the case $f(a) = u$ (in view of 2.2.2) and for the case when $f(a) \in D$, $f(a) \neq f(y)$ (in view of 2.2). Suppose that there is $b \in D - \{z\}$ such that x, y, b, a fulfil the condition that we obtain from (2) of 2.13 (with x, y, b, a instead of x, y, z, u). Then 2.13 implies that either (a) $f(x) = y, f(y) = f(a) = b$, or (b) $f(x) = b, f(a) = f(y) = y$. In the case (b) the elements u, z, x, b, y fulfil the condition that we obtain from (2) of 2.6 (with u, z, x, b, y instead of u, z, u', z', y), which is a contradiction with the fact that the assumption of 3.2 is not satisfied. Hence in this case $f(a) = f(y) = b \in D$, and the proof is complete.

Let us define the following notions. We shall say that the elements x and x' of A behave in the same way, if one of the conditions (a)–(c) is satisfied:

(a) $f(x) = x,\ f(x') = x'$;

(b) $f(x) = f(x'),\ f^{-1}(x) = \emptyset = f^{-1}(x')$;

(c) $x, x' \in D$.

Let (A_1, f_1) be a partial monounary algebra and let $\emptyset \neq B_1 \subseteq A_1$ such that $f_1(t) \in B_1$ whenever $t \in B_1 \cap (A_1 - D_{f_1})$. Suppose that for each $a \in A_1 - B_1$ there exists $b \in B_1$ such that the elements a and b behave in the same way (as elements of the partial algebra (A_1, f_1)). Under these assumptions we shall say that (A_1, f_1) is a c_0-*extension* of the partial algebra (B_1, f_1).

Let the assumptions of 3.1–3.4 be not satisfied.

3.5. Lemma. *Let there exist distinct elements $u, z, u', z' \in A$ fulfilling the condition* (2) *from* 2.4. *Then $R(f)$ consists of two elements, which can be described by means of* Con (A, f). *The partial algebra (A, f) is a c_0-extension of some of the partial algebras given in Fig.* 3.5 (a) *or* 3.5 (b).

Proof. The elements $u, z, u', z' \in A$ fulfil the condition (1) from 2.4 and the assumptions of Lemmas 3.2, 3.3, 3.4 are not satisfied, hence $B = \emptyset$. Further, the assumption of 3.1 is not satisfied, which implies that $f^{-1}(t) = \emptyset$ for each $t \in D - \{z, z'\}$. The fact that the assumption of 3.4 does not hold, implies that $f^{-1}(u') = \emptyset$ and $f^{-1}(u) = \emptyset$. Let $a \in A - \{u, z, u', z'\}$. Then $a \in D$ if and only if $a \in P$ (in view of 2.1). Now let $a \notin D$. Obviously, $f(a) = f(u)$ if and only if $\Theta(a, u) \in S$. Denote $U = \{u\} \cup \{a \in A - \{u, z, u', z'\} : \Theta(a, u) \in S\}$, $U' = \{u'\} \cup \{a \in A - \{u, z, u', z'\} : \Theta(a, u') \in S\}$. Then $A = D \cup U \cup U'$, and we have either $f(a) = z, f(a') = z'$, or $f(a) = z', f(a') = z$ for each $a \in U$, $a' \in U'$. It is obvious that these two cases can not be distinguished by means of Con (A, f).

Let the assumptions of 3.1–3.5 be not satisfied.

3.6. Lemma. *Let there exist distinct elements $x, y, z, u, v \in A$ fulfilling the condition* (2) *from* 2.14 *or* 2.15. *Then f is uniquely determined by* Con (A, f).

Proof. (Cf. Fig. 3.6 (a) and 3.6 (b).) Let $a \in A - \{x, y, z, u, v\}$, $a \notin D = P$. From 3.0 (for u, z, resp. y, z instead of u, z) it follows that we have to characterize by Con (A, f) the case when $f(a) \in \{a\} \cup D$. The case $f(a) = a$ can be described by 2.14. Since the assumption of 3.5 is not satisfied, we have $f(a) \notin D - \{z\}$. Further, $f(a) = z \in D$ if and only if $\Theta(a, y) \in S$ and $a \notin P$. Thus $f(a)$ is determined by Con (A, f).

Let the assumptions of 3.1 – 3.6 be not satisfied.

3.7. Lemma. *Let there exist distinct elements $x, y, z, u, v \in A$ fulfilling the condition* (2) *from* 2.12. *Then f is uniquely determined by* Con (A, f).

