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GENERALIZED VOLTERRA INTEGRAL EQUATIONS

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In this paper the basic theory of Volterra integral equations is developed in the case when the generalized Perron integral is used. The generalized Perron integral was introduced in 1957 by J. Kurzweil in his fundamental paper [6] and it was used in the theory of ordinary differential equations, especially for considerations concerning continuous dependence of solutions of an ordinary differential equation on a parameter. The concept of a generalized ordinary differential equation was also useful for deriving results from topological dynamics (see e.g. the paper [2] of Z. Artstein). Generalized ordinary differential equations are used for the description of systems with discontinuous solutions, systems with impulses, etc.

In the present work we define the concept of a generalized nonlinear Volterra integral equation. The way in which this is done follows the original work of J. Kurzweil. The results concern basic equations as existence of solutions, continuous dependence, connectedness of the solution funnel, conditions for uniqueness. The results are compared with recent ones for the classical theory as they are presented in the works of R. K. Miller [10], W. G. Kelley [5], Z. Artstein [1].

We are interested in Volterra integral equations having continuous solutions although it is possible to obtain results also for discontinuous solutions similarly as in the case of ordinary differential equations.

1. THE GENERALIZED PERRON INTEGRAL

In this section we give a short survey of the generalized Perron integral which will be used for our theory of Volterra integral equations. This concept of integral was developed by J. Kurzweil to a great extent in the middle of fifties in connection with his theory of generalized ordinary differential equations. The Kurzweil theory of integral is very interesting also from the viewpoint of integration theory. It represents a self-contained theory of a simply defined general integral.

This type of integral was discovered independently by R. Henstock (see [4]). It was used e.g. by J. Mawhin in his university course of analysis [9] and by many others.

The results needed for our purposes are mostly contained in the original paper of Kurzweil [6] and in his recent book [8]. A short treatise of this integral is given also in the paper [2] by Z. Artstein.

Let $a, b \in \mathbb{R}$, $-\infty < a \leqq b < +\infty$ be given. A decomposition of the interval $[a, b]$ is a finite sequence

$$A = \{\alpha_0, \sigma_1, \alpha_1, \ldots, \alpha_{k-1}, \sigma_k, \alpha_k\}$$

such that

$$a = \alpha_0 < \alpha_1 < \ldots < \alpha_k = b\,, \quad \alpha_{j-1} \leqq \sigma_j \leqq \alpha_j\,, \quad j = 1, \ldots, k\,.$$

An arbitrary positive function $\delta : [a, b] \to (0, +\infty)$ is a gauge on $[a, b]$. The decomposition $A = \{\alpha_0, \sigma_1, \alpha_1, \ldots, \alpha_{k-1}, \sigma_k, \alpha_k\}$ is subordinate to the gauge δ (shortly we write $A \in A(\delta)$) if

$$[\alpha_{j-1}, \alpha_j] \subset [\sigma_j - \delta(\sigma_j), \sigma_j + \delta(\sigma_j)]\,, \quad j = 1, \ldots, k\,.$$

Assume that $U : [a, b] \times [a, b] \to \mathbb{R}^N$. Given a decomposition $A = \{\alpha_0, \sigma_1, \alpha_1, \ldots$ $\ldots, \alpha_{k-1}, \sigma_k, \alpha_k\}$ of $[a, b]$ we associate with A the integral sum

$$S(A) = \sum_{j=1}^{k} [U(\alpha_j, \sigma_j) - U(\alpha_{j-1}, \sigma_j)]\,.$$

The function U is integrable over $[a, b]$ if there exists $I \in \mathbb{R}^N$ such that for every $\varepsilon > 0$ there exists a gauge $\delta : [a, b] \to (0, +\infty)$ such that for every decomposition A subordinate to $\delta(A \in A(\delta))$ the inequality

$$|S(A) - I| < \varepsilon$$

holds. $I \in R^N$ is called the generalized Perron integral of U over $[a, b]$ and will be denoted by $\int_a^b DU(\tau, s)$.

If $\int_a^b DU(\tau, s)$ exists we define $\int_b^a DU(\tau, s) = -\int_a^b DU(\tau, s)$ and we set $\int_a^b DU(\tau, s) = 0$ if $a = b$.

The set of all functions U integrable over $[a, b]$ in the above sense is denoted by $K([a, b])$.

Remark. For the definition of the integral $\int_a^b DU(\tau, s)$ it is not necessary to have the function U defined on the whole square $[a, b] \times [a, b]$. It is sufficient to know the values of $U(\tau, s)$ close to the diagonal $\tau = s$. Let us mention that if the function U is of the form $U(\tau, s) = f(s)\, g(\tau)$ where $g \in BV([a, b])$, $f : [a, b] \to \mathbb{R}$ then the integral $\int_a^b DU(\tau, s)$ exists if and only if the Perron-Stieltjes integral $\int_a^b f\, dg$ exists and both the integrals have the same value (see [6]).

We give a list of fundamental properties of this concept of integral. The reader can find them in [6] or in the book [8].

(1.1) *If* $U \in K([a, b])$, $\beta \in \mathbb{R}$ *then* $\beta U \in K([a, b])$ *and*

$$\int_a^b D[\beta U(\tau, s)] = \beta \int_a^b DU(\tau, s)\,.$$

(1.2) *If* $U, V \in K([a, b])$ *then* $U + V \in K([a, b])$ *and*

$$\int_a^b \mathrm{D}(U + V)(\tau, s) = \int_a^b \mathrm{D}U(\tau, s) + \int_a^b \mathrm{D}V(\tau, s).$$

(1.3) *If* $U \in K([a, b])$ *then* $U \in K([c, d])$ *for every* $[c, d] \subset [a, b]$.

(1.4) *If* $c \in (a, b)$, $U \in K([a, c])$, $U \in K([c, b])$ *then* $U \in K([a, b])$ *and*

$$\int_a^c \mathrm{D}U(\tau, s) + \int_c^b \mathrm{D}U(\tau, s) = \int_a^b \mathrm{D}U(\tau, s).$$

(1.5) *If* $U : [a, b] \times [a, b] \to \mathbb{R}^N$ *is such that* $U \in K([a, c])$ *for every* $c \in (a, b)$ *and the limit*

$$\lim_{c \to b} \left[\int_a^c \mathrm{D}U(\tau, s) - U(c, b) + U(b, b)\right] = L$$

exist, then $U \in K([a, b])$ *and* $\int_a^b \mathrm{D}U(\tau, s) = L$.

(1.6) *If* $U \in K([a, b])$, $c \in [a, b]$, *then*

$$\lim_{\substack{\alpha \to c \\ \alpha \in [a,b]}} \left[\int_a^\alpha \mathrm{D}U(\tau, s) - U(\tau, c) + U(c, c) = \int_a^c \mathrm{D}U(\tau, s).\right.$$

For some estimates of the integral the following result is useful (see [7]).

(1.7) *Let* $U : [a, b] \times [a, b] \to \mathbb{R}^N$ *be such that* $\int_a^b \mathrm{D}U(\tau, s)$ *exists and let* $V :$ $: [a, b] \times [a, b] \to \mathbb{R}$ *be a real function for which* $\int_a^b \mathrm{D}V(\tau, s)$ *exists. If for every* $s \in [a, b]$ *there is* $\delta(s) > 0$ *such that* $|\tau - s| \, |U(\tau, s) - U(s, s)| \leqq$ $\leqq (s - \tau)(V(\tau, s) - V(s, s))$ *for* $\tau \in [a, b] \cap [s - \delta(s), s + \delta(s)]$, *then*

$$\left|\int_a^b \mathrm{D}U(\tau, s)\right| \leqq \int_a^b \mathrm{D}V(\tau, s).$$

A fundamental statement used in the theory of generalized nonlinear Volterra integral equations is the following theorem which replaces the well-known Lebesgue dominated convergence theorem in the case of generalized Perron integral. For the proof see [8].

(1.8) (Dominated convergence theorem.) *Assume that* U_1, $l = 1, 2, \ldots$ *are such real functions that* $U_l \in K([a, b])$ *and that there exist functions* $V, W \in K([a, b])$ *such that*

$$V(\tau_2, s) - V(\tau_1, s) \leqq U_l(\tau_2, s) - U_l(\tau_1, s) \leqq W(\tau_2, s) - W(\tau_1, s)$$

for every $l = 1, 2, \ldots$, $s \in [a, b]$ *and* $[\tau_1, \tau_2] \subset [s - \delta(s), s + \delta(s)]$, $\tau_1 \leqq s \leqq$ $\leqq \tau_2$ *where* $\delta : [a, b] \to (0, +\infty)$ *is a suitable gauge on* $[a, b]$. *Let* $U :$ $: [a, b] \times [a, b] \to \mathbb{R}$ *be such a function that for every* $\varepsilon > 0$ *there exists a nondecreasing function* $\mu : [a, b] \to \mathbb{R}$, *a gauge* $\xi : [a, b] \to (0, +\infty)$ *and* $p : [a, b] \to \mathbb{N}$ ($\mathbb{N}$ *stands for the natural numbers* $1, 2, \ldots$) *such that*

$$\mu(b) - \mu(a) < \varepsilon$$

and $|U_l(\tau_2, s) - U_l(\tau_1, s) - U(\tau_2, s) + U(\tau_1, s)| \leqq \mu(\tau_2) - \mu(\tau_1)$

for $\tau_1 \leqq s \leqq \tau_2$, $[\tau_1, \tau_2] \in [s - \xi(s), s + \xi(s)]$, $l \geqq p(s)$.

Then $U \in K([a, b])$ *and*

$$\int_a^b \mathrm{D}U(\tau, s) = \lim_{l \to \infty} \int_a^b \mathrm{D}U_l(\tau, s) .$$

2. GENERALIZED VOLTERRA INTEGRAL EQUATIONS. EXISTENCE OF SOLUTIONS AND THEIR CONTINUATION

We consider the nonlinear Volterra integral equation

$$\text{(V)} \qquad x(t) = f(t) + \int_0^t \mathrm{D}G(t, \tau, x(s)) ,$$

where $t \in I = [0, c)$, $0 < c \leqq +\infty$, the values of x, f, G belong to the N-dimensional Euclidean space $\mathbb{R}^N$ and the integral occuring in (V) is the generalized Perron integral.

