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Bernd Stöckert

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REMARKS ON THE INTERPOLATION OF ANISOTROPIC SPACES OF BESOV-HARDY-SOBOLEV TYPE

BERND STÖCKERT, Jena

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This paper deals with the interpolation of anisotropic spaces $B^s_{p,q}$ and $F^s_{p,q}$ of Besov-Hardy-Sobolev type, which are introduced in [11], [9] (cf. also [12]). These spaces generalize the classical anisotropic Besov spaces, Sobolev spaces and Hardy spaces (cf. [2]), as we shall show in Section 3. The fundamental ideas are the same as those used when approaching the interpolation of the corresponding isotropic spaces of Besov-Hardy-Sobolev type, which are due to J. Peetre (cf. [5]) and H. Triebel (cf. [11], [13]).

1. DEFINITIONS AND BASIC PROPERTIES OF THE SPACES

The definition of the anisotropic spaces of Besov-Hardy-Sobolev type is based on an anisotropic decomposition in the Fourier image of the distributions considered with the aid of decomposition-functions. For these functions we need an anisotropic decomposition of $\mathbb{R}^n$ (Euclidean n-space).

Let $\boldsymbol{a} := (a_1, \ldots, a_n)$ be a fixed n-tuple of positive numbers. Then we subdivide the corridors

(1) $$K_k := \{x \mid |x_j| \leqq 2^{ka_j},\ j = 1, \ldots, n\} - \{x \mid |x_j| < 2^{(k-1)a_j},\ j = 1, \ldots, n\}\,,$$
$$k = 1, 2, 3, \ldots\,, \quad \boldsymbol{x} := (x_1, \ldots, x_n) \in \mathbb{R}^n\,,$$

in a natural way by the hyperplanes $\{x \mid x_j = 0\}$, $\{x \mid x_j = 2^{(k-1)a_j}\}$ and $\{x \mid x_j = -2^{(k-1)a_j}\}$, $j = 1, \ldots, n$, into closed rectangles $P_{k,t}$, $t = 1, \ldots, T$ $(T = 4^n - 2^n)$. Furthermore, we set

$$K_0 := \{x \mid |x_j| \leqq 1,\ j = 1, \ldots, n\}$$

and for simplicity, $P_{0,t} := K_0$ for $t = 1, \ldots, T$.

In addition to the $P_{k,t}$ we consider a little larger rectangle $P^*_{k,t}$ with the same centre as $P_{k,t}$ and sides parallel to the corresponding sides of $P_{k,t}$, which are all $\varkappa$-times larger, $1 < \varkappa < 1 + 2^{1-\max a_i}$, so that

$$P^*_{k,t} \subset (K_{k-1} \cup K_k \cup K_{k+1})\,, \quad t = 1, \ldots, T$$

(modification for $k = 0$).

The decomposition

(2) $$\mathscr{P} := \{P_{k,t}\}, \quad \mathbb{R}^n = \bigcup_{\substack{k=0,1,2,\ldots \\ t=1,\ldots,T}} P_{k,t},$$

is a regular covering of $\mathbb{R}^n$ in the sense of [9].

Remark 1. If we take in (1) corridors K_k for $k = 0, \pm 1, \pm 2, \ldots$ then we get by the same construction the so-called anisotropic homogeneous decomposition

(3) $$\mathscr{P}^{\cdot} := \{P_{k,t}\}, \quad \mathbb{R}^n - \{0\} = \bigcup_{\substack{k=0,\pm 1,\pm 2,\ldots \\ t=1,\ldots,T}} P_{k,t}.$$

Remark 2. In addition to the decomposition $\mathscr{P}$ of $\mathbb{R}^n$ we shall later consider the so-called "local modification" $\mathscr{P}^{\sim}$ of this decomposition. This means: $\mathscr{P}^{\sim} := \{\tilde{P}_{k,t}\}$, $\tilde{P}_{k,t}$ are rectangles with sides parallel to the co-ordinate axes, $k = 0, 1, 2, \ldots$, $t = 1, \ldots, \tilde{T}$ ($\tilde{T}$ a fixed natural number),

$$\mathbb{R}^n = \bigcup_{\substack{k=0,1,1,\ldots \\ t=1,\ldots,T}} \tilde{P}_{k,t}, \quad \tilde{P}_{0,t} := K_0$$

and there is a fixed natural number N such that

$$\tilde{P}_{k,t} \subset \bigcup_{h=-N}^{N} K_{k+h}$$

for an arbitrary k and $t = 1, \ldots, \tilde{T}$ ($K_l := \emptyset$ for $l < 0$).

Now we come to the *system of decomposition-functions*. $S = S(\mathbb{R}^n)$ denotes the Schwartz space of all complex-valued infinitely differentiable rapidly decreasing functions on $\mathbb{R}^n$, and $S' = S'(\mathbb{R}^n)$ the corresponding dual space of tempered distributions. $\mathscr{F}$ and $\mathscr{F}^{-1}$ denote the Fourier transform and its inverse transform on $S'(\mathbb{R}^n)$.

