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EMBEDDING GRAPHS IN THEIR COMPLEMENTS

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INTRODUCTION

Several recent papers have dealt with the problem of placing two graphs on n vertices edge disjointly in K_n. This is sometimes called *a mutual placement of the two graphs*. In particular Catlin [5], Sauer and Spencer [7] have shown independently that if the two graphs have maximal degree ϕ and β respectively with $2\phi\beta < n$, then the two graphs are mutually placeable. Also Bollobas and Eldridge [2] have shown that if the graphs have maximal degree $n - 2$ and collectively no more than $2n - 3$ edges then, except for a few special pairs of graphs, the two graphs are again mutually placeable. A summary of many of the recent results appear in [4].

In this paper a more special problem will be considered, namely, mutual placement of two copies of the same graph in K_n. Placing two copies of the same graph G edge disjointly in K_n is really an embedding or an isomorphic mapping of the graph G into its complement $\bar{G}$. Throughout this paper the embedding of a graph G into $\bar{G}$ will be referred to as *an embedding of G*; or it will be said that *the graph G is embeddable.*

There are two principal results in the paper. The first completely characterizes those graphs with n vertices and n edges which are embeddable. The second shows that if a graph G with n vertices is not a star, contains no more than $(6/5)\, n - 2$ edges, and has no cycles of length 3 or 4 as subgraphs, then G is embeddable. It is conjectured that the second result is true when the restriction on the number of edges is completely deleted.

Notation within the paper is kept to a minimum. A graph with p vertices and q edges will be called a (p, q) graph. Also kG, k a positive integer, will refer to k vertex disjoint copies of the graph G. Additional notation will be the usual as found in [1] or [6].

Within the paper frequent reference will be made to three classes of graphs. To define these classes, two classes $\mathscr{F}_1$ and $\mathscr{F}_2$ of forbidden graphs are first defined. Let $\mathscr{F}_1$ be the set of graphs

$$\{K_1 \cup K_3,\ K_1 \cup 2K_3,\ K_1 \cup C_4,\ K_{1,1} \cup K_3,\ K_{1,n-1}(n \geqq 2),\ K_{1,n-4} \cup K_3(n \geqq 8)\}\,.$$

These graphs are shown in Figure 1. To identify the graphs in $\mathscr{F}_2$ several special

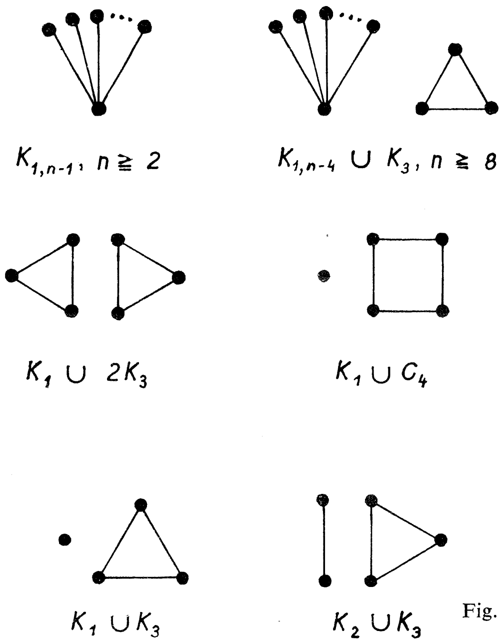

Fig. 1. The Class $\mathscr{F}_1$.

graphs need be constructed. Let $K_3 \# K_{1,n}$ denote the graph obtained by inserting an edge between two end vertices of the star $K_{1,n}$. Let $K_3 \circ K_{1,n}$ denote the graph obtained by identifying a vertex of K_3 with an end vertex of the star $K_{1,n}$. Let $K_3 * * K_{1,n}$ denote the graph obtained by inserting an edge between an end vertex of a $K_{1,n}$ and a fixed vertex of a K_3. Finally let $C_4 \square K_{1,n}$ be the graph obtained by identifying a vertex of C_4 with the central vertex of $K_{1,n}$. The graphs in $\mathscr{F}_2$ include these special graphs. All the graphs in $\mathscr{F}_2$ are shown labeled in Figure 2. It is now appropriate to define the three classes of graphs mentioned earlier. These classes are denoted by $\mathscr{G}_1$, $\mathscr{G}_2$, and $\mathscr{G}_3$ and are defined as follows: The class $\mathscr{G}_1$ consists of those $(n, n-1)$ graphs which do not appear in $\mathscr{F}_1$. The class $\mathscr{G}_2$ consists of those (n, n) graphs which are not members of the collection $\mathscr{F}_2$. The class $\mathscr{G}_3$ consists of graphs with n vertices and no more than $6n/5 - 2$ edges which are not stars and contain no cycles of length 3 or 4.