Proof. Let $a \in A - \{x, y, z, u, v\}$, $a \notin D = P$. From 3.0 (with y, z instead of u, z) it follows that we have to investigate only the case when $f(a) \in \{a, y\} \cup D$. The relation $f(a) = a$ holds if and only if $a \notin P$, $\Theta(a, u) \in S$, $\Theta(a, v) = [\{a, v, u\}]$. Further, we have $f(a) = y$ if and only if $a \notin P$, $\Theta(a, x) \in S$. The assumption of 3.5 is not satisfied, therefore $f(a) \notin D - \{z\}$. Further, $f(a) = z \in D$ if and only if $a \notin P$ and $\Theta(a, y) \in S$, completing the proof.

Let the assumptions of 3.1 – 3.7 be not satisfied. Let us consider the meaning of this fact in the case when there is $z \in D$ with $f^{-1}(z) \neq \emptyset$. Since the assumptions of 3.3 and 3.5 do not hold, we obtain that $f^{-1}(t) = \emptyset$ for each $t \in D - \{z\}$ and that B contains no cycle with two elements. From the fact that the assumption of 3.4 is not satisfied it follows that $\{f(t)\}$ is a one-element cycle for each $t \in B$ and also that $f^{-3}(z) = \emptyset$. Further, the assumption of 3.6 fails to hold, thus if $x, x' \in f^{-2}(z)$, then $f(x) = f(x')$, and if card $f^{-1}(z) \geqq 2$ and $f^{-2}(z) \neq \emptyset$, then $B = \emptyset$. The assumption of 3.7 is not satisfied, hence if the relation $f^{-2}(z) \neq \emptyset$ is valid, then either B consists of one-element cycles or $B = \emptyset$.

3.8. Lemma. *Let there exist distinct elements $x, y, z, u, v \in A$ fulfilling the condition* (2) *from* 2.16. *Then f is uniquely determined by* Con (A, f).

Proof. Let $a \in A - \{x, y, z, u, v\}$, $a \notin D = P$. From 3.0 (for x, z instead of u, z) it follows that we have to characterize $f(a)$ by means of Con (A, f) only in the case $f(a) \in \{a, x\} \cup D$. Further, according to the facts mentioned above, we obtain that if $f(a) \in D$, then $f(a) = z$. The case $f(a) = a$ or $f(a) = z$ can be described by means of Con (A, f) in view of 2.16 (for the elements x, z, u, y, a resp. a, z, u, y, v instead of x, z, u, y, v). The case when $f(a) = x$ is impossible, since we suppose that the assumption of 3.7 does not hold (consider the elements a, x, z, u, y instead of x, y, z, v, u).

Let the assumptions of 3.1 – 3.8 be not satisfied.

3.9. Lemma. *Let there exist distinct elements $x, y, z, u \in A$ fulfilling the condition* (2) *from* 2.13. *Then $R(f)$ consists of two elements, which can be described by means of* Con (A, f). *The partial algebra (A, f) is a c_0-extension of some of the partial algebras given in Fig.* 3.9 (a) *or* 3.9 (b).

Proof. In view of 2.13 we obtain either (a) $f(x) = y$, $f(y) = f(u) = z \in D$, or (b) $f(x) = z \in D$, $f(u) = f(y) = y$. Let $a \in A - \{x, y, z, u\}$, $a \notin D = P$. Consider the case (a). From the facts which were mentioned above when formulating Lemma 3.8 it follows that $f(a) \notin f^{-1}(D - \{z\})$; $f(a) \neq u$ and $f(a) \notin f^{-1}(z) - \{y\}$; $f^{-3}(z) = \emptyset$; $B = \emptyset$. Hence $f(a) \in \{z, y\}$. Now consider the case (b). Then $f(a) \notin$

$\notin f^{-1}(D - \{z\})$; $f^{-2}(z) = \emptyset$ (the assumption of 3.7 is not satisfied); $f(a) \neq a$ (the assumption of 3.8 is not satisfied), and hence $f(t) = y$ for each $t \in B$. Therefore $f(a) \in \{z, y\}$ in the case (b) as well as in the case (a). Further, we have: $f(a) = f(x)$ if and only if $\Theta(a, x) \in S$, and $f(a) = f(u)$ if and only if $\Theta(a, u) \in S$. Denote

$$X = \{x\} \cup \{a \in A - D : \Theta(a, x) \in S\},$$

$$U = \{u\} \cup \{a \in A - D : \Theta(a, u) \in S\}.$$

Then $A = D \cup X \cup U$ and either (a) $f(x_1) = y$, $f(u_1) = f(y) = z$, or (b) $f(x_1) = z$, $f(u_1) = f(y) = y$ is valid for each $x_1 \in X$, $u_1 \in U$. It is obvious that these cases can not be distinguished by means of the system Con (A, f).

