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Ľudovít Niepel; Pavel Tomasta

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ELEVATION OF A GRAPH

ĽUDOVÍT NIEPEL, PAVEL TOMASTA, Bratislava

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I. INTRODUCTION

The purpose of this paper is to introduce the notion of the elevation of a graph and investigate some of its basic properties.

Two rather different problems gave rise to this invariant. First, it is the practical use of matrices in computers. Let M be a binary square matrix. Transform M by replacing rows and columns (the i-th row is replaced by the j-th one if and only if the i-th column is replaced by the j-th one) to obtain a matrix with entries "near" to the diagonal as much as possible. "Near" is often useful to understand as the minimal sum of distances of nonzero entries from the diagonal. Unfortunately, the determining of the exact value for this minimal sum is a very difficult problem in general.

A directed graph G with n vertices corresponds in a natural way to M of type $n \times n$. The question is: how can one label the vertices of G by numbers $1, 2, \ldots, n$ to minimalize the sum of the absolute values of differences of the adjacent labels? We call this minimal sum the elevation of G (the exact definition see in Section 2).

The other reason to study the elevation of G is its relation to the crossing number of a certain infinite class of graphs. This relation was studied in [2].

II. DEFINITIONS AND EXAMPLES

All our considerations can be carried out for directed graphs, but we confine them only to undirected graphs without loops and multiple edges. For undefined concepts see [1].

Definition. Let G be a graph with a vertex set

$$V(G) = \{v_1, v_2, \ldots, v_n\} .$$

To every one-to-one labeling $f : V(G) \to \{1, 2, \ldots, n\}$, a number

$$\mathscr{E}_f(G) = \sum_{(v_i, v_j) \in G} |f(v_i) - f(v_j)|$$

can be assigned.

We shall call the number

$$\mathscr{E}(G) = \min_f \mathscr{E}_f(G)$$

the *elevation* of the graph G, and the number

$$\bar{\mathscr{E}}(G) = \max_f \mathscr{E}_f(G)$$

the *coelevation* of G.

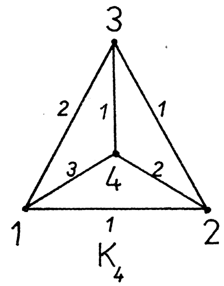

Fig. 1

Figure 1 shows a complete graph K_4 with four vertices. Clearly, for every labeling f of its vertices $\mathscr{E}_f(K_4)$ is the same. Therefore

$$\mathscr{E}(K_4) = \bar{\mathscr{E}}(K_4) = 10 .$$

One can see that the complete graphs and their complements are the only cases for which $\mathscr{E} = \bar{\mathscr{E}}$.

For our further purposes it is useful to establish

Proposition 1. *Let K_n be a complete graph with n vertices. Then*

$$\mathscr{E}(K_n) = \binom{n+1}{3} .$$

Proof. As we pointed before, every labeling of K_n is the minimal one. Let us have the labeling $f(v_i) = i$.

Then we compute:

$$\mathscr{E}(K_n) = \mathscr{E}_f(K_n) = \sum_{i>j} |i - j| = \sum_{i>j} (i - j) \sum_{k=1}^{n-1} (n - k)\, k = n \sum_{k=1}^{n-1} k - \sum_{k=1}^{n-1} k^2 = \binom{n+1}{3}.$$

Hence the result.

From Figures 2 and 3, the following observation is immediate:

$$\mathscr{E}(C_4) + \bar{\mathscr{E}}(K_2 \cup K_2) = 10 = \bar{\mathscr{E}}(C_4) + \mathscr{E}(K_2 \cup K_2).$$

This fact is explained by

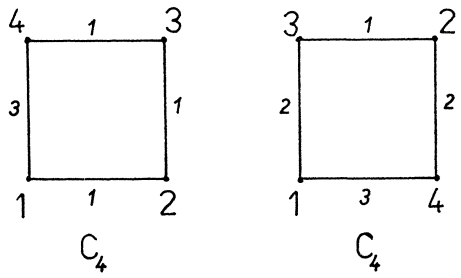

Fig. 2

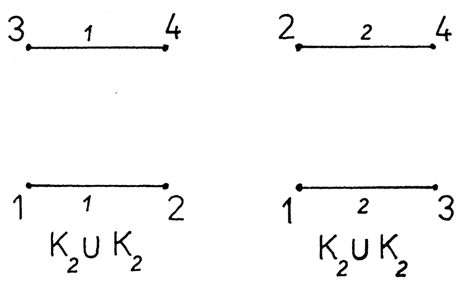

Fig. 3

Proposition 2. *Let G and $\bar{G}$ be an n-vertex graph and its complement, respectively. Then*