We assume that the function f is continuous and our effort is directed to obtaining results for continuous solutions of the equation (V).

The function $G : I \times I \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies the following assumptions.

(G_1) $G(t, 0, x) = 0$ *for every* $(t, x) \in I \times \mathbb{R}^N$. *If* $(t, \tau, x) \in I \times I \times \mathbb{R}^N$ *and* $\tau > t$, *then* $G(t, \tau, x) = G(t, t, x)$.

(G_2) *For each* $b \in I$ *and each* $K > 0$ *there exists* $M : [0, b] \times [0, b] \to \mathbb{R}$, $M(t, \cdot)$ *nondecreasing in* $[0, t]$ *such that*

$$|G(t, \tau_2, x) - G(t, \tau_1, x)| \leqq M(t, \tau_2) - M(t, \tau_1)$$

for $x \in \mathbb{R}^N$, $|x| \leqq K$ *and* $0 \leqq \tau_1 \leqq \tau_2 \leqq t \leqq b$.

(G_3) *For every* $b \in I$, $K > 0$ *and* $\varepsilon > 0$ *there is a function* $\mu(t, \tau)$, $\mu(t, \cdot)$ *nondecreasing in* $[0, t]$, $\mu(t, t) - \mu(t, 0) < \varepsilon$ *and* $\varrho(t) > 0$, $t \in [0, b]$ *such that*

$$|G(t, \tau_2, x) - G(t, \tau_1, x) - G(t, \tau_2, y) + G(t, \tau_1, y)| \leqq$$
$$\leqq \mu(t, \tau_2) - \mu(t, \tau_1)$$

provided $0 \leqq \tau_1 \leqq \tau_2 \leqq t \leqq b$, $|x| \leqq K$, $|y| \leqq K$, $|x - y| < \varrho(t)$.

The first question which has to be dealt with is the question of the existence of the integral $\int_t^0 \mathrm{D}G(t, \tau, x(s))$ from (V) when some information about the function $x : [0, t] \to \mathbb{R}^N$ is available. In order to be able to solve (V) in the space of continuous functions we need the existence of this integral for continuous functions $x : [0, t] \to \mathbb{R}^N$.

2.1. Proposition. *Suppose that* $G : I \times I \times \mathbb{R}^N \to \mathbb{R}^N$ *satisfies the assumptions* (G_1), (G_2) *and* (G_3). *Let* $t \in I$ *be given and assume that the sequence of functions* $\varphi_k : [0, t] \to \mathbb{R}^N$, $k = 1, 2, \ldots$ *is such that* $|\varphi_k(s)| \leqq K$, $s \in [0, t]$, $k = 1, 2, \ldots$ *for* *some $K > 0$ and*

$$\lim_{k\to\infty} \varphi_k(s) = \varphi(s) \in \mathbb{R}^N$$

for all $s \in [0, t]$, i.e. the sequence of functions φ_k converges on $[0, t]$ pointwise to a function $\varphi : [0, t] \to \mathbb{R}^N$. Assume further that $\int_0^t \mathrm{D}G(t, \tau, \varphi_k(s))$ exists for every $k = 1, 2, \ldots$. Then the integral $\int_0^t \mathrm{D}G(t, \tau, \varphi(s))$ exists and the equality

$$\lim_{k\to\infty} \int_0^t \mathrm{D}G(t, \tau, \varphi_k(s)) = \int_0^t \mathrm{D}G(t, \tau, \varphi(s))$$

holds.

Proof. By (G_2) for every component G_l of G we have

$$-(M(t, \tau_2) - M(t, \tau_1)) \leqq G_l(t, \tau_2, \varphi_k(s)) - G_l(t, \tau_1, \varphi_k(s)) \leqq$$
$$\leqq M(t, \tau_2) - M(t, \tau_1)$$

for any $s \in [0, t]$, $k = 1, 2, \ldots$, $0 \leqq \tau_1 \leqq \tau_2 \leqq t$ and the integral $\int_0^t \mathrm{D}M(t, \tau) = M(t, t) - M(t, 0)$ evidently exists. Let $\varepsilon > 0$ be given. Let us mention that for every $s \in [0, t]$ we have $|\varphi(s)| \leqq K$ and that there exists $p(s) \in \mathbb{N}$ such that for every $k \in \mathbb{N}$, $k \geqq p(s)$ we have $|\varphi_k(s) - \varphi(s)| < \varrho(t)$ where $\varrho(t) > 0$ is given by (G_3). By (G_3) there exists $\mu : [0, t] \times [0, t] \to \mathbb{R}$, $\mu(t, \cdot)$ nondecreasing in $[0, t]$, $\mu(t, t) - \mu(t, 0) < \varepsilon$ such that

$$|G(t, \tau_2, \varphi_k(s)) - G(t, \tau_1, \varphi_k(s)) - G(t, \tau_2, \varphi(s)) + G(t, \tau_1, \varphi(s))| \leqq$$
$$\leqq \mu(t, \tau_2) - \mu(t, \tau_1)$$

for every $s \in [0, t]$, $0 \leqq \tau_1 \leqq \tau_2 \leqq t$ and $k \geqq p(s)$.

All the assumptions of the dominated convergence theorem (1.8) being satisfied for the integral $\int_0^t \mathrm{D}G(t, \tau, \varphi(s))$, we immediately get the result stated in the Proposition.

2.2. Corollary. *If G satisfies the assumptions (G_1), (G_2) and (G_3), $t \in I$ is given and the function $\varphi : [0, t] \to \mathbb{R}^N$ is the limit of a sequence of uniformly bounded piecewise constant functions on $[0, t]$, then the integral $\int_0^t \mathrm{D}G(t, \tau, \varphi(s))$ exists.*

Proof. By Proposition 2.1 it is sufficient to show that the integral $\int_0^t \mathrm{D}G(t, \tau, \psi(s))$ exists for any piecewise constant function $\psi : [0, t] \to \mathbb{R}^N$. If $\psi : [T_1, T_2] \to \mathbb{R}^N$, $0 \leqq T_1 \leqq T_2 \leqq t$ is such a function that $\psi(s) = c \in \mathbb{R}^N$ for $s \in (T_1, T_2)$ then by definition the integral

$$\int_{t_1}^{t_2} \mathrm{D}G(t, \tau, \psi(s)) = G(t, t_2, c) - G(t, t_1, c)$$

exists for every t_1, t_2 such that $T_1 < t_1 \leqq t_2 < T_2$.

Since the limit

$$\lim_{t_2 \to T_2-} \left[\int_{t_0}^{t_2} \mathrm{D}G(t, \tau, \psi(s)) - G(t, t_2, \psi(T_2)) + G(t, T_2, \psi(T_2))\right] =$$

$$= G(t, T_2-, c) - G(t, t_0, c) + G(t, T_2, \psi(T_2)) - G(t, T_2-, \psi(T_2))$$

exists for every fixed $t_0 \in (T_1, T_2)$ and similarly

$$\lim_{t_1 \to T_1+} \left[\int_{t_1}^{t_0} DG(t, \tau, \psi(s)) - G(t, t_1, \psi(T_1)) - G(t, T_1, \psi(T_1)) \right] =$$

$$= G(t, t_0, c) - G(t, T_1+, c) + G(t, T_1+, \psi(T_1)) - G(t, T_1, \psi(T_1))$$

exists for $t_0 \in (T_1, T_2)$ we can use (1.5) and the additivity of the integral (1.4) for obtaining the existence of the integral $\int_{T_1}^{T_2} DG(t, \tau, \psi(s))$ and also the equality

$$\int_{T_1}^{T_2} DG(t, \tau, \psi(s)) = G(t, T_2-, c) - G(t, T_1+, c) + G(t, T_2, \psi(T_2)) -$$
$$- G(t, T_2-, \psi(T_2)) + G(t, T_1+, \psi(T_1)) - G(t, T_1, \psi(T_1)).$$

Using again the additivity of the integral (1.4), we get immediately the existence of the integral $\int_0^t DG(t, \tau, \psi(s))$ for every piecewise continuous function $\psi : [0, t] \to R^N$. Let us mention that the existence of the limits used above easily follows from (G_2).

2.3. Corollary. *If G satisfies $(G_1), (G_2)$ and (G_3), $t \in I$ and $\psi : [0, t] \to R^N$ is continuous, then the integral $\int_0^t DG(t, \tau, \psi(s))$ exists.*

Proof. The result follows immediately from Corollary 2.2 and from the well-known fact that every continuous function on a closed interval can be uniformly approximated by piecewise constant functions which are uniformly bounded.

Remark. Corollary 2.3 makes it possible to consider the generalized Volterra integral equation in the space of continuous functions on $[0, b] \subset I$ since under the conditions $(G_1), (G_2)$ and (G_3) the existence of the integral $\int_0^t DG(t, \tau, x(s))$ for every $t \in I$ and every continuous $x : [0, t] \to R^N$ is ensured. On the other hand Corollary 2.2 shows that the integral $\int_0^t DG(t, \tau, \varphi(s))$ exists for every function $\varphi : [0, t] \to \to R^N$ which is the (pointwise) limit of piecewise constant and uniformly bounded functions; this property has e.g. any bounded regulated function and in this way also the consideration of Volterra integral equations for such functions is possible.

Now we introduce an additional assumption, which is essential for the forthcoming considerations.

(G_4) *For each $b \in I$ and $K > 0$ the expression*

$$\sup_{\substack{x \in C([0,b]) \\ \|x\|_C \leq K}} \left\{ \left\| \int_0^b DG(t, \tau, x(s)) - \int_0^b DG(t_0, \tau, x(s)) \right\| \right\}$$

tends to zero for $t \to t_0$ provided $t_0 \in I$.

We are naturally forced to introduce this assumption by the requirement of continuity of solutions of the equation (V). The integrals occuring in (G_4) exist by Corollary 2.3 provided the assumptions $(G_1)-(G_3)$ are satisfied.

Summarizing these preliminary results we can conclude that if the function G,

$G : I \times I \times \mathbb{R}^N \to \mathbb{R}^N$, satisfies $(G_1)-(G_4)$ and the function $f : I - \mathbb{R}^N$ is continuous then we can ask for continuous solutions of the generalized Volterra integral equation (V).