Let the assumptions of 3.1 – 3.9 be not satisfied.

3.10. Lemma. *Let there exist distinct elements $x, x', y, z, u \in A$ fulfilling the condition* (2) *from* 2.9. *Then $R(f)$ consists of two elements, which can be described by means of* Con (A, f). *The partial algebra (A, f) is a c_0-extension of some of the partial algebras given in Fig.* 3.10 (a) *or* 3.10 (b).

Proof. From 2.9 it follows that either (a) $f(x) = f(x') = y$, $f(y) = z \in D$, $f(u) = u$, or (b) $f(u) = y$, $f(y) = z \in D$, $f(x) = x$, $f(x') = x'$. Since the assumptions of 3.1 – 3.9 are not satisfied, we obtain that each element $a \in A - D$, $a \neq y$, behaves in the same way as x or as u (in the case (a) and also in the case (b)). We denote by the symbol X or U the set of all elements $a \in A - D$ such that a and x or a and u, respectively, behave in the same way. We have to characterize the sets X and U by the system Con (A, f). It is easy to see that

$$X = \{x\} \cup \{a \in A - D : \Theta(a, x) \in S,\ \Theta(a, u) = [\{a, u, y, z\}]\},$$

$$U = \{u\} \cup \{a \in A - D : \Theta(a, u) \in S,\ \Theta(a, x) = [\{a, x, y, z\}]\}.$$

Then $A = D \cup \{y\} \cup X \cup U$ and either (a) $f(x_1) = y$, $f(y) = z$, $f(u_1) = u_1$, or (b) $f(u_1) = y$, $f(y) = z$, $f(x_1) = x_1$ for each $x_1 \in X$, $u_1 \in U$. It is obvious that these two cases cannot be distinguished by means of congruence relations.

Let the assumptions of 3.1 – 3.10 be not satisfied.

3.11. Lemma. *Let there exist distinct elements $x, x', y, y', z \in A$ fulfilling the condition* (2) *from* 2.10. *Then $R(f)$ consists of two elements, which can be described by means of* Con (A, f). *The partial algebra (A, f) is a c_0-extension of some of the partial algebras given in Fig.* 3.11 (a) *or* 3.11 (b).

Proof. In view of 2.10 we have either (a) $f(x) = f(x') = z \in D$, $f(y) = y$, $f(y') = y'$, or (b) $f(y) = f(y') = z \in D$, $f(x) = x$, $f(x') = x'$. Let $a \in A - D$. The assumptions of 3.1 – 3.10 are not valid, hence $a \in X$ or $a \in Y$, where X is the set of all elements of A behaving in the same way as x, and Y is the set of all elements of A which behave in the same way as y. It is obvious that

$$X = \{x\} \cup \{a \in A - D : \Theta(a, x) \in S,\ \Theta(a, y) = [\{a, y, z\}]\}\,,$$
$$Y = \{y\} \cup \{a \in A - D : \Theta(a, y) \in S,\ \Theta(a, x) = [\{a, x, z\}]\}\,.$$

Hence $A = D \cup X \cup Y$ and either (a) $f(x_1) = z$, $f(y_1) = y_1$, or (b) $f(y_1) = z$, $f(x_1) = x_1$ for each $x_1 \in X$, $y_1 \in Y$. These two cases cannot be distinguished by means of Con (A, f).

Let the assumptions of 3.1 – 3.11 be not satisfied. Hence, in particular, if there exists $z \in D$ with $f^{-1}(\emptyset) \neq \emptyset$, then we have: (i) B consists of one-element cycles or $B = \emptyset$ (cf. 3.9, 3.3 and 3.4); (ii) card $f^{-1}(z) \leqq 1$ or $f^{-2}(z) = \emptyset$ (cf. 3.9); (iii) if card $f^{-2}(z) \geqq 2$, then $B = \emptyset$, and if card $B \geqq 2$, then $f^{-2}(z) = \emptyset$ card $f^{-1}(z) \leqq 1$ (cf. 3.10, 3.11); (iv) if card $f^{-1}(z) \geqq 2$, then card $B \leqq 1$ (cf. 3.11).