$$\mathscr{E}(G) + \bar{\mathscr{E}}(\bar{G}) = \mathscr{E}(K_n).$$

Proof. Let f be a labeling of G for which

$$\mathscr{E}(G) = \sum_{(v_i,v_j)\in G} |f(v_i) - f(v_j)|.$$

Then

$$\mathscr{E}_f(G) + \mathscr{E}_f(\bar{G}) = \sum_{(v_i,v_j)\in G} |f(v_i) - f(v_j)| +$$

$$+ \sum_{(v_i,v_j)\in \bar{G}} |f(v_i) - f(v_j)| = \sum_{(v_i,v_j)\in K_n} |f(v_i) - f(v_j)| = \mathscr{E}(K_n).$$

Since $\sum_{(v_i,v_j)\in G} |f(v_i) - f(v_j)|$ is the minimum, $\sum_{(v_i,v_j)\in \bar{G}} |f(v_i) - f(v_j)|$ is the maximum and the proposition follows.

Corollary. *The determining of the elevation and the coelevation of a graph are equivalent problems.*

The corollary enables us to restrict our considerations to the case of the elevation.

For the graph in Figure 3a, the identity

$$\mathscr{E}(K_2 \cup K_2) = \mathscr{E}(K_2) + \mathscr{E}(K_2) \quad \text{holds}\,.$$

Hence the elevation of $K_2 \cup K_2$ is equal to the sum of the elevations of its components.

Prove the following

Proposition 3. *Let G be a graph and $G = G_1 \cup G_2$.*

Then

$$\mathscr{E}(G) = \mathscr{E}(G_1) + \mathscr{E}(G_2).$$

Proof. Let $V(G_1) = \{v_1, v_2, \ldots, v_k\}$ and $V(G_2) = \{v_{k+1}, v_{k+2}, \ldots, v_n\}$.

1° First we prove

$$\mathscr{E}(G) \leqq \mathscr{E}(G_1) + \mathscr{E}(G_2)\,.$$

Let f_1 and f_2 be minimal labelings of G_1 and G_2, respectively. Put

$$f(v_i) = \begin{cases} f_1(v_i)\,, & \text{if} \quad v_i \in G_1\,, \\ f_2(v_i) + k\,, & \text{if} \quad v_i \in G_2\,. \end{cases}$$

Then clearly

$$\mathscr{E}(G) \leqq \mathscr{E}_f(G) = \mathscr{E}(G_1) + \mathscr{E}(G_2)\,.$$

2° Now it is sufficient to prove

$$\mathscr{E}(G) \geqq \mathscr{E}(G_1) + \mathscr{E}(G_2)\,.$$

Suppose f to be a minimal labeling of G.

Put

$$M_1 = \{f(v_i);\, v_i \in G_1\} \quad \text{and} \quad M_2 = \{f(v_i);\, v_i \in G_2\}\,.$$

Consider M_1 and M_2 in the form of sequences

$$f(v_{i,1}) < f(v_{i,2}) < \ldots < f(v_{i,k})$$

and

$$f(v_{i,k+1}) < f(v_{i,k+2}) < \ldots < f(v_{i,n})$$

Define a function $g : V(G) \to \{1, 2, \ldots, n\}$ in the following manner:

$$g(v_{i,j}) = j\,.$$

Obviously, for any edge $(v_{i,p}, v_{i,q})$,

$$|f(v_{i,p}) - f(v_{i,q})| \geqq |p - q| = |g(v_{i,p}) - g(v_{i,q})|\,.$$

Evidently we have

$$\mathscr{E}_f(G) \geqq \mathscr{E}_g(G) \geqq \mathscr{E}(G_1) + \mathscr{E}(G_2)\,.$$

Hence the result.

Proposition 3 implies that we can confine ourselves without loss of generality to the connected graphs.

III. ENUMERATION FOR SOME CLASSES OF GRAPHS

First we prove some general facts. By a factor we mean a subgraph of G with the same vertex set as G.

Theorem 1. *Let G be a graph and G_i, $i = 1, 2, \ldots, k$, its disjoint factors the union of which covers G. If there exists a labeling f with the property:*

$$\mathscr{E}_f(G_i) = \mathscr{E}(G_i)\,, \quad i = 1, 2, \ldots, k\,,$$

then

$$\mathscr{E}_f(G) = \mathscr{E}(G)\,.$$

Proof. Compute:

$$\mathscr{E}_f(G) = \sum_{i=1}^{k} \mathscr{E}_f(G_i) = \sum_{i=1}^{k} \mathscr{E}(G_i) \leqq \mathscr{E}(G)\,.$$

Since $\mathscr{E}(G) \leqq \mathscr{E}_f(G)$, the proof follows.