2.4. Theorem. (Existence of a continuous local solution.) *If $f : I \to \mathbb{R}^N$ is continuous and if $G : I \times I \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies the assumptions $(G_1)-(G_4)$ then there exists a positive number $d \in I$ and a continuous function $x : [0, d] \to \mathbb{R}^N$ which for every $t \in [0, d]$ satisfies the equation*

$$x(t) = f(t) + \int_0^t DG(t, \tau, x(s)) .$$

Proof. The basic tool for proving this theorem is the known Schauder-Tichonov fixed point theorem.

Let $b \in (0, c)$ be given. Denote $K_1 = \sup_{t \in [0,b]} |f(t)|$ and set $K = K_1 + 1$. Since by (G_4) we have

$$\lim_{t \to 0} \sup_{\substack{x \in C([0,b]) \\ \|x\|_C \leqq K}} \left| \int_0^b DG(t, \tau, x(s)) - \int_0^b DG(0, \tau, x(s)) \right| =$$

$$= \lim_{t \to 0} \sup_{\substack{x \in C([0,b]) \\ \|x\|_C \leqq K}} \left| \int_0^b DG(t, \tau, x(s)) \right| = 0 ,$$

there exists $d \in (0, b]$ such that for any $t \in [0, d]$ the inequality

$$\left| \int_0^b DG(t, \tau, x(s)) \right| < 1 \tag{2.1}$$

holds for every $x \in C([0, b])$, $\|x\|_C \leqq K$.

Denote

$$A = \{x \in C([0, d]);\ \|x - f\|_{C([0,d])} \leqq 1\} ;$$

A is evidently a closed convex subset of the convex F-space $C([0, d])$. For $x \in A$ and $t \in [0, d]$ define the mapping

$$Tx(t) = f(t) + \int_0^t DG(t, \tau, x(s)) .$$

Since f is continuous on $[0, d]$ and by (G_4) the function

$$t \in [0, d] \mapsto \int_0^t DG(t, \tau, x(s)) \in \mathbb{R}^N$$

is also continuous for every $x \in A$ we conclude that $Tx(t)$ is a continuous function, i.e. $Tx \in C([0, d])$. If $x \in A$ then

$$\|x\|_C \leqq \|f\|_C + \|x - f\|_C \leqq K_1 + 1 = K .$$

Hence by (2.1)

$$|Tx(t) - f(t)| = \left|\int_0^t DG(t, \tau, x(s)\right| = \left|\int_0^b DG(t, \tau, x(s))\right| < 1$$

for every $t \in [0, d]$, i.e. we have $Tx \in A$ for every $x \in A$ or $T(A) \subset A$ and moreover $|Tx(t)| \leqq |f(t)| + 1 \leqq K_1 + 1 = K$ for every $x \in A$ and $t \in [0, d]$. This means that the set $T(A)$ consists of equibounded functions belonging to $C([0, d])$. Since for every $t_1, t_2 \in [0, d]$ and $x \in A$ we have

$$|Tx(t_2) - Tx(t_1)| \leqq |f(t_2) - f(t_1)| +$$

$$+ \left|\int_0^{t_2} DG(t_2, \tau, x(s)) - \int_0^{t_1} DG(t_1, \tau, x(s))\right| =$$

$$= |f(t_2) - f(t_1)| + \left|\int_0^d DG(t_2, \tau, x(s)) - \int_0^d DG(t_1, \tau, x(s))\right| \leqq$$

$$\leqq |f(t_2) - f(t_1)| + \sup_{\substack{x \in C([0,b]) \\ \|x\|_C \leqq K}} \left|\int_0^b DG(t_2, \tau, x(s)) - \int_0^b DG(t_1, \tau, x(s))\right|$$

we obtain that all functions belonging to $T(A)$ are equicontinuous. Hence the set $T(A)$ is precompact in $C([0, d])$. It remains to show that $T: A \to A$ is a continuous mapping. Assume that $x_k \in A$, $k = 1, 2, \ldots$ and $x_k \to x$ in $C([0, d])$ for $k \to \infty$. Since A is closed we have $x \in A$ and by Proposition 2.1 we have

$$\int_0^t DG(t, \tau, x_k(s)) \to \int_0^t DG(t, \tau, x(s))$$

for every $t \in [0, d]$ if $k \to \infty$. Since all the functions belonging to $T(A)$ are equicontinuous and $Tx_k(t) \to Tx(t)$, $k \to \infty$ for any $t \in [0, d]$, we conclude that $\lim_{k\to\infty} Tx_k = Tx$ in $C([0, d])$ and the mapping $T: A \to A$ is continuous. All the assumptions of the Schauder-Tichonov theorem being satisfied, we obtain that there exists at least one $x \in A$ such that $x = Tx$, i.e.

$$x(t) = f(t) + \int_0^t DG(t, \tau, x(s))$$

for all $t \in [0, d]$ and this is the result stated in the theorem.

2.5. Theorem. (Continuation of bounded solutions.) *Let $f: I \to \mathbb{R}^N$ be a continuous function and let $G: I \times I \times \mathbb{R}^N \to \mathbb{R}^N$ satisfy the assumptions* (G_1)–(G_4). *If $x: [0, d) \to \mathbb{R}^N$, $0 < d < c$ is a bounded (continuous) solution of the equation* (V) *then x can be extended to a continuous solution of the equation* (V) *on an interval $[0, \tilde{d}]$ where $\tilde{d} > d$.*

Proof. Assume that $|x(t)| \leqq K$ for every $t \in [0, d)$. Let t_k, $k = 1, 2, \ldots$ be an arbitrary sequence such that $t_k \in [0, d)$, $t_{k+1} \geqq t_k$ and $\lim_{k\to\infty} t_k = d$. Since x is a solution of (V) we have for $m \geqq n$, $m, n \in \mathbb{N}$

$$|x(t_m) - x(t_n)| \leqq |f(t_m) - f(t_n)| +$$

$$+ \left| \int_0^{t_m} DG(t_m, \tau, x(s)) - \int_0^{t_n} DG(t_n, \tau, x(s)) \right| =$$

$$= |f(t_m) - f(t_n)| + \left| \int_0^{t_m} DG(t_m, \tau, x(s)) - \int_0^{t_m} DG(t_n, \tau, x(s)) \right| \leqq$$

$$\leqq |f(t_m) - f(t_n)| + \sup_{\substack{\varphi \in C([0,d]) \\ \|\varphi\|_C \leqq K}} \left| \int_0^d DG(t_m, \tau, \varphi(s)) - \int_0^d DG(t_n, \tau, \varphi(s)) \right| .$$

Using (G_4) we obtain from this inequality $|x(t_m) - x(t_n)| \to 0$ for $m, n \to \infty$. (It is apparent that the assumption of monotonicity of the sequence t_k, $k = 1, 2, \ldots$ is done for technical reasons only.) This means that the limit $\lim_{t\to d-} x(t)$ exists. If we define $x(d) = \lim_{t\to d-} x(t)$ we obtain a continuous function $x(t)$ defined on the closed interval $[0, d]$, which satisfies equation (V) on the closed interval $[0, d]$.

Using the continuity of f and (G_4) we obtain that the function

$$t \in [0, c - d) \mapsto f(t + d) + \int_0^d DG(t + d, \tau, x(s))$$

is continuous on $[0, c - d)$.

It is a matter of routine to show that the function $\tilde{G}(t, \tau, x) = G(t + d, \tau + d, x)$ defined for $(t, \tau, x) \in [0, c - d) \times [0, c - d) \times \mathbb{R}^N$ satisfies the assumptions $(G_1)-(G_4)$. Hence by Theorem 2.4 for $h : [0, c - d) \to \mathbb{R}^N$ continuous there exists a local solution of the Volterra integral equation

$$y(t) = h(t) + \int_0^t D\tilde{G}(t, \tau, y(s)), \tag{2.2}$$

i.e. we can find $d_0 \in (0, c - d)$ such that on $[0, d_0]$ there exists a solution y of the equation (2.2) where $h(t) = f(t + d) + \int_0^d DG(t + d, \tau, x(s))$. Let us put $\tilde{d} = d + d_0$ (evidently $d < \tilde{d} < c$) and set $x(s) = y(s - d)$ for $s \in (d, \tilde{d}]$. This gives a continuous extension of the function $x : [0, d] \to \mathbb{R}^N$ onto the interval $[0, \tilde{d}]$ with $0 < d < \tilde{d} < < c$.

For $t \in [0, d]$ the function $x(t)$ satisfies (V). For $t \in (d, \tilde{d}]$ we have for the extension $x(t)$

$$x(t) = y(t - d) = f(t) + \int_0^d DG(t, \tau, x(s)) + \int_0^{t-d} D\tilde{G}(t - d, \tau, y(s)) =$$

$$= f(t) + \int_0^d DG(t, \tau, x(s)) + \int_0^{t-d} DG(t, \tau + d, y(s)) =$$

$$= f(t) + \int_0^d DG(t, \tau, x(s)) + \int_0^{t-d} DG(t, \tau + d, x(s + d)) =$$

$$= f(t) + \int_0^d DG(t, \tau, x(s)) + \int_d^t DG(t, \tau, x(s)) = f(t) + \int_0^t DG(t, \tau, x(s)) .$$

Hence the extended function $x : [0, d] \to \mathbb{R}^N$ satisfies the equation (V) and the statement is proved.

2.6. Corollary. *If $x : [0, b) \to \mathbb{R}^N$ is a continuous solution of (V) which cannot be extended, then either $b = c$ or*

$$\limsup_{t \to b-} |x(t)| = +\infty .$$

3. COMPARISON OF THE GENERALIZED VOLTERRA INTEGRAL EQUATION WITH THE CLASSICAL ONE

We consider the classical Volterra equation

$$x(t) = f(t) + \int_0^t g(t, s, x(s))\, ds , \quad t \in I = [0, c) \tag{3.1}$$

where the function g will be assumed to satisfy the following assumptions:

(g_1) *The function $g(t, \cdot, x)$ is measurable for every $(t, x) \in I \times \mathbb{R}^N$, the function $g(t, s, \cdot)$ is continuous for every $(t, s) \in I \times I$ and $g(t, s, x) = 0$ if $t < s$.*

(g_2) *For every $b \in I$ and $K > 0$ there is a function $m(t, s)$, $m(t, \cdot)$ Lebesgue integrable for every $t \in I$ such that*

$$|g(t, s, x)| \leqq m(t, s)$$

if $|x| \leqq K$ and $0 \leqq t \leqq b$.