3.12. Lemma. *Let there exist distinct elements $x, x', y, z \in A$ fulfilling the condition* (2) *from* 2.8. *Then $R(f)$ consists of three elements, which can be described by means of* Con (A, f). *The partial algebra (A, f) is a c_0-extension of some of the partial algebras given in Fig.* 3.12 (a) – (c) *such that whenever $a \in A - \{x, x', y, z\}$, then a behaves in the same way as some of the elements x, x', z.*

Proof. From 2.8 it follows that either (a) $f(x) = f(x') = y$, $f(y) = z \in D$, or (b) $f(x) = f(x') = z \in D$, $f(y) = y$, or (c) $f(y) = z \in D$, $f(x') = x'$. The assumptions of 3.1 – 3.11 are not satisfied, hence if $a \in A - D$, $a \neq y$, then a and x behave in the same way. By the symbol X we denote the set of all elements of A which behave in the same way as x. Then

$$X = \{x\} \cup \{a \in A - D : \Theta(a, x) \in S,\ \Theta(a, y) = [\{a, y, z\}]\}\,.$$

Thus $A = D \cup X \cup \{y\}$ and one of the following possibilities is valid: (a) $f(x_1) = y$, $f(y) = z$, or (b) $f(x_1) = z$, $f(y) = y$, or (c) $f(y) = z$, $f(x_1) = x_1$, for each $x_1 \in X$. It is obvious that these three cases cannot be distinguished by means of Con (A, f).

Let the assumptions of 3.1 – 3.12 be not satisfied.

3.13. Lemma. *Let there exist distinct elements $x, y, z, u \in A$ fulfilling the condttion* (2) *from* 2.11. *Then $R(f)$ consists of two elements, which can be described by means of* Con (A, f). *The partial algebra (A, f) is a c_0-extension of some of the partial algebras given in Fig.* 3.13 (a) *or* (3.13) (b), *such that whenever $a \in A - \{x, y, z, u\}$, then a behaves in the same way as z.*

Proof. In view of 2.11 we have either (a) $f(x) = y, f(y) = z \in D, f(u) = u$, or (b) $f(u) = y, f(y) = z \in D, f(x) = x$. Since none of the assumptions applied in 3.1 – 3.12 is valid, we obtain that $A = D \cup \{x, y, u\}$. The cases (a) and (b) cannot be distinguished by means of congruence relations.

Let the assumptions of 3.1 – 3.13 be not satisfied. If there is $z \in D$ with $f^{-1}(z) \neq \emptyset$, then we obtain: (i) if $f^{-2}(z) \neq \emptyset$, then $B = \emptyset$ (cf. 3.13); (ii) if $B \neq \emptyset$, then $f^{-2}(z) = \emptyset$ and $\operatorname{card} f^{-1}(z) \leqq 1$ (cf. 3.13 and 3.12); (iii) $\operatorname{card} f^{-2}(z) \leqq 1$, $\operatorname{card} B \leqq 1$ (cf. 3.12).

3.14. Lemma. *Let there exist distinct elements $x, y, z \in A$ fulfilling the condition (2) from 2.7. Then $R(f)$ consists of four elements, which can be described by means of* Con (A, f). *The partial algebra (A, f) is a c_0-extension of some of the partial algebras given in Figs.* 3.14 (a)–(d) *such that whenever $a \in A - \{x, y, z\}$, then a and z behave in the same way.*

Proof. From 2.7 it follows that one of the following possibilities holds: (a) $f(x) = y, f(y) = z \in D$, (b) $f(y) = x, f(x) = z \in D$, (c) $f(y) = z \in D, f(x) = x$, (d) $f(x) = z \in D$, $f(y) = y$. Since the assumptions of 3.1–3.13 are not satisfied, we obtain that $A = D \cup \{x, y\}$. These four cases cannot be distinguished by Con (A, f).

We introduce the following two notions.

Let (B_1, g_1) and (A_1, f_1) be partial monounary algebras. Then (A_1, f_1) will be said to be a *c-extension* of (B_1, g_1), if there is an isomorphism φ of (B_1, g_1) into (A_1, f_1) such that

(*) for each $a \in A_1 - \varphi(B_1)$ there exists $b \in \varphi(B_1)$ such that a and b behave in the same way.

A partial monounary algebra (A_1, f_1) is said to be a *d-extension* of a monounary algebra (B_1, g_1), if $B_1 \subseteq A_1$, $D_{f_1} = A_1 - B_1$ and $g_1(x) = f_1(x)$ for each $x \in B_1$. Denote $f_1 = g_1'$.