If L is a natural number then $\mathscr{A}_L^0 = \mathscr{A}_L^0[\mathscr{P}]$ is the set of all systems $\varphi := \{\varphi_{k,t}\}$, such that

(4) $$\varphi_{k,t} \in S, \quad \operatorname{supp} \varphi_{k,t} \subset P_{k,t}^*, \quad k = 0, 1, 2, \ldots, \quad t = 1, \ldots, T,$$

and

(5) $$\sup_{\substack{k=0,1,2,\ldots \\ t=1,\ldots,T \\ |\alpha| \leq L}} \sup_{x \in R^n} \Big[\sum_{j=1}^{n} (1 + x_j^2)^{1/2a_j}\Big]^{a.\alpha} \left|D^\alpha \varphi_{k,t}(x)\right| = c_\varphi < \infty,$$

where

$$a.\alpha := \sum_{j=1}^{n} a_j \alpha_j, \quad \alpha := (\alpha_1, \ldots, \alpha_n), \quad \alpha_j \geqq 0 \quad \text{integer},$$

$$D^\alpha \varphi_{k,t} := \frac{\partial^{|\alpha|} \varphi_{k,t}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}.$$

Furthermore, let

(6) $$\sum_{\substack{k=0,1,2,\ldots \\ t=1,\ldots,T}} \varphi_{k,t}(x) = 1 \quad \text{for all} \quad x \in \mathbb{R}^n.$$

It is easy to see that $\mathscr{A}_L^0$ is not empty.

After these preliminaries we are able to define the anisotropic spaces $B^{s}_{p,q}$ and $F^{s}_{p,q}$. We use the following abbreviations: For measurable functions f on $\mathbb{R}^n$,

$$\|f\|_{L_p} := \left(\int_{R^n} |f(x)|^p \, dx\right)^{1/p} \quad \text{if} \quad 0 < p < \infty ,$$

$$\|f\|_{L_\infty} := \operatorname{ess\,sup}_{x \in R^n} |f(x)|$$

and for sequences $\{g_{k,t}(x)\}_{\substack{k=0,1,2,\dots \\ t=1,\dots,T}}$ of measurable functions,

$$\|g_{k,t}\|_{l_q(L_p)} := \Big(\sum_{k=0}^{\infty} \sum_{t=1}^{T} \|g_{k,t}\|^q_{L_p}\Big)^{1/q} \quad \text{if} \quad 0 < q < \infty , \quad 0 < p \leqq \infty ,$$

$$\|g_{k,t}\|_{l_\infty(L_p)} := \sup_{\substack{k=0,1,2,\dots \\ t=1,\dots,T}} \|g_{k,t}\|_{L_p} , \quad 0 < p \leqq \infty ,$$

$$\|g_{k,t}\|_{L_p(l_q)} := \Big\|\Big(\sum_{k=0}^{\infty} \sum_{t=1}^{T} |g_{k,t}(x)|^q\Big)^{1/q}\Big\|_{L_p} \quad \text{if} \quad 0 < q < \infty , \quad 0 < p \leqq \infty ,$$

$$\|g_{k,t}\|_{L_p(l_\infty)} := \Big\| \sup_{\substack{k=0,1,2,\dots \\ t=1,\dots T}} |g_{k,t}(x)| \Big\|_{L_p} , \quad 0 < p \leqq \infty .$$

Definition. Let $\boldsymbol{a} := (a_1, \dots, a_n)$ be the n-tuple of positive numbers which characterizes the decomposition $\mathscr{P}$ and $\boldsymbol{s} := (s/a_1, \dots, s/a_n)$, $-\infty < s < \infty$.

(i) If $0 < p \leqq \infty$, $0 < q \leqq \infty$ and if L is a fixed natural number, $L > L^B(s, p, q)$ [1]), then

$$B^{\boldsymbol{s}}_{p,q} := \{f \mid f \in S', \ \|f\|^{\varphi}_{B^{\boldsymbol{s}}_{p,q}} := \|2^{sk}\mathscr{F}^{-1}\varphi_{k,t}\mathscr{F}f\|_{l_q(L_p)} < \infty \ \text{for all} \ \varphi \in \mathscr{A}^0_L\} .$$

(ii) If $0 < p < \infty$, $0 < q \leqq \infty$ and if L is a fixed natural number, $L > L^F(s, p, q)$ [1]), then

$$F^{\boldsymbol{s}}_{p,q} := \{f \mid f \in S', \ \|f\|^{\varphi}_{F^{\boldsymbol{s}}_{p,q}} := \|2^{sk}\mathscr{F}^{-1}\varphi_{k,t}\mathscr{F}f\|_{L_p(l_q)} < \infty \ \text{for all} \ \varphi \in \mathscr{A}^0_L\} .$$

(iii) If $0 < p < \infty$, then

$$H^{\boldsymbol{s}}_p := F^{\boldsymbol{s}}_{p,2} .$$

Remark 3. In our definition we have used the sets $S(\mathbb{R}^n)$ and $S'(\mathbb{R})$ instead of $Z(\mathbb{R}^n)$ and $Z'(\mathbb{R}^n)$, respectively, which were used in the paper [9]. Both definitions coincide, cf. [13], p. 27. It is easy to see that the definition of $B^{\boldsymbol{s}}_{p,q}$ and $F^{\boldsymbol{s}}_{p,q}$ depends only on the quotients $s/a_1, \dots, s/a_n$; this justifies our notation.