(g_3) *For every $b \in I$ and $K > 0$ the expression*

$$\sup_{\substack{x \in C([0,b]) \\ \|x\|_C \leqq K}} \left| \int_0^b (g(t, s, x(s)) - g(t_0, s, x(s)))\, ds \right|$$

tends to zero when t tends to t_0, $t_0 \in I$.

The assumptions (g_1)–(g_3) are the basic assumptions under which the equation (3.1) was studied by Arstein in [1]. In [1] Artstein gives also a comparison of these assumptions with the assumptions given by R. K. Miller in the book [10].

Let us define

$$G(t, \tau, x) = \int_0^\tau g(t, s, x)\, ds \tag{3.2}$$

for $(t, \tau, x) \in I \times I \times \mathbb{R}^N$.

The measurability of $g(t, \cdot, x)$ stated in (g_1) and (g_2) ensures that the function G can be defined in this way.

For $(t, x) \in I \times \mathbb{R}^N$ we clearly have $G(t, 0, x) = 0$. If $\tau > t$ then $G(t, \tau, x) = \int_0^t g(t, s, x)\, ds = G(t, t, x)$ since $g(t, s, x) = 0$ for $s > t$. Hence G satisfies the assumption (G_1).

Let $b \in I$, $K > 0$. Using (g_2) we have for $t \in [0, b)$, $0 \leqq \tau_1 \leqq \tau_2 \leqq b$, $|x| \leqq K$

$$|G(t, \tau_2, x) - G(t, \tau_1, x)| = \left|\int_{\tau_1}^{\tau_2} g(t, s, x)\, ds\right| \leqq$$

$$\leqq \int_{\tau_1}^{\tau_2} |g(t, s, x)|\, ds \leqq \int_{\tau_1}^{\tau_2} m(t, s)\, ds = M(t, \tau_2) - M(t, \tau_1)$$

where $M(t, \tau) = \int_0^\tau m(t, s)\, ds$, $M(t, \cdot)$ is nondecreasing in $[0, t]$ and $M(t, t) - M(t, 0) = \int_0^t m(t, s)\, ds < \infty$.

Hence G satisfies the assumption (G_2).

Let us prove that the function G given by (3.2) satisfies the assumption (G_3). Let $b \in I$, $K > 0$ and $\varepsilon > 0$ be given. By (g_2) there is a function $m(t, s) : [0, b] \times [0, b] \to [0, +\infty)$ such that $m(t, \cdot)$ is integrable and $|g(t, s, x)| \leqq m(t, s)$ provided $|x| < K + 1$, $t, s \in [0, b]$.

For $k \in \mathbb{N}$, $t, s \in [0, b]$ define

$$g_k(t, s) = \sup_{\substack{x, y \in R^N \\ |x| \leqq K, |y - x| \leqq 1/k}} |g(t, s, x) - g(t, s, y)| \,;$$

for every fixed $t \in [0, b]$ the functions $g_k(t, \cdot)$ are measurable, $g_{k+1}(t, s) \leqq g_k(t, s)$ and $0 \leqq g_k(t, s) \leqq 2m(t, s)$ for $k \in \mathbb{N}$. Since for every fixed $(t, s) \in I \times I$ the function $g(t, s, x)$ is uniformly continuous on compact sets in the third variable x, we have $\lim_{k \to \infty} g_k(t, s) = 0$ for every $t, s \in [0, b]$. Hence by the Lebesgue dominated convergence theorem for every $t \in [0, b]$ we get $\lim_{k \to \infty} \int_0^t g_k(t, s)\, ds = 0$ and consequently there exists $k_0 \in \mathbb{N}$ (depending on t) such that $\int_0^t g_{k_0}(t, s)\, ds < \varepsilon$.

Let us set $\varrho = 1/k_0$ and $\mu(t, \tau) = \int_0^\tau g_{k_0}(t, s)\, ds$. For $y \in \mathbb{R}^N$, $|y - x| < \varrho$, $0 \leqq \tau_1 \leqq \tau_2 \leqq t \leqq b$ we have

$$|G(t, \tau_2, y) - G(t, \tau_1, y) - G(t, \tau_2, x) + G(t, \tau_1, x)| =$$

$$= \left|\int_{\tau_1}^{\tau_2} (g(t, s, y) - g(t, s, x))\, ds\right| \leqq \int_{\tau_1}^{\tau_2} |g(t, s, y) - g(t, s, x)|\, ds \leqq$$

$$\leqq \int_{\tau_1}^{\tau_2} g_{k_0}(t, s)\, ds = \mu(t, \tau_2) - \mu(t, \tau_1)\,.$$

Since by definition we have $\mu(t, t) - \mu(t, 0) < \varepsilon$ all the requirements of the assumption (G_3) are satisfied for the function G. In this way we have obtained the following result:

3.1. Proposition. *Assume that the function $g : I \times I \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies* (g_1),

(g_2). *If the function* $G : I \times I \times \mathbb{R}^N \to \mathbb{R}^N$ *is defined by the relation* (3.2) *then it satisfies the assumptions* (G_1), (G_2) *and* (G_3).

Further, let us prove the following statement.

3.2. Proposition. *Assume that the function* $g : I \times I \times \mathbb{R}^N \to \mathbb{R}^N$ *satisfies* (g_1), (g_2); *let* $G : I \times I \times \mathbb{R}^N \to \mathbb{R}^N$ *be defined by* (3.2). *If* $t \in I$ *and* $\varphi : [0, t] \to \mathbb{R}^N$ *is a continuous function, then both the integrals* $\int_0^t g(t, s, \varphi(s))\, ds$, $\int_0^t DG(t, \tau, \varphi(s))$ *exist and have the same value.*

Proof. Since the function φ is uniformly continuous on $[0, t]$, there exists a sequence of piecewise constant functions $\varphi_k : [0, t] \to \mathbb{R}^N$ such that $\lim_{k \to \infty} \varphi_k(s) = \varphi(s)$ uniformly on $[0, t]$. Evidently the integrals $\int_0^t g(t, s, \varphi_k(s))\, ds$ and $\int_0^t DG(t, \tau, \varphi_k(s))$ exist and are equal for every $k \in \mathbb{N}$.

It is easy to check that if (g_1) and (g_2) hold, then the sequence of functions $g(t, s, \varphi_k(s))$ satisfies all the required assumptions of the Lebesgue dominated convergence theorem; the sequence $G(t, \tau, \varphi_k(s))$ similarly satisfies the assumptions of the theorem (1.8). Hence the integrals $\int_0^t g(t, s, \varphi(s))\, ds$, $\int_0^t DG(t, \tau, \varphi(s))$ exist by the corresponding convergence theorems and by the same theorems they equal each other.

Now we can state the following simple result.

3.3. Proposition. *If* $g : I \times I \times \mathbb{R}^N \to \mathbb{R}^N$ *satisfies* (g_1), (g_2) *and* (g_3) *then the function* $G : I \times I \times \mathbb{R}^N \to \mathbb{R}^N$ *defined by* (3.2) *satisfies* (G_4). *Moreover, every solution* $x(s)$ *of the equation* (3.1) *is also a solution of the generalized Volterra integral equation*

$$x(t) = f(t) + \int_0^t DG(t, \tau, x(s)) \tag{V}$$

and vice versa.

Proof. The condition (G_4) for G follows immediately from (g_3) and from the fact stated in Proposition 3.2. Since every solution of (3.1) and (V) is a continuous function (this is apparent in the case of (3.1) and can be shown in the case of (V) by an approximation argument and by the convergence theorem 1.8), we obtain the required equivalence of the equations again by Proposition 3.2.

In this way we obtain that the classical theory of Volterra integral equations (3.1) with g satisfying the (probably weakest) conditions $(g_1)-(g_3)$ is covered by the theory of generalized Volterra integral equations described in this note.

In particular, by Theorem 2.4 we obtain the local existence of solutions of the equation (3.1) and the existence of maximal solutions of this equation together with their properties described by Theorem 2.5 and Corollary 2.6.

4. CONTINUOUS DEPENDENCE RESULTS

In this section we will consider the dependence of solutions of the equation (V) on the functions f and G, where f belongs to $C(I)$ – the space of continuous $\mathbb{R}^N$-valued functions on I equipped with the usual topology of uniform convergence on compact sets and $G : I \times I \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies the assumptions $(G_1)-(G_4)$.

The main result is an Artstein-type theorem which was stated for the classical case in [1]. In the same way as Artstein did, we restrict our considerations to a subspace of functions G satisfying $(G_1)-(G_3)$ and a uniform version of (G_4).

Let $U(t, t_0, b, K)$ be a function defined on $I \times I \times I \times (0, +\infty)$ with values in $(0, +\infty)$ such that

$$\lim_{t \to t_0} U(t, t_0, b, K) = 0 .$$

The set $\mathscr{G}(U)$ associated with the function U consists of all functions $G : I \times I \times \mathbb{R}^N \to \mathbb{R}^N$ that satisfy $(G_1)-(G_3)$ and

(G_4, U) *If* $x \in C([0, b])$, $\|x\|_{C([0,b])} \leqq K$ *then*

$$\left| \int_0^b \mathrm{D}[G(t, \tau, x(s)) - G(t_0, \tau, x(s))] \right| \leqq U(t, t_0, b, K) .$$

(The function $U(t, t_0, b, K)$ represents a common "modulus of continuity" for all functions of the type

$$t \in I \mapsto \int_0^b \mathrm{D}G(t, \tau, x(s)) \in \mathbb{R}^N$$

where $x \in C([0, b])$, $\|x\|_C \leqq K$ and $G \in \mathscr{G}(U)$.

A topology $\mathscr{T}$ on $\mathscr{G}(U)$ is called *jointly continuous* if for every fixed $t \in I$ the mapping

$$(G, x) \in \mathscr{G}(U) \times C([0, t]) \mapsto \int_0^t \mathrm{D}G(t, \tau, x(s)) \in \mathbb{R}^N$$

is continuous with respect to the product topology on $\mathscr{G}(U) \times C([0, t])$ given by $\mathscr{T}$ and the sup – norm topology on $C([0, t])$.

Let $\mathscr{T}$ be and arbitrary topology on $\mathscr{G}(U)$. Let us define the property of continuous dependence (C) for $\mathscr{G}(U)$ with the topology $\mathscr{T}$.