Now suppose that none of the assumptions applied in 3.1–3.14 is satisfied. From 3.1–3.14 we obtain that this holds if and only if $f^{-1}(D) = \emptyset$. Thus the condition $f^{-1}(D) = \emptyset$ is characterized merely by the system Con (A, f). Then we have $A = D \cup B$, where $D = P$ (i.e., D can be described by means of Con (A, f)). Moreover, (A, f) is a d-extension of $(B, f/B)$ and

$$\begin{aligned} R(f) &= \{g \in F : \operatorname{Con}(A, f) = \operatorname{Con}(A, g)\} = \\ &= \{g \in F : \operatorname{Con}(B, f/B) = \operatorname{Con}(B, g/B) \text{ and } D_f = D_g\} = \\ &= \{g_1' : g_1 \text{ is a unary operation on } B \text{ and } \operatorname{Con}(\mathrm{B}, g_1) = \\ &= \operatorname{Con}(B, f/B)\}\,. \end{aligned}$$

Hence in the case $f^{-1}(D) = \emptyset$ the investigation of $R(f)$ can be reduced to the investigation of $R(f/B)$, i.e., to the analogous question concerning the complete unary operation f/B.

The considerations performed in this section can be summarized as follows:

Let (A, f) be a partial monounary algebra such that Con $(A, f) \neq E(A)$. *Then by using merely the system* Con (A, f) *(without applying explicitly the operation f)*

1) *we can decide whether or not $f^{-1}(D) \neq \emptyset$;*

2) *in the case when $f^{-1}(D) \neq \emptyset$ we can describe all partial unary operations g on A having the property that* Con $(A, f) =$ Con (A, g).

In particular, as a consequence of 3.1–3.14 we have

3.15. Proposition. *Let* (A, f) *be a partial monounary algebra such that* $\operatorname{Con}(A, f) \neq E(A)$. *Then we have*:

(i) $\operatorname{card} R(f) = 2$ *if and only if* (A, f) *is a c-extension of some of partial monounary algebras described in Figs.* 3.5, 3.9, 3.13 *and* 3.11.

(ii) $\operatorname{card} R(f) = 3$ *if and only if* (A, f) *is a c-extension of some of partial monounary algebras described in Fig.* 3.12, *such that whenever* $a \in A - \{\varphi(x), \varphi(x'), \varphi(y), \varphi(z)\}$, *then a behaves in the same way as some of the elements* $\varphi(x), \varphi(x'), \varphi(z)$.

(iii) $\operatorname{card} R(f) = 4$ *if and only if* (A, f) *is a c-extension of some of partial monounary algebras described in Fig.* 3.14, *such that whenever* $a \in A - \{\varphi(x), \varphi(y), \varphi(z)\}$, *then a and* $\varphi(z)$ *behave in the same way.*

3.16. Proposition. *Let* (A, f) *be a partial monounary algebra such that* $\operatorname{Con}(A, f) \neq E(A)$. *Assume that* $f^{-1}(D) \neq \emptyset$. *Then* $\operatorname{card} R(f) \leqq 4$.

3.17. Proposition. *Let A be a set,* $\operatorname{card} A \geqq 4$. *Then for each* $i \in \{1, 2, 3, 4\}$ *there exists a partial unary operation* f_i *on A such that* $\operatorname{card} R(f_i) = i$.

References

[1] *Bartol W.*: On the existence of machine homomorphisms I, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astr. Phys. *19* (1971), 856—869; II, ibid., *20* (1972), 773—777.

[2] *Bartol W.*: Algebraic complexity of machines, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astr. Phys. *22* (1974), 851—856.

[3] *Bartol W.*: Dynamics programs of computations, (Polish), Warszawa 1974.

[4] *Jónsson* B.: Topics in universal algebra, Berlin 1972.

[5] *Kopeček O.*: Homomorphisms of partial unary algebras, Czech. Math. J. *26* (*101*) (1976), 108—127.

[6] *Kopeček O.*: Construction of all machine homomorphisms, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astr. Phys. *8* (1976), 655—658.

[7] *Kopeček O.*: The category of connected partial unary algebras, Czech. Math. J. *27* (*102*) (1977), 415—423.

[8] *Kopeček O.*: Homomorphisms of machines I, Arch. Math. (Brno) *1* (1978), 45—50; II, ibid., *2* (1978), 99—108.

[9] *Kopeček O.*: Existence of monomorphisms of partial unary algebras, Czech. Math. J. *28* (*103*) (1978), 462—473.

[10] *Kopeček O., Novotný M.*: On some invariants of unary algebras, Czech. Math. J. *24* (*99*) (1974), 219—246.

[11] *Novotný M.*: On a problem from the theory of mappings, (Czech), Publ. Fac. Sci. Univ. Masaryk, No *344* (1953), 53—64.

[12] *Novotný M.*: Über Abbildungen von Mengen, Pacif. J. Math. *13* (1963), 1359—1369.

[13] *Novotný M.*: Construction of all homomorphisms of unary algebras, (Czech), Dept. Math. Ped. Fac. Univ. J. E. Purkyně, Seminar on new directions in mathematical education, Brno, 1973.

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