We recall the basic properties of $B^{\boldsymbol{s}}_{p,q}$ and $F^{\boldsymbol{s}}_{p,q}$ which are proved in [9]. $B^{\boldsymbol{s}}_{p,q}$ equipped with the quasi-norm $\|f\|^{\varphi}_{B^{\boldsymbol{s}}_{p,q}}$ is a quasi-Banach space (Banach space if $1 \leqq p \leqq \infty$

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[1]) The numbers L^F and L^B can be chosen (cf. [12], p. 82) so that

$$L^B(s, p, q) := |s| + \frac{6n}{p} + n + 4 , \quad L^F(s, p, q) := |s| + \frac{6n}{\min(p, q)} + n + 4 .$$

and $1 \leqq q \leqq \infty$). All the quasi-norms $\|f\|^{\varphi}_{B^s_{p,q}}$ with $\varphi \in \bigcup_{L > L^B} \mathscr{A}^0_L$ are mutually equivalent; so we omit the symbol φ in $\|\cdot\|^{\varphi}_{B^s_{p,q}}$.

The corresponding assertions hold for the spaces $F^s_{p,q}$ ($p < \infty$). For all admissible values of parameters we have

$$S \subset B^s_{p,q} \subset S' , \quad S \subset F^s_{p,q} \subset S' , \tag{7}$$

$$B^s_{p,\min(p,q)} \subset F^s_{p,q} \subset B^s_{p,\max(p,q)} , \tag{8}$$

where the sign $\subset$ means a topological embedding.

Remark 4. If we replace the nonhomogeneous decomposition $\mathscr{P}$ of $\mathbb{R}^n$ by the homogeneous decomposition $\dot{\mathscr{P}}$ of $\mathbb{R}^n - \{0\}$), take the system of decomposition-functions from $D(\mathbb{R}^n - \{0\})$ instead from $S(\mathbb{R}^n)$ and replace the condition (5) by

$$\sup_{\substack{k=0,\pm1,\pm2,\ldots \\ t=1,\ldots,T \\ |\alpha| \leqq L}} \sup_{x \in R^n} \left[\sum_{j=1}^{n} |x_j|^{1/a_j}\right]^{a.\alpha} |D^\alpha \varphi_{k,t}(x)| = c_\varphi < \infty ,$$

then we get in the same way as above the homogeneous anisotropic spaces

$$\dot{B}^s_{p,q},\ \dot{F}^s_{p,q},\ \dot{H}^s_p := \dot{F}^s_{p,2}$$

consisting of distributions from $Z'(\mathbb{R}^n - \{0\})$ instead of $S'(\mathbb{R}^n)$. Here $Z'(\mathbb{R}^n - \{0\})$ is the strong topological dual of the Fourier image $Z(\mathbb{R}^n - \{0\})$ of the space $D(\mathbb{R}^n - \{0\})$ (complex-valued infinitely differentiable functions with compact supports in $\mathbb{R}^n - \{0\}$ equipped in the usual way with a locally convex topology). For these spaces we also have the corresponding basic properties as above, cf. [9].

Remark 5. Let $\tilde{B}^s_{p,q}$, $\tilde{F}^s_{p,q}$ be the spaces which are analogously defined as $B^s_{p,q}$, $F^s_{p,q}$, based on a local modified decomposition $\mathscr{P}^{\sim}$ of $\mathscr{P}$ (cf. Definition 4 in [9]). Then it is easy to see that also in the anisotropic case the identities

$$\tilde{B}^s_{p,q} = B^s_{p,q} , \quad \tilde{F}^s_{p,q} = F^s_{p,q}$$

hold for all admissible values of s, p, q (for the isotropic case cf. [13], p. 43).

Finally, we remark that the definitions for $a_1 = a_2 = \ldots = a_n > 0$ yield the corresponding isotropic spaces. If the above relation is not fulfiled, then the result are anisotropic spaces even for $s = 0$. It is nontrivial to get isotropic spaces in the same cases, for instance: $H^0_p = L_p$, $1 < p < \infty$.

2. INTERPOLATION THEOREMS

The symbol $(\cdot, \cdot)_{\theta,q}$ denotes the K-method of interpolation (cf. [11], Sec. 1.3.2 or [1], Sec. 3.1, and for the extension to the quasi-Banach spaces [1], Sec. 3.11).

For general (anisotropic) H^p spaces (including the usual Hardy spaces), A. P.

Calderon and A. Torchinsky proved in [2, II] the following interpolation theorem:

$$(H^{p_0}, H^{p_1})_{\theta,p} = H^p , \tag{9}$$

where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad 0 < p_0 < p_1 < \infty , \quad 0 < \theta < 1 .$$

H. Triebel showed in [14] that in a special case these general H^p spaces coincide with the anisotropic spaces $\dot{H}^0_p$ from our definition (cf. Remark 4). So (9) yields the result for $0 < p_0 < p_1 < \infty$, $0 < \theta < 1$,

$$(\dot{H}^0_{p_0}, \dot{H}^0_{p_1})_{\theta,p} = \dot{H}^0_p , \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \tag{10}$$

where all the spaces have the same underlying decomposition of $\mathbb{R}^n - \{0\}$.

This result for homogeneous spaces implies the corresponding result for non-homogeneous spaces.

Proposition 1. *If* $0 < p_0 < p_1 < \infty$ *and* $0 < \theta < 1$, *then*

$$(H^0_{p_0}, H^0_{p_1})_{\theta,p} = H^0_p , \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \tag{11}$$

where all the spaces have the same underlying decomposition of $\mathbb{R}^n$.

Proof. Let ψ be a function belonging to S with the properties:

$\operatorname{supp} \psi \subset K_2 := \{x \mid |x| \leqq 2\}$ and $\psi(x) = 1$ for all $x \in \{x \mid |x_j| \leqq 1, j = 1, \ldots, n\}$.