(C) *Suppose that the net G_k converges to G in the topology $\mathscr{T}$. Then for every net f_k converging to f in $C(I)$ the following assertion holds. Let $x_k(t)$ be a maximally defined solution of*

$$(\mathrm{V}_k) \qquad x(t) = f_k(t) + \int_0^t \mathrm{D}G_k(t, \tau, x(s)) .$$

Then there exists a maximally defined solution $x(t)$ of

$$(\mathrm{V}) \qquad x(t) = f(t) + \int_0^t \mathrm{D}G(t, \tau, x(s))$$

with domain $[0, \alpha)$ *and a subnet* x_m *of the net* x_k *such that* x_m *converges to* x *uniformly on compact subintervals of* $[0, \alpha)$. *(In particular if* $[0, \alpha_m)$ *is the domain of* x_m *and* $0 < d < \alpha$ *then for* m *sufficiently large we have* $d \leqq \alpha_m$.*)*

The main result is the following

4.1.a. Theorem. (Artstein's continuous dependence theorem.) *Let* $\mathcal{T}$ *be a topology on* $\mathcal{G}(U)$. *Then* $\mathcal{T}$ *has the property* (C) *if and only if it is jointly continuous.*

The detailed proof of this theorem is given in [1] by Artstein for the classical Volterra equation and it can be used with slight changes for our case too.

Let us mention that if we are speaking generally about a topology $\mathcal{T}$, then generalized sequences (nets) with the Moore-Smith concept of convergence are to be used.

Here we give a weaker sequential form of Artstein's theorem where we are speaking about convergence of ordinary sequences instead of topologies.

For an (ordinary) sequence G_k, $k = 1, 2, \ldots$ of functions in $\mathcal{G}(U)$ we say that it converges jointly to a function $G \in \mathcal{G}(U)$ for $k \to \infty$ if for every fixed $t \in I$ and every uniformly convergent sequence $\varphi_k \in C([0, t])$, $\varphi_k \to \varphi$ in $C([0, t])$ for $k \to \infty$ we have

$$\lim_{k \to \infty} \int_0^t \mathrm{D}G_k(t, \tau, \varphi_k(s)) = \int_0^t \mathrm{D}G(t, \tau, \varphi(s)) .$$

4.1.b. Theorem. *Suppose that the sequence* G_k, $k = 1, 2, \ldots$ *converges in some sense to* G *for* $k \to \infty$ $(G, G_k \in \mathcal{G}(U), k = 1, 2, \ldots)$. *Then the following property* (C_s) *holds if and only if* G_k *converges jointly to* G *for* $k \to \infty$.

(C_s) *For every* $f_k \in C(I)$, $k = 1, 2, \ldots, f_k \to f$ *in* $C(I)$ *the following assertion holds. If* x_k *is a maximally defined continuous solution of the equation* (V_k) *then there exists a maximally defined solution* x *of the equation* (V) *with domain* $[0, \alpha)$ *and a subsequence* x_m *of* x_k *such that* x_m *converges to* x *uniformly on compact subintervals of* $[0, \alpha)$.

Proof. Sufficiency. Let us define α by

$$\alpha = \sup \{d;\ x_k \text{ are defined and equicontinuous on } [0, d] \text{ for } k \text{ sufficiently large}\} .$$

Let $b \in I$. Since $f_k \to f$ for $k \to \infty$ in $C([0, b])$ there is $K > 0$ such that $|f_k(t)| \leqq K$ for $t \in [0, b]$ and f_k are equicontinuous on $[0, b]$. Assume that $d > 0$ is such that $U(t, 0, b, K + 1) < 1$ for $t \in [0, d]$. The solutions x_k exist on $[0, d]$ and $|x_k(t)| \leqq K + 1$ for $t \in [0, d]$. Assume that $[0, \alpha_m) \subset [0, d]$. If we assume that the estimate $|x_k(t)| < K + 1$ does not hold, then there exists a $t_0 \in [0, \alpha_m)$ such that $|x_k(t)| \leqq K + 1$ for $t \in [0, t_0)$ and $|x(t_0)| = K + 1$. In this case we get a contradictory inequality

$$|x_k(t_0)| \leqq |f_k(t_0)| + \left| \int_0^{t_0} \mathrm{D}G(t_0, \tau, x_k(s)) \right| \leqq K + U(t_0, 0, b, K + 1) < K + 1 .$$

Hence $|x_k(t)| < K + 1$ for all $t \in [0, \alpha_k) \subset [0, d]$ and this contradicts by Theorem 2.5 and Corollary 2.6 the maximality of x_k. This implies $d < \alpha_k$.

By (G_4, K) we get further

$$|x_k(t) - x_k(t_0)| \leqq$$

$$\leqq |f_k(t) - f_k(t_0)| + \left|\int_0^d D[G_k(t, \tau, x_k(s)) - G_k(t_0, \tau, x_k(s))]\right| \leqq$$

$$\leqq |\omega(t) - \omega(t_0)| + U(t, t_0, d, N + 1).$$

where ω is the modulus of continuity for the sequence f_k on $[0, b]$. Hence the sequence x_k is equicontinuous on $[0, d]$. This yields $\alpha > 0$.

Assume that $\alpha < c$ ($I = [0, c)$). We prove the following statement:

(S) *There exists a subsequence $\{x_l\}$ of $\{x_k\}$ such that either for every l the function x_l is not defined on $[0, \alpha]$ or no subsequence of $\{x_l\}$ is equicontinuous on $[0, \alpha]$.*

Assume that this statement is false, i.e. x_k is defined on $[0, \alpha]$ for every k and x_k are equicontinuous functions on $[0, \alpha]$. Since f_k are equibounded and $x_k(0) = f_k(0)$, the sequence $\{x_k\}$ is bounded by a constant $K > 0$. Let $b \in (\alpha, c)$; we can assume that $|f_k(t)| < K$ for $t \in [0, b]$. Let $\tilde{d} \in (\alpha, b)$ be such that $U(t, \alpha, b, 3K + 1) < 1$ for $t \in [\alpha, \tilde{d}]$.

In the same way as above we can show that $x_k(t)$ is defined on $[0, \tilde{d}]$ and $|x_k(t)| < 3K + 1$ for $t \in [0, \tilde{d}]$. If we assume that there is a value $t_0 \in (\alpha, \tilde{d}]$ such that $|x_k(t)| < 3K + 1$ for $t \in [\alpha, t_0)$ and $|x_k(t_0)| = 3K + 1$ then

$$|x_k(t_0) - x_k(\alpha)| \leqq |f_k(t_0) - f_k(\alpha)| +$$

$$+ \left|\int_0^b D^2 G_k(t, \tau, x_k(s)) - G_k(\alpha, \tau. x_k(s))]\right| \leqq 2K + U(t_0, \alpha, b, 3K + 1) < 2K + 1$$

and we get a contradiction $|x_k(t_0)| < |x_k(\alpha)| + 2K + 1 \leqq 3K + 1$. The boundedness of $x_k(t)$ on $[0, \tilde{d}]$ implies by (G_4, U) the equicontinuity of the sequence $\{x_k(t)\}$ on $[0, \tilde{d}]$ and by the definition of α we sould have $\tilde{d} \leqq \alpha$ which would contradict $\tilde{d} \in (\alpha, b)$. Hence the statement (S) holds.

Let now $\{x_l(t)\}$ be a sequence with the properties given in (S). For every $d \in (0, \alpha)$ the sequence $\{x_l\}$ is bounded and equicontinuous on $[0, d]$. Let us assume $d_m \to \alpha-$. Let $\{x_{l_1}(t)\}$ be a subsequence of $\{x_l(t)\}$ which converges uniformly on $[0, d_1]$ to a certain $x(t)$; let further $\{x_{l_2}(t)\}$ be a subsequence of $\{x_{l_1}(t)\}$ which converges uniformly to $x(t)$ on $[0, d_2]$. Continuing in this way we get by the usual diagonalization a subsequence $\{x_m\}$ of $\{x_l\}$ which uniformly converges on every compact interval $[0, d] \subset [0, \alpha)$ to a function $x(t)$ defined on $[0, \alpha)$.

For $t \in [0, \alpha)$ we have

$$x_m(t) = f_m(t) + \int_0^t DG_m(t, \tau, x_m(s)).$$

Since $x_m \to x$ in $C([0, t])$ and $G_m \to G$ jointly we obtain

$$\lim_{m\to\infty} \int_0^t DG_m(t, \tau, x_m(s)) = \int_0^t DG(t, \tau, x(s))$$

and consequently – passing to the limit $m \to \infty$ – $x(t)$ is a solution of (V) for every $t \in [0, \alpha)$ because $f_m \to f$ in $C([0, t])$.

Finally we have to show that the solution $x(t)$ cannot be continued to a solution on $[0, \beta]$ where $\beta > \alpha$. Assume that $\alpha < c$ (if $\alpha = c$ then x is a maximal solution and we have nothing to prove). Let us suppose that the solution $x(t)$ of (V) can be continued to $[0, \beta]$ with $\beta > \alpha$. Then $|x(t)| \leqq K$ for $t \in [0, \alpha)$ and let also $|f_m(t)| \leqq K$ hold for $t \in [0, \alpha]$. Further, let $t_0 < \alpha$ be such that for $t \in [t_0, \alpha]$ we have

$$U(t, t_0, \alpha, 3K + 3) < 1 .$$

For sufficiently large m we have $|x_m(t)| < K + 1$ for $t \in [0, t_0)$. By definition of a solution we have

$$|x_m(t) - x_m(t_0)| \leqq$$

$$\leqq |f_m(t) - f_m(t_0)| + \left|\int_0^\alpha D[G_m(t, \tau, x_m(s)) - G_m(t_0, \tau, x_m(s))]\right|$$

for $t > t_0$. Assume that there exists a $t \in (t_0, \alpha]$ such that $s \in [0, t) \Rightarrow |x_m(s)| < < 3K + 3$ and $|x_m(t)| = 3K + 3$ then

$$|x_m(t) - x_m(t_0)| \leqq 2K + U(t, t_0, \alpha, 3K + 3) < 2K + 1$$

and $|x_m(t)| < 3K + 2$. This contradiction yields $|x_m(t)| < 3K + 2$ for every $t \in \in [0, \alpha]$.