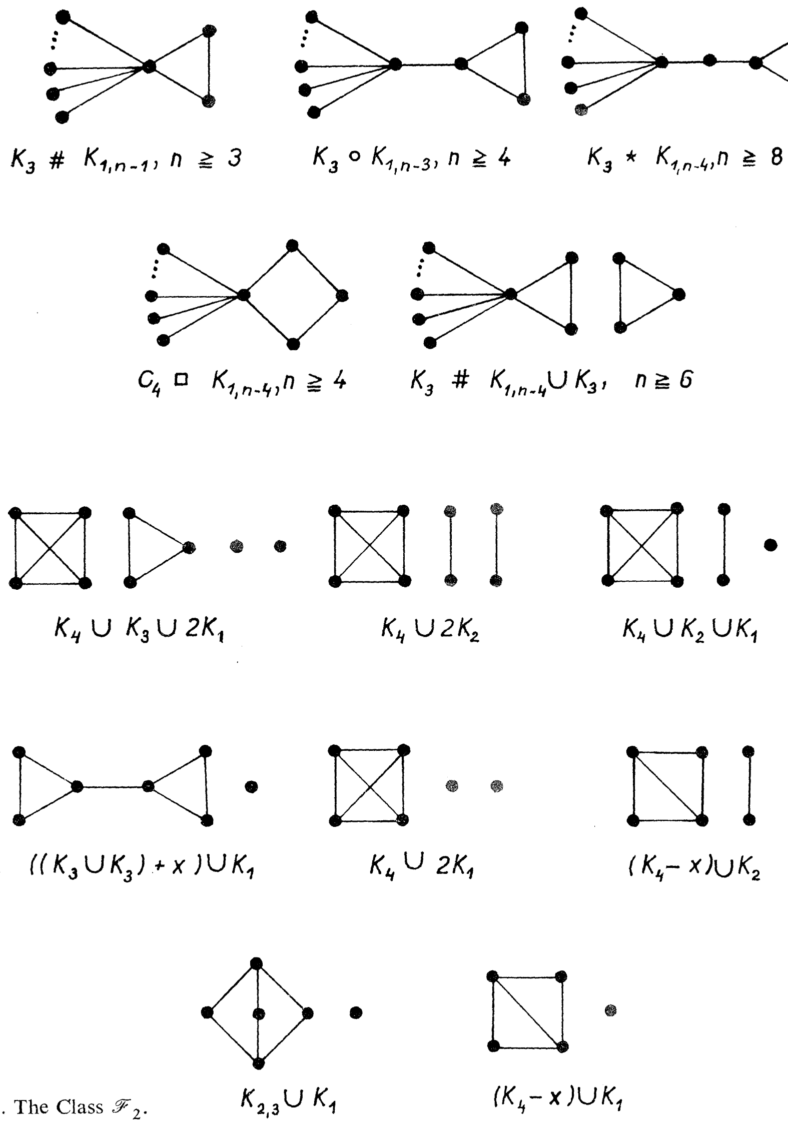

Fig. 2. The Class $\mathscr{F}_2$.

It is clear from the results proved in the sequel that the classes $\mathscr{G}_1$, $\mathscr{G}_2$, and $\mathscr{G}_3$ are descriptions of classes of embeddable graphs. The first two of these classes are precisely those $(n, n-1)$ graphs and (n, n) graphs that are embeddable. Thus the graphs in $\mathscr{F}_1$ are the only $(n, n-1)$ graphs which fail to be embeddable and those in $\mathscr{F}_2$ are the only non-embeddable (n, n) graphs. It should be noted that except for a few small order cases all the forbidden (non-embeddable) graphs are obtained from a K_3, a C_4, and a star, frequently identifying vertices of the cycle with those of the star. For this reason, it is conjectured that each non-star graph which contains no cycles of length 3 or 4 as subgraphs is always embeddable.

RESULTS

To prove the two main theorems mentioned in the introduction, two embedding results are needed. They are stated without proof, with proofs found in the given references.