Then we split the functions $f \in H^0_p$ into

$$f = \mathscr{F}^{-1}\psi\mathscr{F}f + \mathscr{F}^{-1}(1-\psi)\mathscr{F}f = f^{(0)} + f^{(1)} ,$$

where $f^{(0)} \in H^0_p$ and $f^{(1)} \in H^0_p \cap \dot{H}^0_p$.

For the $\mathscr{K}$-functional of the interpolation method

$$\mathscr{K}(t, f, H^0_{p_0}, H^0_{p_1}) := \inf_{f = f_0 + f_1} \{\|f_0\|_{H^0_{p_0}} + t\|f_1\|_{H^0_{p_1}}\} , \quad 0 < t < \infty ,$$

we have by this splitting of $f = f_0 + f_1 \in (H^0_{p_0} + H^0_{p_1})$,

$$f = f_0^{(0)} + f_0^{(1)} + f_1^{(0)} + f_1^{(1)} = (f_0^{(0)} + f_1^{(0)}) + (f_0^{(1)} + f_1^{(1)}) = f^{(0)} + f^{(1)} ,$$

$$\mathscr{K}(t, f, H^0_{p_0}, H^0_{p_1}) \sim \mathscr{K}(t, f^{(0)}, H^0_{p_0}, H^0_{p_1}) + \mathscr{K}(t, f^{(1)}, H^0_{p_0}, H^0_{p_1}) . \tag{12}$$

Consider the term on the right-hand side. It is clear that

$$\mathscr{K}(t, f^{(1)}, H^0_{p_0}, H^0_{p_1}) = \mathscr{K}(t, f^{(1)}, \dot{H}^0_{p_0}, \dot{H}^0_{p_1}) \tag{13}$$

and

$$\mathscr{K}(t, f^{(0)}, H^0_{p_0}, H^0_{p_1}) \sim \mathscr{K}(t, f^{(0)}, L^{K_2}_{p_0}, L^{K_2}_{p_1}) , \tag{14}$$

where $L^\Omega_p := \{g \mid g \in S', \operatorname{supp} \mathscr{F}g \subset \Omega, \|g\|_{L_p^\Omega} := \|g\|_{L^p} < \infty\}$.

Further, for an appropriate $h \in \mathbb{R}^n$ we have

$$\tilde{f}^{(0)}(x) := f^{(0)}(x)\, e^{ixh} \in (L^{\Omega}_{p_0} + L^{\Omega}_{p_1}) \quad \text{with} \quad \Omega \cap K_2 = \emptyset\,,$$

and

$$(15) \qquad \|f^{(0)}\|_{L_{p_i}} \sim \|\tilde{f}^{(0)}\|_{L_{p_i}} \sim \|\tilde{f}^{(0)}\|_{H^0_{p_i}} \sim \|\tilde{f}^{(0)}\|_{\dot{H}^0_{p_i}}\,, \quad i = 0, 1\,.$$

So we have

$$(16) \qquad \mathcal{K}(t, f^{(0)}, H^0_{p_0}, H^0_{p_1}) \sim \mathcal{K}(t, \tilde{f}^{(0)}, L^{\Omega}_{p_0}, L^{\Omega}_{p_1}) \sim \mathcal{K}(t, \tilde{f}^{(0)}, \dot{H}^0_{p_0}, \dot{H}^0_{p_1})\,.$$

From (12), (13), (14) and (10), (15) we finally get

$$\begin{aligned}\|f\|_{(H^0_{p_0}, H^0_{p_1})_{\theta,p}} &\sim \|\tilde{f}^{(0)}\|_{(\dot{H}^0_{p_0}, \dot{H}^0_{p_1})_{\theta,p}} + \|f^{(1)}\|_{(\dot{H}^0_{p_0}, \dot{H}^0_{p_1})_{\theta,p}} \\ &\sim \|\tilde{f}^{(0)}\|_{\dot{H}^0_p} + \|f^{(1)}\|_{\dot{H}^p_0} \sim \|f^{(0)}\|_{L_p} + \|f^{(1)}\|_{H^0_p} \\ &\sim \|f\|_{H^0_p}\,, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}\,.\end{aligned}$$

The construction of a retraction from the space $B^s_{p,q}$ into the vector-valued sequence space $l^s_q(A)$ [2]), the known interpolation theorems for $l^s_q(A)$ and the proposition above yield the following interpolation theorem for the spaces $B^s_{p,q}$ and $F^s_{p,q}$.

Theorem 1. *Let $\boldsymbol{a} := (a_1, \ldots, a_n)$ be a fixed n-tuple of positive numbers which characterizes the anisotropic decomposition for all considered spaces, $\boldsymbol{s}_i := (s_i/a_1, \ldots, s_i/a_n)$, $-\infty < s_i < \infty$ $(i = 0, 1)$ and $0 < \theta < 1$.*

(i) *If $0 < p \leqq \infty$, $0 < q_0, q_1, q \leqq \infty$, $s_0 \neq s_1$, then*

$$(17) \qquad (B^{\boldsymbol{s}_0}_{p,q_0}, B^{\boldsymbol{s}_1}_{p,q_1})_{\theta,q} = B^{\boldsymbol{s}}_{p,q}\,,$$

and for $p < \infty$,

$$(18) \qquad (F^{\boldsymbol{s}_0}_{p,q_0}, F^{\boldsymbol{s}_1}_{p,q_1})_{\theta,q} = (F^{\boldsymbol{s}_0}_{p,q_0}, B^{\boldsymbol{s}_1}_{p,q_1})_{\theta,q} = B^{\boldsymbol{s}}_{p,q}\,,$$

where $\boldsymbol{s} := (1-\theta)\, \boldsymbol{s}_0 + \theta \boldsymbol{s}_1$.