Hence

$$|x_m(t) - x_m(t_0)| \leqq |f_m(t) - f_m(t_0)| + U(t, t_0, \alpha, 3K + 3)$$

for $t, t_0 \in [0, \alpha]$ and consequently $\{x_m\}$ is equicontinuous on $[0, \alpha]$ since the sequence $\{f_m\}$ is equicontinuous on $[0, \alpha]$. But this contradicts the property of the sequence $\{x_m\}$ stated in (S).

Necessity. Assume that there exists $t \in I$ and a sequence $\varphi_k \in C([0, t])$ such that $\varphi_k \to \varphi$ in $C([0, t])$ and such that $\int_0^t DG_k(t, \tau, \varphi_k(s))$ does not tend to $\int_0^t DG(t, \tau, \varphi(s))$ for $k \to \infty$, i.e. we assume that the sequence $G_k \in \mathcal{G}(U)$, $k = 1, 2, \ldots$ does not converge jointly to the function $G \in \mathcal{G}(U)$. This means that there exists a fixed $\varepsilon > 0$ and a subsequence $\{G_l\}$ of $\{G_k\}$ such that

$$(+) \qquad \left|\int_0^t DG_l(t, \tau, \varphi_l(s)) - \int_0^t DG(t, \tau, \varphi(s))\right| \geqq \varepsilon .$$

Let us set

$$f_l(r) = \varphi_l(r) - \int_0^r DG_l(r, \tau, \varphi_l(s))$$

for $r \in [0, t]$. Evidently $f_l \in C([0, t])$. It can be easily shown that the sequence $\{f_l\}$ is bounded and equicontinuous on $[0, t]$ (to this end (G_4, U) and the uniform convergence of φ_l to φ is used together with the equality $f_l(0) = \varphi_l(0)$). Hence there exists a subsequence $\{f_m\}$ of $\{f_l\}$ which converges uniformly on $[0, t]$ to a certain function f. The functions φ_m are solutions of the equation (V_m) and by the continuous dependence property (C) there exists a subsequence of φ_m which converges uniformly to a solution of equation (V). Hence φ is a solution of (V), but by (+) we get

$$\left|\varphi(t) - f(t) - \int_0^t DG(t, \tau, \varphi(s))\right| =$$

$$= \lim_{m\to\infty} \left|\varphi_m(t) - f_m(t) - \int_0^t DG(t, \tau, \varphi(s))\right| =$$

$$= \lim_{m\to\infty} \left|\int_0^t DG_m(t, \tau, \varphi_m(s)) - \int_0^t DG(t, \tau, \varphi(s))\right| \geqq \varepsilon ,$$

i.e., φ cannot be a solution of (V) and this is a contradiction. The property (C) does not hold under these circumstances.

In the general case of assumptions given for the validity of Artstein's theorem 4.1.a it is not easy to give suitable sufficient conditions for the joint continuity of the topology $\mathscr{T}$ on the space $\mathscr{G}(U)$.

If the assumption (G_3) is replaced by its "uniform" version which will be exactly formulated below, then the topology of uniform convergence on compact sets in the space of functions G is jointly continuous.

Let $\mu(t, \tau, b, K, \varepsilon) : I \times I \times I \times (0, +\infty) \times (0, +\infty) \to \mathbb{R}$ be such a function that $\mu(t, \cdot, b, K, \varepsilon)$ is nondecreasing in $[0, t]$ and $\mu(t, t, b, K, \varepsilon) - \mu(t, 0, b, K, \varepsilon) < < \varepsilon$.

The set $\mathscr{G}(U, \mu)$ associated with this function μ consists of all functions from $\mathscr{G}(U)$ that satisfy the condition (G_3) with the given function μ and with the same $\varrho(t)$ from (G_3).

On $\mathscr{G}(U, \mu)$ let us define the well-known topology $\mathscr{T}_1$ of uniform convergence on compact sets. This topology is given by the following seminorms.

For each $b \in I$, $K > 0$ let

$$p(G; b, K) = \sup\{|G(t, \tau, x)|;\ 0 \leqq t \leqq b,\ 0 \leqq \tau \leqq b,\ |x| \leqq K\} .$$

The collection of sets $s(\varepsilon; b, K) = \{G; p(G; b, K) < \varepsilon\}$ defines a subbasis of neighbourhoods of the origin. It is known that if we set $b_n \to c-$, $K_n \to +\infty$ and define $p_n(G) = p(G; b_n, K_n)$, then the countable subsets $s_n(\varepsilon) = s(\varepsilon; b_n, K_n)$ generate the same topology and the space in question is a metric space.

4.2. Proposition. *The topology $\mathscr{T}_1$ (of uniform convergence on compact sets) is jontly continuous on $\mathscr{G}(U, \mu)$.*

Proof. By definition we have to show that for every $t \in I$ the mapping

$$(G, x) \mapsto \int_0^t DG(t, \tau, x(s))$$

from $\mathcal{G}(U, \mu) \times C([0, t])$ into $\mathbb{R}^N$ is continuous with respect to the product topology of $\mathcal{T}_1$ and the sup-norm topology on $C([0, t])$. Let $t \in I$ be fixed. Assume that $(G_n, x_n) \to (G_0, x_0)$ with respect to the product topology on $\mathcal{G}(U, \mu) \times C([0, t])$.

Let us consider the difference

$$\int_0^t DG_n(t, \tau, x_n(s)) - \int_0^t DG_0(t, \tau, x_0(s)) =$$

$$= \int_0^t D[G_n(t, \tau, x_n(s)) - G_n(t, \tau, x_0(s))] +$$

$$+ \int_0^t D[G_n(t, \tau, x_0(s)) - G_0(t, \tau, x_0(s))] = I_1 + I_2 .$$

Since $\lim_{n \to \infty} x_n(s) = x_0(s)$ uniformly on $[0, t]$, there exists $K > 0$ such that $|x_n(s)| \leqq K$ for every $n = 0, 1, \ldots$.

Let $\varepsilon > 0$ be given and let $\varrho(t) > 0$ correspond to this $\varepsilon > 0$ by the assumption (G_3). To this $\varrho(t) > 0$ there is $n_0 \in \mathbb{N}$ such that for $n > n_0$ we have $|x_n(s) - x_0(s)| < \varrho(t)$ for $s \in [0, t]$. By (G_3) used for $n > n_0$ we get for every $s \in [0, t]$ and $0 \leqq \tau_1 \leqq \tau_2 \leqq t$ the inequality

$$|G_n(t, \tau_2, x_n(s)) - G_n(t, \tau_1, x_n(s)) - G_n(t, \tau_2, x_0(s)) + G_n(t, \tau_1, x_0(s))| \leqq$$
$$\leqq \mu(t, \tau_2, t, K, \varepsilon) - \mu(t, \tau_1, t, K, \varepsilon) .$$

Hence for $n > n_0$ we get by (1.7) the inequality

$$|I_1| \leqq \mu(t, t, t, K, \varepsilon) - \mu(t, 0, t, K, \varepsilon) < \varepsilon .$$

Since $x_0 : [0, t] \to \mathbb{R}^N$ is uniformly continuous, there exists a piecewise constant function $\varphi : [0, t] \to \mathbb{R}^N$ such that $|\varphi(s) - x_0(s)| < \varrho(t)$ for every $s \in [0, t]$. Using (G_3) we have

$$|G_n(t, \tau_2, \varphi(s)) - G_n(t, \tau_1, \varphi(s)) - G_n(t, \tau_2, x_0(s)) + G_n(t, \tau_1, x_0(s))| \leqq$$
$$\leqq \mu(t, \tau_2, \ldots) - \mu(t, \tau_1, \ldots)$$

for every $s \in [0, t]$, $0 \leqq \tau_1 \leqq \tau_2 \leqq t$, $n = 0, 1, \ldots$.

Hence

$$\left| \int_0^t D(G_n(t, \tau, x_0(s)) - G_n(t, \tau, \varphi(s))) \right| \leqq \varepsilon$$

for $n = 0, 1, \ldots$ and this yields

$$|I_2| = \left| \int_0^t D^2 G_n(t, \tau, x_0(s)) - G_0(t, \tau, x_0(s))] \right| \leqq$$

$$\leqq 2\varepsilon + \left| \int_0^t D[G_n(t, \tau, \varphi(s)) - G_0(t, \tau, \varphi(s))] \right| .$$

Since $\varphi : [0, t] \to \mathbb{R}^N$ is piecewise constant, the last term (by definition of the integral) is a finite sum of terms of the form $G_n(t, \tau, c) - G_0(t, \tau, c)$, $G_n(t, \tau+, c) - G_0(t, \tau+, c)$ or $G_n(t, \tau-, c) - G_0(t, \tau-, c)$ where $0 \leqq \tau \leqq t$ and $c \in \mathbb{R}^N$, $|c| \leqq K$ for a suitably chosen constant K.

Since $G_n \to G_0$ in the topology $\mathscr{T}_1$ of uniform convergence on compact sets, there is an $n_1 \in \mathbb{N}$ such that for $n > n_1$ we have

$$\left| \int_0^t D[G_n(t, \tau, \varphi(s)) - G_0(t, \tau, \varphi(s))] \right| < \varepsilon .$$

In this way we have shown that for every $\varepsilon > 0$ there is an $n_2 \in \mathbb{N}$ ($n_2 = \max(n_0, n_1)$) such that for $n > n_2$ we have

$$\left| \int_0^t D[G_n(t, \tau, x_n(s)) - G_0(t, \tau, x_0(s))] \right| \leqq |I_1| + |I_2| < \varepsilon + 3\varepsilon = 4\varepsilon ,$$

i.e., the topology $\mathscr{T}_1$ on $\mathscr{G}(U, \mu)$ is jointly continuous.

5. SOME RESULTS WITH A LIPSCHITZ-TYPE CONDITION

For functions $G : I \times I \times \mathbb{R}^N \to \mathbb{R}^N$ determining the integral term in the equation (V) the assumptions $(G_1)-(G_4)$ played an essential role for deriving basic results concerning the solutions of (V).

In the assumption (G_3) the type of continuity of the differences $G(t, \tau_2, x) - G(t, \tau_1, x)$ in the variable x is specified. Let us now replace the condition (G_3) by a stronger condition which reads as follows.