Theorem 1. [2, 3] *Each $(n, n - 2)$ graph, is embeddable.*

Theorem 2. [8] *Each $(n, n - 1)$ graph, in $\mathcal{G}_1$ is embeddable.*

With these results the first principal result can be proved.

Theorem 3. *All graphs in class $\mathcal{G}_2$ are embeddable.*

The proof of this theorem will follow by induction on the order of the graphs in $\mathcal{G}_2$. For simplicity four lemmas will be proved. Observing that all graphs of small order in $\mathcal{G}_2$ are embeddable, the first, third, and fourth of these lemmas give an immediate inductive proof of the theorem. Thus the proof follows from the next four lemmas. For convenience throughout the statements and proofs of these lemmas G will always denote a graph in $\mathcal{G}_2$ with n vertices.

Lemma 4. *If G has an isolated vertex, then G is embeddable.*

Proof. Let v be an isolated vertex of G and let $H = G - v$. Since G is a (n, n) graph, H is a $(n - 1, n)$ graph. Thus H contains a vertex of degree 3 or more. Let w be a vertex of maximal degree. The graph $H - w$ is a $(n - 2, q)$ graph with $q \leqq n - 3$. By Theorem 1 if $q \leqq n - 4$, then there exists an embedding σ of $H - w$. This embedding can be extended to an embedding of G by mapping v to w and w to v. Hence the proof is complete unless $q = n - 3$. By Theorem 2, $H - w$ is again embeddable unless $H - w \in \mathcal{F}_1$. Therefore the only case that remains is when $G \in \mathcal{G}_2$, each vertex of H is of degree 3 or less, and $H - w \in \mathcal{F}_1$. Although there are several cases to consider, it is easily checked that such graphs are embeddable completing the proof of the lemma.

Lemma 5. *Let G contain two vertices of degree 1 with disjoint neighborhoods. If every graph in $\mathcal{G}_2$ with fewer than n vertices is embeddable, then G is embeddable.*

Proof. Let v and w be vertices of degree 1 in G with disjoint neighborhoods and $H = G - v - w$. If H is embeddable, then since the neighborhoods of v and w are disjoint, this embedding can be extended to an embedding of G. Thus the proof is complete unless $H \in \mathscr{F}_2$. This reduces G to a graph such that the deletion of each pair of vertices of degree 1 with disjoint neighborhoods gives an element of $\mathscr{F}_2$. Although it is somewhat tedious, one can check that such graphs G are always embeddable. Hence the result follows.

Lemma 6. *If G is disconnected, has a tree as a component, and every graph in $\mathscr{G}_2$ with less than n vertices is embeddable, then $\mathscr{G}$ is embeddable.*

Proof. From Lemma 4 there is no loss of generality in assuming that G has no isolated vertices. Let T be a tree which is a nontrivial component of G and $H = G - T$. Thus T is a $(t, t-1)$ graph and H is a $(n-t, n-t+1)$ graph. By Lemma 5 if T is not a star or H has a vertex of degree 1, then G is embeddable. Hence the proof is complete unless $T = K_{1,t-1}$, $t \geqq 2$, and H has no vertices of degree 1. Let v be the central vertex of $T = K_{1,t-1}$ and w be a vertex of maximal degree in H. Clearly the degree of w in H (or G) is at least 3. If the degree of w is at least 4, $H - w$ is a $(n-t-1, q)$ graph with $q \leqq n-t-3$, so by Theorem 1 is embeddable. For this case and more generally when $H - w$ is embeddable the embedding is extendable to an embedding σ for G by defining $\sigma(v) = w$, $\sigma(w) = v$, and $\sigma(x) = x$ for each vertex x in $T - v$. Thus the proof is complete unless $T = K_{1,t-1}$, $t \geqq 2$, each vertex of H is of degree 2 or 3, and $H - w \in \mathscr{F}_1$. For each of these possibilities one can easily check that G is embeddable so that the result follows.