(ii) *If $0 < p_0 < p_1 < \infty$, $0 < q_0 < q_1 < \infty$, then*

$$(19) \qquad (B^{\boldsymbol{s}_0}_{p_0,q_0}, B^{\boldsymbol{s}_1}_{p_1,q_1})_{\theta,p} = B^{\boldsymbol{s}}_{p,p}\,,$$

where

$$\boldsymbol{s} := (1-\theta)\, \boldsymbol{s}_0 + \theta \boldsymbol{s}_1 \quad \textit{and} \quad \frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}\,.$$

Proof. Step 1. First we recall a useful fact, given by H. Triebel in [15].

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[2]) $l^s_q(A) := \{\xi \mid \xi = (\xi_{k,t})_{k=0,1,2,\ldots,\; t=1,\ldots,T}\,,\ \|\xi\|_{l_q{}^s(A)} := (\sum_{k=0}^{\infty} \sum_{t=1}^{T} 2^{ksq} \|\xi_{k,t}\|^q_A)^{1/q} < \infty\}\,.$

For $0 < p < \infty$, $\varphi = \{\varphi_{k,t}\} \in \mathscr{A}_L^0$ (L large enough) and all $f \in S'$, the inequality

$$(20)\qquad c_1 \|\mathscr{F}^{-1}\varphi_{k,t}\mathscr{F}f\|_{L_p} \leqq \|\mathscr{F}^{-1}\varphi_{k,t}\mathscr{F}f\|_{H_p^0} \leqq c_2 \|\mathscr{F}^{-1}\varphi_{k,t}\mathscr{F}f\|_{L_p}, \quad k = 0, 1, 2, \ldots, \quad t = 1, \ldots, T,$$

holds with constants $c_1, c_2 > 0$ independent of k, t.

Hence in the definition of $B^s_{p,q}$ we can replace the space $l_q(L_p)$ by the space $l_q(H_p^0)$, $0 < p < \infty$, which is very useful, because for H_p^0 we have the Fourier transform and multiplier theorems.

Step 2. Now we establish that for $0 < p_0, p_1 < \infty$, $0 < q_0 \leqq q_1 \leqq \infty$, the following relation holds:

$$(21)\qquad \|f\|_{(B^{s_0}_{p_0,q_0}, B^{s_1}_{p_1,q_1})_{\theta,q}} \sim \|\mathscr{F}^{-1}\varphi_{k,t}\mathscr{F}f\|_{(l^{s_0}_{q_0}(A_0), l^{s_1}_{q_1}(A_1))_{\theta,q}},$$

where $A_i := H^0_{p_i}$ if $p_i < \infty$ and $A_i := L_\infty$ if $p_i = \infty$ $(i = 0, 1)$.

The equivalence of the quasi-norms in (21) follows from Theorem 1.2.4 in [11] (which also holds for quasi-Banach spaces), if we have a retraction $\mathscr{R}$ from $B^s_{p,q}$ into $l^s_q(H^0_p)$ or $l^s_q(L_\infty)$, respectively. Now we construct such an operator $\mathscr{R}$.

Let $\varphi = \{\varphi_{k,t}\}$ be a system of decomposition functions corresponding to the decomposition $\mathscr{P}$, $\varphi \in \mathscr{A}^0_L[\mathscr{P}]$. Then there is a local modification $\mathscr{P}^\sim$ of $\mathscr{P}$ and a corresponding system $\psi = \{\psi_{k,t}\} \in \mathscr{A}^0_L[\mathscr{P}^\sim]$, cf. [9], Definition 2, such that

$$(22)\qquad \psi_{k,t}(x) = 1 \quad \text{for} \quad x \in \operatorname{supp} \varphi_{k,t}$$

and

$$\psi_{k,t}(x) = \psi(\{2^{-ka_j}(x_j - \tilde{x}_j^{k,t})\}) \quad \text{for all} \quad x \in \mathbb{R}^n$$

($\tilde{x}^{k,t}$ is the centre of $\tilde{P}_{k,t} \in \mathscr{P}^\sim$, ψ an appropriate function from S). With the aid of these systems we define the linear operators

$$(23)\qquad \mathscr{S}f := \{\mathscr{F}^{-1}\varphi_{k,t}\mathscr{F}f\}_{\substack{k=0,1,2,\ldots\\ t=1,\ldots,T}}, \quad f \in B^s_{p,q},$$

and

$$(24)\qquad \mathscr{R}\{g_{l,n}\} := \sum_{\substack{l=0,1,2,\ldots\\ n=1,\ldots,T}} \mathscr{F}^{-1}\psi_{l,n}\mathscr{F}g_{l,n}, \quad \{g_{l,n}\} \in l^s_q(H^0_p).$$

i) $\mathscr{S}$ is a bounded linear operator from $B^s_{p,q}$ into $l^s_q(H^0_p)$. This follows immediately from (20).