This simple theorem has interesting consequences.

Corollary 1. *Let P_n be a path of length $n - 1$. Then*

$$\mathscr{E}(P_n) = n - 1\,.$$

Proof. Decompose P_n into $n - 1$ factors containing exactly one edge each. The proof follows immediately from Theorem 1.

Denote by $K_{m,n}$ a complete bipartite graph. The graph $K_{1,n}$ is called *a star* and denoted by S_n.

Proposition 4.

$$\mathscr{E}(S_n) = \frac{n}{2}\left(\frac{n}{2} + 1\right), \quad \text{if } n \text{ is even}\,,$$

$$\mathscr{E}(S_n) = \left(\frac{n+1}{2}\right)^2, \quad \text{if } n \text{ is odd}\,.$$

Proof. Let f be a labeling of S_n. Compute the differences $|f(v_i) - f(v_j)|$ for every edge (v_i, v_j) of S_n.

Any natural number can occur at most twice. Summing up the sequences

$$\left\{1, 1, 2, 2, \ldots, \frac{n}{2}, \frac{n}{2}\right\} \quad \text{and} \quad \left\{1, 1, 2, 2, \ldots, \frac{n-1}{2}, \frac{n-1}{2}, \frac{n+1}{2}\right\}$$

we obtain

$$\mathscr{E}_f(S_n) \geqq \frac{n}{2}\left(\frac{n}{2} + 1\right), \quad \text{if } n \text{ is even},$$

$$\mathscr{E}_f(S_n) \geqq \left(\frac{n+1}{2}\right)^2, \quad \text{if } n \text{ is odd}.$$

Since f was an arbitrary labeling we have the same ineuqalites for $\mathscr{E}(S_n)$.

On the other hand, take any labeling f of S_n with the property $f(v_c) = [n/2] + 1$ for the central vertex v_c of S_n. Obviously

$$\mathscr{E}_f(S_n) = \frac{n}{2}\left(\frac{n}{2} + 1\right), \quad \text{if } n \text{ is even},$$

$$\mathscr{E}_f(S_n) = \left(\frac{n+1}{2}\right)^2, \quad \text{if } n \text{ is odd}.$$

Hence the result.

As another consequence of Theorem 1 we have

Corollary 2. *Let $G = K_1 + (G_1 \cup G_2 \cup \ldots \cup G_k)$, $k \geqq 2$, be an n-vertex graph, let there exist p such that $1 \leqq p \leqq k$ and*

$$\sum_{i=1}^{p} |V(G_i)| = \left[\frac{n}{2}\right].$$

Then

$$\mathscr{E}(G) = \mathscr{E}(K_{1,n-1}) + \sum_{i=1}^{p} \mathscr{E}(G_i).$$

Proof. Let $K_{1,n-1} \cup G_1 \cup G_2 \cup \ldots \cup G_k$ be a decomposition of G into disjoint factors. Using Theorem 1 and Proposition 4 we obtain the required result. The condition

$$\sum_{i=1}^{p} |V(G_i)| = \left[\frac{n}{2}\right]$$

guarantees a minimal labeling on $K_{1,n-1}$.

Proposition 5. *Let C_n be a cycle with $n \geqq 3$ vertices. Then*

$$\mathscr{E}(C_n) = 2(n-1).$$

Proof. Let f be any labeling of C_n. Without loss of generality we can assume

$$f(v_1) = 1, \quad f(v_n) = n.$$

Both vertices v_1 and v_n are connected by a two disjoint paths p_1 and p_2. Clearly

$$\mathscr{E}_f(p_1) \geqq n - 1, \quad \mathscr{E}_f(p_2) \geqq n - 1.$$

Hence

$$\mathscr{E}_f(C_n) \geqq 2(n - 1).$$

Since f was an arbitrary labeling we have

$$\mathscr{E}(C_n) \geqq 2(n - 1).$$

Using a cyclic labeling c of C_n we have

$$\mathscr{E}_c(C_n) = 2(n - 1).$$

Hence the result.

The elevation of $P_n \times P_2$ is obtained by using a skilful decomposition of this graph.