(G_3+) *For every* $b \in I$, $K > 0$ *there exists a bounded function* $L(t, \tau)$, $L(t, \cdot)$ *nondecreasing in* $[0, t]$, *such that*

$$|G(t, \tau_2, x) - G(t, \tau_1, x) - G(t, \tau_2, y) - G(t, \tau_1, y)| \leqq$$

$$\leqq |x - y| (L(t, \tau_2) - L(t, \tau_1))$$

whenever

$$(t, \tau_1, x), (t, \tau_2, x), (t, \tau_1, y), (t, \tau_2, y) \in [0, b] \times [0, b] \times \mathbb{R}^N$$

such that $0 \leqq \tau_1 \leqq \tau_2 \leqq t \leqq b$ *and* $|x|, |y| \leqq K$.

It is easy to see that (G_3+) implies (G_3).

Remark. If a function $g : I \times I \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies (g_1), (g_2) given in Section 3 and the Lipschitz condition:

for every $b \in I$, $K > 0$ there exists a real valued function $l(t, s)$ defined for $0 \leqq s \leqq t \leqq b$, $l(t, \cdot)$ integrable on $[0, t]$ such that

$$|g(t, s, x) - g(t, s, y)| \leqq l(t, s)\,|x - y|$$

if $0 \leqq s \leqq t \leqq b$, $|x|, |y| \leqq K$,

then the function $G : I \times I \times \mathbb{R}^N \to \mathbb{R}^N$ can be defined by

$$G(t, \tau, x) = \int_0^\tau g(t, s, x)\,\mathrm{d}s$$

and it can be easily shown that the function G satisfies (G_3+) with the function $L(t, \tau)$ given by

$$L(t, \tau) = \int_0^\tau l(t, s)\,\mathrm{d}s\,.$$

A fundamental result using (G_3+) is a uniqueness result for the equation (V).

5.1. Theorem. *If $f : I \to \mathbb{R}^N$ is continuous and if $G : I \times I \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies the assumptions (G_1), (G_2), (G_3+), (G_4), where the function L from (G_3+) satisfies*

(5.1) $$\lim_{h \to 0+} [L(t + h, t + h) - L(t + h, t)] = 0$$

for every $t \in [0, b]$, then the integral equation (V) has a uniquely determined continuous solution.

Proof. The existence of a continuous solution of the equation (V) is an immediate consequence of Theorem 2.4. Assume that we have two continuous solutions x, y of the equation (V) both defined on the interval $[0, d)$. Let us define

$$\hat{d} = \sup \{t \in [0, d);\ x(s) = y(s) \text{ for } s \leqq t\}$$

and assume that $\hat{d} < d$. Evidently $\hat{d} \geqq 0$ since $x(0) = y(0) = f(0)$ by definition of a solution. From the continuity of solutions y, x we have $x(t) = y(t)$ for every $t \in [0, \hat{d}]$. Further we have also

$$x(t) - y(t) = \int_0^t \mathrm{D}[G(t, \tau, x(s)) - G(t, \tau, y(s))] =$$

$$= \int_{\hat{d}}^t \mathrm{D}[G(t, \tau, x(s)) - G(t, \tau, y(s))]$$

for $t \in [\hat{d}, \hat{d} + h]$, $h > 0$, $\hat{d} + h < d$.

Using (G_3+) and the estimate (1.7) we get

$$(5.2) \qquad |x(t) - y(t)| \leqq \left| \int_a^t \mathrm{D}[G(t, \tau, x(s)) - G(t, \tau, y(s))] \right| \leqq$$

$$\leqq \sup_{s \in [\hat{a}, \hat{a}+h]} |x(s) - y(s)| \int_a^t \mathrm{D}L(t, \tau) = \|x - y\|_{C([\hat{a}, \hat{a}+h])} (L(t, t) - L(t, \hat{a}))$$

for $t \in [\hat{a}, \hat{a} + h]$.

By (5.1) we can assume that $h > 0$ is such that $L(t, t) - L(t, \hat{a}) < \frac{1}{2}$ for every $t \in [\hat{a}, \hat{a} + h]$. By (5.2) we obtain

$$\|x - y\|_{C([\hat{a}, \hat{a}+h])} < \tfrac{1}{2} \|x - y\|_{C([\hat{a}, \hat{a}+h])} \, .$$

Hence $x(s) = y(s)$ for $s \in [\hat{a}, \hat{a} + h]$, consequently $\hat{a} = d$ and the theorem is proved.

Remark. Theorem 5.1 generalizes the known classical uniqueness result for Volterra integral equations as stated e.g. in Miller's book [10].

In fact if we restrict ourselves to the classical Volterra equation

$$x(t) = f(t) + \int_0^t g(t, s, x(s)) \, \mathrm{d}s$$

then by the above remark the additional condition (5.1) has the form

$$\lim_{h \to 0} \int_t^{t+h} l(t + h, s) \, \mathrm{d}s = 0$$

for $t \in [0, b]$, i.e., we obtain exactly the condition for uniqueness stated in [10] by R. K. Miller.

Let $\mathscr{G}(U)$ be the space given in Section 4. Let $\mathscr{G}(U, L)$ be the subspace of functions belonging to $\mathscr{G}(U)$ which satisfy (G_3+) with a given function $L(t, \tau, b, K)$. Then in the considerations in Sec. 4 the subspace $\mathscr{G}(U, \mu)$ can be replaced by $\mathscr{G}(U, L)$.

Moreover, in a similar way as Artstein did in [1], it can be shown that a topology $\mathscr{T}$ on $\mathscr{G}(U, L)$ is jointly continuous if and only if for every t and every continuous function $x : [0, t] \to \mathbb{R}^N$ the expression

$$\int_0^t \mathrm{D}G(t, \tau, x(s))$$

is continuous in G.

6. A KNESER-TYPE THEOREM FOR GENERALIZED VOLTERRA EQUATIONS

In the paper [5] W. G. Kelley has shown that for the solution funnel of the classical Volterra integral equation the Kneser theorem holds. Here we prove the same theorem for generalized Volterra integral equation (V).

For any $[a, b] \subset I$ let $B([a, b])$ be the metric space of bounded functions x :

$: [a, b] \to \mathbb{R}^N$ with the supremal metric

$$\varrho(x, y) = \sup_{t\in[a,b]} |x(t) - y(t)| \quad \text{for} \quad x, y \in B([a, b]),$$

$$\|x\|_B = \sup_{t\in[a,b]} |x(t)| \quad \text{for} \quad x \in B([a, b]).$$

$B_c([a, b])$ will denote the subspace of piecewise constant functions belonging to $B([a, b])$. Clearly, $C([a, b])$ is a subspace of $B([a, b])$.

Let us observe the following. If a function $G : I \times I \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies $(G_1)-(G_4)$ then for each $b \in I$ and $K > 0$ the expression

$$\sup_{\substack{\varphi\in B_c([a,b]) \\ \|\varphi\|_B \leqq K}} \left| \int_0^b D[G(t, \tau, \varphi(s)) - G(t_0, \tau, \varphi(s))] \right|$$

tends to zero for $t \to t_0$ provided $t_0 \in I$.

This can be shown as follows. Let $b \in I$, $K > 0$ be given. By (G_4) for every $\varepsilon > 0$, $t_0 \in I$ there exists $\delta > 0$ such that

$$\left| \int_0^b D[G(t, \tau, x(s)) - G(t_0, \tau, x(s))] \right| < \varepsilon$$

for $|t - t_0| < \delta$, $x \in C([0, b])$, $\|x\|_C \leqq K$.

Let an arbitrary $\varphi \in B_c([0, b])$, $\|\varphi\|_B \leqq K$ be given. Then there exists a sequence $x_n \in C([0, b])$, $\|x_n\|_C \leqq K$ such that $\lim_{n\to\infty} x_n(s) = \varphi(s)$ for every $s \in [0, b]$.

By $(G_1)-(G_3)$ we obtain that for the functions $G(t, \tau, x_n(s))$, $G(t, \tau, \varphi(s))$ the assumptions of the convergence theorem (1.8) are satisfied. Hence for every $t \in I$ we have

$$\lim_{n\to\infty} \int_0^b DG(t, \tau, x_n(s)) = \int_0^b DG(t, \tau, \varphi(s)).$$

If t_0, $t \in I$ are fixed and such that $|t - t_0| < \delta$ then

$$\left| \int_0^b DG(t, \tau, \varphi(s)) - \int_0^b DG(t_0, \tau, \varphi(s)) \right| \leqq \left| \int_0^b D[G(t, \tau, \varphi(s)) - G(t, \tau, x_n(s))] \right| +$$

$$+ \left| \int_0^b D[G(t_0, \tau, \varphi(s) - G(t_0, \tau, x_n(s))] \right| + \left| \int_0^b D[G(t, \tau, x_n(s)) - G(t_0, \tau, x_n(s))] \right| \leqq$$

$$\leqq \varepsilon + \left| \int_0^b D[G(t, \tau, \varphi(s)) - G(t, \tau, x_n(s))] \right| + \left| \int_0^b D[G(t_0, \tau, \varphi(s)) - G(t_0, \tau, x_n(s))] \right| \leqq 2\varepsilon$$

if we choose a sufficiently large n. This yields our statement. Let us mention that the integral $\int_0^b DG(t, \tau, x(s))$ exists for every $x \in B_c([0, b])$. This can be verified easily by the definition of the generalized Perron integral (see the proof of Corollary 2.2).

Now we prove the generalized Kneser theorem for integral equations of the form (V).

6.1. Theorem. *Assume that* $G : I \times I \times \mathbb{R}^N \to \mathbb{R}^N$ *satisfies* $(G_1)-(G_4)$. *Let* $f : I \to \mathbb{R}^N$ *be continuous. Suppose all solutions of*

$$\text{(V)} \qquad x(t) = f(t) + \int_0^t DG(t, \tau, x(s))$$

exist on $[0, d]$, *where* $d \in I$. *Denote*

$$S = \{x \in C([0, d]);\ x \text{ is a continuous solution of (V) on } [0, d]\}.$$

Then S *is compact and connected in* $B([0, d])$.