ii) $\mathscr{R}$ is a bounded linear operator from $l^s_q(H^0_p)$ into $B^s_{p,q}$. We get this from the estimate

$$(25)\qquad \begin{aligned} \|\mathscr{R}\{g_{l,n}\}\|_{B^s_{p,q}} &\leqq \|\{2^{sk}\mathscr{F}^{-1}\psi_{k,t}\mathscr{F}(\mathscr{R}\{g_{l,n}\})\}_{k,t}\|_{l_q(H^0_p)} \leqq \\ &\leqq c\|\{2^{sk}\mathscr{F}^{-1}\sum_{l,n}\psi_{k,t}\psi_{l,n}\mathscr{F}g_{l,n}\}_{k,t}\|_{l_q(H^0_p)} \leqq \\ &\leqq c'\|\{\|2^{sk}\sum_{\substack{h=-N,\ldots,N\\ i=1,\ldots,T}}\psi_{k,t}\psi_{k+h,t+i}\mathscr{F}g_{k+h,t+i}\|_{H^0_p}\}_{k,t}\|_{l_q} \leqq \\ &\leqq c''\|\{\sum_{\substack{h=-N,\ldots,N\\ i=1,\ldots,T}}\|2^{sk}g_{k+h,t+i}\|_{H^0_p}\}_{k,t}\|_{l_q} \leqq c'''\|g_{k,t}\|_{l_q^s(H^0_p)}, \end{aligned}$$

where (25) is established by the multiplier property of $\psi_{k,t}$ (cf. [9], Theorem 5) for the spaces $\tilde{F}^0_{p,2} = F^0_{p,2}$ (cf. Remark 5) with a constant which is independent of k, t.

iii) $\mathscr{R}\mathscr{S}f = \sum_{\substack{l=0,1,2,\ldots \\ n=1,\ldots,T}} \mathscr{F}^{-1}\psi_{l,n}\varphi_{l,n}\mathscr{F}f = \sum \mathscr{F}^{-1}\varphi_{l,n}\mathscr{F}f = f$ for all $f \in B^s_{p,q}$.

This means $\mathscr{R}$ and $\mathscr{S}$ are a retraction and a coretraction for the spaces $B^s_{p,q}$ and $l^s_q(H^0_p)$, where $0 < p < \infty$, $0 < q \leqq \infty$, $-\infty < s < \infty$. In the case $p = \infty$ the operators $\mathscr{R}$ and $\mathscr{S}$ are a retraction and a coretraction for the spaces $B^s_{\infty,q}$ and $l^s_q(L_\infty)$. Instead of estimating by a multiplier theorem in (25), we use now the estimate

$$|\mathscr{F}^{-1}(\psi_{k,t}\psi_{k+h,t+i}\mathscr{F}g_{k+h,t+i})(x)| =$$

$$= |[\mathscr{F}^{-1}(\psi(\{2^{-ka_j}\cdot\})\,\psi(\{2^{-(k+h)a_j}\cdot\})) * g_{k+h,t+i}](x)| =$$

$$= \left|2^{k\Sigma a_j}\int_{R^n} \mathscr{F}^{-1}(\psi(\cdot)\,\psi(\{2^{-ka_j}\cdot\}))(\{2^{ka_j}(x_j - \tilde{x}^{k,t}_j)\})\, g_{k+h,t+i}(y)\,\mathrm{d}y\right| \leqq c\|g_{k+h,t+i}\|_{L_\infty}.$$

Step 3. The concrete interpolation formula (17) follows now from (21) and the known interpolation theorem (cf. [1], p. 122, extended to quasi-Banach spaces A)

$$(l^{s_0}_{q_0}(A),\, l^{s_1}_{q_1}(A))_{\theta,q} = l^s_q(A)\,, \quad s = (1-\theta)\, s_0 + \theta s_1\,,$$

with $s_0 \neq s_1$, $0 < q_0, q_1 \leqq \infty$, $0 < \theta < 1$ and $A := H^0_p$ for $0 < p < \infty$ and $A := L_\infty$ for $p = \infty$. Formula (18) is obtained from (17) and (8) by the reiteration theorem (cf. [1], p. 50, extended to quasi-Banach spaces). Formula (19) follows from (21) and the interpolation theorem (cf. [1], p. 123, extended to quasi-Banach spaces A_k)

$$(l^{s_0}_{q_0}(A_0),\, l^{s_1}_{q_1}(A_1))_{\theta,q} = l^s_q((A_0, A_1)_{\theta,q})\,, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}\,, \quad s = (1-\theta)\, s_0 + \theta s_1\,,$$

with $0 < q_0, q_1 < \infty$, $0 < \theta < 1$ and $A_i := H^0_{p_i}$ $(i = 0, 1)$, and the interpolation formula (11).

3. CLASSICAL ANISOTROPIC FUNCTION SPACES

Now we compare the spaces $B^s_{p,q}$ and $F^s_{p,q}$ with the classical anisotropic spaces.

If $\boldsymbol{s} := (s_1, \ldots, s_n)$, $s_j > 0$, then the well-known anisotropic Lebesgue spaces (= Bessel-potential spaces) are defined by

$$H^{\boldsymbol{s}}_p := \{f \mid f \in S',\ \|f\|_{H^{\boldsymbol{s}}_p} := \|\mathscr{F}^{-1}(\sum_{j=1}^n (1 + x_j^2)^{s_j/2}\, \mathscr{F}f\|_{L_p} < \infty\}\,,$$

$$1 < p < \infty\,.$$

Denote as usual by $D_j := \partial/\partial x_j$, $D^\alpha := D_1^{\alpha_1} \ldots D_n^{\alpha_n}$, $\alpha := (\alpha_1, \ldots, \alpha_n)$, $\alpha_j \geqq 0$ integer, the derivatives of functions or distributions on $\mathbb{R}^n$. For $\boldsymbol{m} := (m_1, \ldots, m_n)$, $m_j > 0$ integer, we have the anisotropic Sobolev spaces