Proposition 6. *Let P_n be a path of length $n - 1$ $(n \geqq 2)$. Then*

$$\mathscr{E}(P_n \times P_2) = 5n - 4.$$

Proof. Let f be any labeling of $P_n \times P_2$ pictured in Figure 4. Decompose this graph into a hamiltonian cycle C_{2n}

$$\{v_1, v_2, v_4, \ldots, v_{2n}, v_{2n-1}, \ldots, v_3, v_1\}$$

and $n - 2$ disjoint edges. From Proposition 5 we have

$$\mathscr{E}_f(C_{2n}) \geqq 4n - 2.$$

Hence $\mathscr{E}_f(P_n \times P_2) \geqq \mathscr{E}_f(C_{2n}) + n - 2 \geqq 5n - 4$.

Since f was an arbitrary labeling we have

$$\mathscr{E}(P_n \times P_2) \geqq 5n - 4.$$

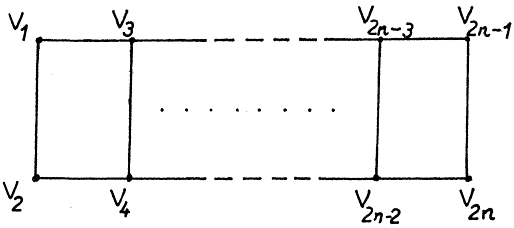

Fig. 4

In Figure 4 take a labeling $f(v_i) = i$. Then by easy computation we obtain

$$\mathscr{E}_f(P_n \times P_2) = 5n - 4.$$

Hence the result.

While this special case is solved, the general case $P_n \times P_m$ is still open.

Conjecture. *Let $m \leqq n$ be natural numbers. Then*

$$\mathscr{E}(P_n \times P_m) = n(m^2 + m - 1) - m^2.$$

In the conclusion we shall give two general results which are useful for possible further investigation of the elevation.

Theorem 2. *Let G be a graph and V_1, V_2 two disjoint subsets of its vertex set such that for every vertex $v_i \in V_1$ there exist exactly p vertices of V_2 adjacent to it and every vertex of V_2 is adjacent to the same number of vertices of V_1. Then for every labeling f with the properties:*

(i) $f(v_i) \leqq f(v_j)$, $v_i \in V_1$, $v_j \in V_2$,

(ii) *the sets fV_1 and fV_2 are intervals in $N = \{1, 2, \ldots\}$,*

$$(*) \qquad \mathscr{E}_f([V_1 \cup V_2]) - (\mathscr{E}_f([V_1]) + \mathscr{E}_f([V_2])) = K \quad \text{holds},$$

where $[H]$ denotes the subgraph of G induced by H, and K is a constant independent of f.

Proof. Let f be a labeling of G with the properties (i) and (ii). Put

$$m = \max \{f(v_i);\ v_i \in V_1\}, \quad M = \min \{f(v_j);\ v_j \in V_2\}.$$

The left hand side of $(*)$ can be expressed as a sum

$$S = \sum_{(v_i, v_j) \in G} |f(v_i) - f(v_j)|, \quad \text{where} \quad v_i \in V_1, \quad v_j \in V_2.$$

Compute:

$$S = \sum_{(v_i, v_j) \in G} |f(v_i) - m + m - M + M - f(v_j)| =$$

$$= p \sum_{v_i \in V_1} |m - f(v_i)| + (M - m)\,|V_1|\, p + p \sum_{v_j \in V_2} |f(v_j) - M| = K,$$

where K is a constant independent of f, since (ii) holds. Hence the result.

Theorem 2 suggests a method of finding a minimal labeling (elevation) by repeated use of permutations of labels.

Theorem 3. *Let G be a graph and d_i the degree of a vertex $v_i \in G$. Then*

$$\mathscr{E}(G) \geqq \tfrac{1}{2} \sum_{v_i} \mathscr{E}(S_{d_i}).$$

Proof. The closed neighborhood $N[v_i]$ of the vertex v_i is a star S_{α_i}. Since every edge is contained in two of such stars, summing over all stars and dividing by two we obtain the required result.

This lower bound is exact. It is attained for paths.

Remark. *In [2] the elevation of $K_{n,n}$ is determined and it is used for a conjecture about the crossing number of $K_{p,p} \times C_n$.*

References

[1] *F. Harary:* Graph theory, Addison-Wesley P.C., 1969.
[2] *P. Tomasta:* Elevation and crossing numbers (to appear).

Author's address: Ľudovít Niepel, Mlynská dolina, 816 31 Bratislava, ČSSR (Katedra geometrie MFF UK); Pavel Tomasta, Obrancov mieru 49, 886 25 Bratislava, ČSSR (Matematický ústav SAV).