Proof. a) There is a $K > 0$ such that $|x(s)| \leqq K$ for every $x \in S$ and $s \in [0, d]$, i.e., S is uniformly bounded. Assume that this statement does not hold. Then there exists a sequence $x_n \in S$, $n = 1, 2, \ldots$ such that $\sup_{t \in [0,d]} |x_n(t)| > n$. Using the continuous dependence theorem 4.1.b in this special case we obtain that there exists a subsequence $\{x_k\}$ of $\{x_n\}$ which converges uniformly on $[0, d]$ to a solution x_0 of (V); the continuous function x_0 is defined on $[0, d]$, hence $\{x_k\}$ is a bounded sequence but this contradicts the properties of the sequence $\{x_n\}$.

Let us set

$$U(t_1, t_2) = U(t_1, t_2, K) = \\ = \sup_{\substack{x \in C([0,d]) \cup B_c([0,d]) \\ \|x\|_B \leqq K}} \left| \int_0^d D[G(t_1, \tau, x(s)) - G(t_2, \tau, x(s))] \right|$$

for $t_1, t_2 \in [0, d]$ where $K > 0$ is the uniform bound for S.

Let $x \in S$ be arbitrary. Then by the definition of a solution of (V) we have

$$|x(t_2) - x(t_1)| \leqq |f(t_2) - f(t_1)| + \\ + \left| \int_0^d D[G(t_2, \tau, x(s)) - G(t_1, \tau, x(s))] \right| \leqq |f(t_2) - f(t_1)| + U(t_2, t_1).$$

Hence S is a set of equicontinuous functions.

Let $x_n \in S$, $n = 1, 2, \ldots$ be such a sequence that $x_n \to x_0$ in $B([0, d])$. Evidently $x_0 \in C([0, d])$ and by Proposition 2.1 we have

$$\lim_{n \to \infty} \int_0^t DG(t, \tau, x_n(s)) = \int_0^t DG(t, \tau, x_0(s))$$

for every $t \in [0, d]$. Using this and the equality $x_n(t) = f(t) + \int_0^t DG(t, \tau, x_n(s))$ which holds for every $n = 1, 2, \ldots$ and $t \in [0, d]$ we obtain that x_0 is a solution of (V) and consequently $x_0 \in S$. This yields the closedness of S in $B([0, d])$ and also in $C([0, d])$. Hence S is compact in $B([0, d])$.

b) Let us prove that S is connected in $B([0, d])$. Without loss of generality we can assume that

$$G(t, \tau, x) = G\left(t, \tau, \frac{Kx}{|x|}\right)$$

for $x \in \mathbb{R}^N$, $|x| > K$ because any solution $x \in S$ remains in the ball $|x| \leqq K$, and the values of G for $|x| > K$ do not affect the solutions of (V) belonging to S.

By (G_2) we obtain the existence of $M : [0, d] \times [0, d] \to \mathbb{R}$, $M(t, \cdot)$ nondecreasing in $[0, t]$, such that

$$|G(t, \tau_2, x) - G(t, \tau_1, x)| \leqq M(t, \tau_2) - M(t, \tau_1)$$

for every $x \in \mathbb{R}^N$ and $0 \leqq \tau_1 \leqq \tau_2 \leqq t \leqq d$. Similarly, by (G_3) we have for every $\varepsilon > 0$ a function $\mu(t, \tau)$, $\mu(t, \cdot)$ nondecreasing in $[0, t]$, $\mu(t, t) - \mu(t, 0) < \varepsilon$, and $\varrho(t) > 0$, $t \in [0, d]$ such that

$$|G(t, \tau_2, x) - G(t, \tau_1, x) - G(t, \tau_2, y) - G(t, \tau_1, y)| \leqq$$
$$\leqq \mu(t, \tau_2) - \mu(t, \tau_1)$$

for all $x, y \in \mathbb{R}^N$, $|x - y| < \varrho(t)$ and $0 \leqq \tau_1 \leqq \tau_2 \leqq t \leqq d$.

Assume that S is not connected in $B([0, d])$. Then $S = A \cup B$ where $A, B \in B([0, d])$ are nonempty compact sets such that $\varrho(A, B) = \gamma > 0$. For $x \in B([0, d])$ we define

$$F(x) = \varrho(x, A) - \varrho(x, B) ;$$

F is a continuous real valued function. Let $x_A \in A$ and $x_B \in B$. Then $F(x_B) \geqq \gamma$, $F(x_A) \leqq -\gamma$.

For $k = 1, 2, \ldots$ we define $t_{k_j} = jd/k, j = 0, 1, \ldots, k$ and define $x_{B,k} : [0, d] \to \mathbb{R}^N$ by the relations

$$x_{B,k}(t) = x_B(t_{k_j}) \quad \text{for} \quad t \in (t_{k_j}, t_{k_{j+1}}] , \quad j = 0, 1, \ldots, k - 1 ,$$
$$x_{B,k}(0) = x_B(0) = f(0) ,$$

and similarly for $x_{A,k} : [0, d] \to \mathbb{R}^N$.

Evidently $x_{B,k} \to x_B$ and $x_{A,k} \to x_A$ in $B([0, d])$.

For $u \in [0, 1]$ and $k = 1, 2, \ldots$ define further

$$x_{u,k}(0) = f(0) ,$$

$$x_{u,k}(t) = f(t_{k_j}) + \int_0^{t_{k_j}} \mathrm{D}\{G(t_{k_j}, \tau, x_{u,k}(s)) +$$
$$+ (1 - u) [G(t_{k_j}, \tau, x_B(s)) - G(t_{k_j}, \tau, x_{B,k}(s))] +$$
$$+ u[G(t_{k_j}, \tau, x_A(s)) - G(t_{k_j}, \tau, x_{A,k}(s))]\}$$

for $t \in (t_{k_j}, t_{k_{j+1}}]$, $j = 0, 1, \ldots, k - 1$.

By definition we have $x_{A,k}, x_{B,k}, x_{u,k} \in B_c([0, d])$ for every $k = 1, 2, \ldots$ and $u \in [0, 1]$. It can be easily checked that the equalities $x_{0,k} = x_{B,k}$ and $x_{1,k} = x_{A,k}$ hold for every $k = 1, 2, \ldots$.

By an induction argument using Proposition 2.1 we can show that if $u_l \to u_0$, $l \to \infty$ is a sequence of points in $[0, 1]$ then $x_{u_l,k} \to x_{u_0,k}$, $l \to \infty$ uniformly on

$[0, d]$ for every fixed $k = 1, 2, \ldots$. Hence the function $F(x_{u,k}) : [0, 1] \to \mathbb{R}$ is a continuous function of u for every fixed k.

Since $x_{0,k} \to x_B$ and $x_{1,k} \to x_A$ in $B([0, d])$ for $k \to \infty$ and $F(x_B) \geqq \gamma > 0$, $F(x_A) \leqq -\gamma < 0$, there is $k_0 \in \mathbb{N}$ such that for $k > k_0$ we have $F(x_{0,k}) > 0$ and $F(x_{1,k}) < 0$. By the continuity of $F(x_{u,k})$ for every $k > k_0$ there exists $u_k \in (0, 1)$ such that $F(x_{u_k,k}) = 0$.

Let us now define

$$(*) \qquad x_k(t) = f(t) + \int_0^t \mathrm{D}\{G(t, \tau, x_{u_k,k}(s)) + \\ + (1 - u_k)\,[G(t, \tau, x_B(s)) - G(t, \tau, x_{B,k}(s))] + \\ + u_k[G(t, \tau, x_A(s)) - G(t, \tau, x_{A,k}(s))]\}$$

for $t \in [0, d]$ and $k > k_0$.

Since $x_{u_k,k}, x_{B,k}, x_{A,k}, x_B, x_A \in C([0, d]) \cup B_c([0, d])$ and all these functions are bounded by the constant $K > 0$, we obtain the inequality

$$(*, *) \qquad |x_k(t_2) - x_k(t_1)| \leqq |f(t_2) - f(t_1)| + 3U(t_2, t_1)$$

for every $k > k_0$ and $t_1, t_2 \in [0, d]$. Since $\lim_{t_2 \to t_1} U(t_2, t_1) = 0$ the functions $x_k : [0, d] \to \mathbb{R}^N$ are equicontinuous. We have $x_k(0) = f(0)$ for every $k = 1, 2, \ldots$ and this yields by $(*, *)$

$$|x_k(t)| \leqq |f(0)| + |f(t) - f(0)| + 3U(t, 0) \leqq \\ \leqq |f(0)| + \sup_{t \in [0,d]} \{|f(t) - f(0)| + 3U(t, 0)\},$$

i.e., the functions x_k are equibounded.

Hence there exists a subsequence of $\{x_k\}$ which uniformly converges to a function x on $[0, d]$. We denote this subsequence again by $\{x_k\}$.

Further $x_k(0) = x_{u_k,k}(0)$ and for $t \in (t_{k_j}, t_{k_{j+1}}]$ we have

$$|x_k(t) - x_{u_k,k}(t)| \leqq |f(t) - f(t_{k_j})| + \\ + \left|\int_0^t \mathrm{D}G(t, \tau, x_{u_k,k}(s)) - \int_0^{t_{k_j}} \mathrm{D}G(t_{k_j}, \tau, x_{u_k,k}(s))\right| + \\ + (1 - u_k)\left|\int_0^t \mathrm{D}[G(t, \tau, x_B(s)) - G(t, \tau, x_{B,k}(s))] - \right. \\ \left. - \int_0^{t_{k_j}} \mathrm{D}[G(t_{k_j}, \tau, x_B(s)) - G(t_{k_j}, \tau, x_{B,k}(s))]\right| + \\ + u_k\left|\int_0^t \mathrm{D}[G(t, \tau, x_A(s)) - G(t, \tau, x_{A,k}(s))] - \right. \\ \left. - \int_0^{t_{k_j}} \mathrm{D}[G(t_{k_j}, \tau, x_A(s)) - G(t_{k_j}, \tau, x_{A,k}(s))]\right| \leqq |f(t) - f(t_{k_j})| + 3U(t, t_{k_j})$$

and this yields easily $\lim_{k \to \infty} \|x_k - x_{u_k,k}\|_{B([0,d])} = 0$. Hence $x_{u_k,k} \to x$ in $B([0, d])$ and $F(x) = 0$ because $F(x_{u_k,k}) = 0$ and F is continuous in $B([0, d])$. Passing to the limit $k \to \infty$ in $(*)$ and using Proposition 2.1 we obtain that the function x belongs to S but at the same time $x \notin S$ because $F(x) = 0$. Hence S is connected and the theorem is proved.

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