$$W^{\boldsymbol{m}}_p := \{f \mid f \in S',\ \|f\|_{W^{\boldsymbol{m}}_p} := \sum_{0 \leqq \Sigma\alpha_j/m_j \leqq 1} \|D^\alpha f\|_{L_p} < \infty\}\,, \quad 1 \leqq p < \infty\,,$$

and the anisotropic Hölder spaces

$$C^{\boldsymbol{m}} := \{f \mid f \in C,\ \|f\|_{C^{\boldsymbol{m}}} := \sum_{0 \leqq \Sigma \alpha_j/m_j \leqq 1} \|D^\alpha f\|_C < \infty\},$$

where C is the set of all bounded uniformly continuous functions on $\mathbb{R}^n$ and

$$\|f\|_C := \sup_{x \in R^n} |f(x)|.$$

The following theorem shows the connection between the spaces $F^{\boldsymbol{s}}_{p,2} = H^{\boldsymbol{s}}_p$ and the anisotropic Lebesgue- and Sobolev spaces.

Theorem 2. ([9], p. 266). *If* $1 < p < \infty$ *and* $\boldsymbol{s} := (s_1, \ldots, s_n)$, $s_j > 0$, *then*

(26) $$H^{\boldsymbol{s}}_p = \mathbb{H}^{\boldsymbol{s}}_p,$$

and, provided s_j $(j = 1, \ldots, n)$ *are integers,*

(27) $$H^{\boldsymbol{s}}_p = W^{\boldsymbol{s}}_p,$$

where the corresponding norms are equivalent.

Before we give a classical interpretation for the spaces $B^{\boldsymbol{s}}_{p,q}$ $(1 \leqq p, q \leqq \infty, s > 0)$ we need the following proposition.

Proposition 2. *Let* $\boldsymbol{a} := (a_1, \ldots, a_n)$, $a_j > 0$, $\boldsymbol{s} := (s/a_1, \ldots, s/a_n)$, $-\infty < s < \infty$, *be the numbers which characterize the anisotropic smoothness. Then*

(28) $$B^{\boldsymbol{0}}_{p,1} \subset L_p \subset B^{\boldsymbol{0}}_{p,\infty}, \quad 1 \leqq p \leqq \infty,$$

(29) $$B^{\boldsymbol{0}}_{\infty,1} \subset C \subset B^{\boldsymbol{0}}_{\infty,\infty},$$

and if $s/a_j = m_j$ $(j = 1, \ldots, n)$ *are integers,*

(30) $$B^{\boldsymbol{m}}_{p,1} \subset W^{\boldsymbol{m}}_p \subset B^{\boldsymbol{m}}_{p,\infty}, \quad 1 \leqq p < \infty,$$

(31) $$B^{\boldsymbol{m}}_{\infty,1} \subset C^{\boldsymbol{m}} \subset B^{\boldsymbol{m}}_{\infty,\infty}.$$

Proof. The proof of the proposition is the same as in the isotropic case (cf. [13], p. 68), if we use the following identity that holds for $0 < p \leqq \infty$, $0 < q \leqq \infty$ and $s/a_j = m_j$ $(j = 1, \ldots, n)$ integers:

$$B^{\boldsymbol{m}}_{p,q} = \{f \mid f \in S',\ \|f\|^{*\boldsymbol{s}}_{B_{p,q}} := \sum_{0 \leqq \Sigma \alpha_j/m_j \leqq 1} \|D^\alpha f\|_{B^{\boldsymbol{0}}_{p,q}} < \infty\}.$$

This statement follows from the lifting property for $B^{\boldsymbol{s}}_{p,q}$ (analogous to $F^{\boldsymbol{s}}_{p,q}$ $(p < \infty)$ [9], p. 265) by standard arguments (cf. [13], p. 67) using the anisotropic multiplier theorems ([9], p. 264).

Theorem 3. *Let* $\boldsymbol{a} := (a_1, \ldots, a_n)$, $a_j > 0$, $s > 0$, $\boldsymbol{s} := (s/a_1, \ldots, s/a_n) = (s_1, \ldots, s_n)$ *be the multiindices which characterize the anisotropic smoothness and* $\boldsymbol{\delta} :=$ $:= (\delta_1, \ldots, \delta_n)$, $0 < \delta_j \leqq \infty$, $\boldsymbol{\beta} := (\beta_1, \ldots, \beta_n)$, $\boldsymbol{l} := (l_1, \ldots, l_n)$, $0 \leqq \beta_j$, l_j *integers*

($j = 1, \ldots, n$), *such that*

$$0 \leqq \beta_j < s_j, \quad s_j - \beta_j < l_j \quad (j = 1, \ldots, n),$$

then

(32) $$B^s_{p,q} = \mathscr{B}^s_{p,q},$$

holds for $1 \leqq p < \infty$, $1 \leqq q \leqq \infty$, *where*

$$\mathscr{B}^s_{p,q} := \{f \mid f \in L_p,\ \|f\|^s_{p,q,l,\beta,\delta} < \infty\},$$

and the expressions

(33) $$\|f\|^s_{p,q,l,\beta,\delta} := \|f\|_{L_p} + \sum_{j=1}^{n} \left(\int_0^{\delta_j} t^{(\beta_j - s_j)} \|\Delta^{l_j}_{t,j} D^{\beta_j}_j f\|^q_{L_p} \frac{dt}{t} \right)^{1/q} \ ^{3)}$$

are equivalent norms in $\mathscr{B}^s_{p,q}$ *and* $B^s_{p,q}$ *for all admissible values of parameters.*

For $p = q = \infty$ *we have*

(34) $$B^s_{\infty,\infty} = \mathscr{B}^s_{\infty,\infty},$$

where

$$\mathscr{B}^s_{\infty,\infty} := \{f \mid f \in C,\ \|f\|^s_{\infty,l,\beta,\delta} < \infty\}$$

and the expressions

(35) $$\|f\|^s_{\infty,l,\beta,\delta} := \|f\|_C + \sum_{j=1}^{n} \sup_{0 < t < \delta_j} t^{(\beta_j - s_j)} \|\Delta^{l_j}_{t,j} D^{\beta_j}_j f\|_C,$$

are equivalent norms in $\mathscr{B}^s_{\infty,\infty}$ *and* $B^s_{\infty,\infty}$ *for all admissible values of parameters.*

Proof. The theorem is based on interpolation theorems for classical anisotropic spaces and on Theorem 1.

For $1 \leqq p < \infty$, $0 < \theta < 1$, $1 \leqq q \leqq \infty$, H.-J. Schmeisser and H. Triebel have proved in [6] (cf. also [7], [8]) the identities

(36) $$(L_p, W^l_p)_{\theta,q} = \mathscr{B}^{\theta l}_{p,q}$$

and

(37) $$(C, C^l)_{\theta,\infty} = \mathscr{B}^{\theta l}_{\infty,\infty}.$$

We remark that the space C (completion of $C^\infty_0(\mathbb{R}^n)$) in [6], p. 120, can be replaced by our space C.

Now we obtain the statements (32) and (34) from our Theorem 1, the formulas (28)–(31), (36), (37) and the reiteration theorem (cf. [1], p. 50).

Remark 6. Theorem 3 shows that our anisotropic $B^s_{p,q}$ spaces for $p, q \geqq 1$, $s > 0$ coincide with the classical ones.

(i) If $s_j > 0$ are arbitrary numbers and $s_j = [s_j]^- + \{s_j\}^+$ ($[s_j]^-$ integer, $0 <$

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3) $$(\Delta_{t,j} f)(x) := f(x_1, \ldots, x_{j-1}, x_j + t, x_{j+1}, \ldots, x_n) - f(x),$$
$$\Delta^l_{t,j} f := \Delta_{t,j}(\Delta^{l-1}_{t,j} f), \quad l = 2, 3, \ldots.$$

$< \{s_j\}^- \leqq 1$), then (33) and (35) for $\beta_j := [s_j]^-$, $s_j - \beta_j := \{s_j\}^+$, $l_j := 2$, $\delta_j := \infty$ ($j = 1, \ldots, n$) give the well-known norms for the classical anisotropic Besov spaces $\mathbb{B}^s_{p,q}$ and Zygmund spaces $\mathscr{C}^s$, respectively (cf. [3], [4]) and we get from (34) and (36):

$$B^s_{p,q} = \mathbb{B}^s_{p,q}\,, \quad B^s_{\infty,\infty} = \mathscr{C}^s\,,$$

($1 \leqq p < \infty$, $1 \leqq q \leqq \infty$, $s = (s_1, \ldots, s_n)$, $s_j > 0$).

(ii) If $s_j > 0$ is not an integer and $s_j = [s_j] + \{s_j\}$ ($[s_j]$ integer, $0 \leqq \{s_j\} < 1$), then (33) and (35) for $\beta_j := [s_j]$, $s_j - \beta_j := \{s_j\}$, $l_j := 1$, $\delta_j := \infty$ ($j = 1, \ldots, n$) give the well-known norms for the classical anisotropic Slobodeckij spaces $\mathbb{W}^s_p$ ($p = q$) and Hölder spaces $\mathbb{C}^s$, respectively, and (34), (36) imply

$$B^s_{p,p} = \mathbb{W}^s_p\,, \quad B^s_{\infty,\infty} = \mathbb{C}^s\,,$$

($1 \leqq p < \infty$, $s = (s_1, \ldots, s_n)$, $0 < s_j \neq$ integer).

Remark 7. Together with the classical interpretations of the spaces $B^s_{p,q}$ and $F^s_{p,q}$, Theorem 1 yields new interpolation formulas, e.g. for $a := (a_1, \ldots, a_n)$, $a_j > 0$; $s_i > 0$, $s_i := (s_i/a_1, \ldots, s_i/a_n)$, $i = 0, 1$; $1 \leqq p_0 < p_1 < \infty$, $1 \leqq q_0$; $q_1 < \infty$ we get

$$(\mathbb{B}^{s_0}_{p_0,q_0}, \mathbb{B}^{s_1}_{p_1,q_1})_{\theta,p} = \mathbb{B}^s_{p,p}\,, \quad s = (1-\theta)\, s_0 + \theta s_1\,,$$

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$

and

$$(L_{p_0}, \mathbb{B}^{s_1}_{p_1,q_1})_{\theta,p} = \mathbb{B}^s_{p,p}\,, \quad s = \theta s_1\,, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}\,,$$

which complete the interpolation results on the classical anisotropic function spaces in [11], [7].

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Author's address: Universität Jena, Sektion Mathematik, Universitäts-Hochhaus, DDR-69, Jena.