Lemma 7. *Let G be such that each of its components contains a cycle. If each graph in $\mathscr{G}_2$ with fewer than n vertices is embeddable, then G is embeddable.*

Proof. Since all components of G contain a cycle, each component must be a (k, k) graph. Thus each component is a cycle or contains a vertex of degree 1. Observe that G may be connected. By Lemma 5 the result follows unless all but at most one of the components of G is a cycle and this component on k vertices and k edges has all its vertices of degree 1 adjacent to a single vertex. To describe G precisely let $H(r, s-1, t)$ be the graph obtained by identifying a fixed vertex of the cycle C_r with an end vertex of the path P_s on s vertices, and identifying the other end vertex of P_s with the central vertex of the star $K_{1,t}$. Note that $H(r, 0, 0)$ is simply the cycle C_r. Therefore G has one component which is a $H(r, s, t)$ graph with the remaining components cycles. It is straightforward to check that the graph $H(r, s, t)$ is embeddable if $r \geqq 5$, or $r = 4$ and $s, t \geqq 1$, or $r = 3$, $s \geqq 3$, and $t \geqq 0$. Also observe that each of the following graphs are embeddable: $C_r (r \geqq 5)$, $H(r, s, t) \cup C_k (s, t \geqq 0, r \geqq 4, k \geqq 3$ or $r, k \geqq 3,\ s, t \geqq 1)$, $H(r, s, t) \cup C_k \cup C_l (r, k, l \geqq 3,\ s, t \geqq 0)$. With this information it is easy to check that G is always embeddable. This completes the proof of the lemma and consequently the proof of Theorem 3.

The next objective is to prove the second main result of the paper. This proof is quite involved and requires several intermediate results.

Theorem 8. *If $G \in \mathcal{G}_3$, then G is embeddable.*

This result, as that of Theorem 3, is proved by induction on the order of G. If G has 10 vertices or less, then G has no more than n edges and is embeddable by Theorem 3. The inductive proof will be a consequence of the next set of seven lemmas. In particular it will follow from Lemmas 12, 14 and 15. Throughout the proof of these lemmas the following be assumed.

(1) *G is a graph on n vertices in $\mathcal{G}_3$, and*

(2) *each graph with fewer than n vertices in $\mathcal{G}_3$ is embeddable.*

The next lemma is a simple embedding result which is used repeatedly in the lemmas that follow.

Lemma 9. *Let H' be a subgraph of G' and $K' = G' - H'$. Further let B' be the set of those vertices of H' adjacent to at least one vertex of K' and A' the set of the remaining vertices of H'. If there exist embeddings τ of H' and σ of K' such that $\tau(B') \subseteq A'$, then there exists an embedding θ of G' which extends both τ and σ.*

Proof. Simply let $\theta(x) = \sigma(x)$ for all vertices x in K' and $\theta(x) = \tau(x)$ for all vertices x in H'. The map θ is an embedding since τ is an embedding of H', σ is an embedding of K', and $\bar{G}'$ contains all edges between A' and the vertices of K'.

Lemma 10. *If G has an isolated vertex, then G is embeddable.*

Proof. Let v be an isolated vertex in G and w a vertex of maximal degree in G. If the degree of w is no more than 2, then G has at most $n - 1$ edges. Since G contains no cycles of length 3 or 4, $G \notin F_1$ so it is embeddable. Hence it is assumed that the degree of w is at least 3. Let H be the graph with vertex set $\{v, w\}$ and empty edge set and let $K = G - H$. Since $G \in \mathcal{G}_3$, it is clear that K contains no cycles of length 3 or 4. Also K is not a star, otherwise w is adjacent to at least 3 vertices of the star implying that G contains a cycle of length 3 or 4. The graph K has $n - 2$ vertices and no more than $6n/5-5$ edges. Since $6n/5-5 \leqq (6/5)(n - 2) - 2$, it follows that $K \in \mathcal{G}_3$ and is embeddable. Thus by mapping v to w and w to v the embedding of K is clearly extendable to an embedding of G. This completes the proof.

Lemma 11. *If G has adjacent vertices v and w, both of degree 2, such that the neighbors of v and w are of degree greater than 1, then G is embeddable.*

Proof. Let the neighbors of v be $\{v_1, w\}$ and those of w be $\{v, w_1\}$. Also let $H = \langle v, w, v_1, w_1 \rangle$, the subgraph induced by $\{v, w, v_1, w_1\}$, and define $L = G - H$. If L is a star, each of v_1 and w_1 are adjacent to at most one vertex of L, otherwise G would contain a cycle of length 3 or 4. But then L is a $K_{1,n-5}$ graph so that G is an (n, n) graph in G_2 and hence embeddable. Thus we may assume that L is not a star and contains no cycles of length 3 or 4. Further L contains at most $(6/5)\, n - 7 \leqq$

$\leqq (6/5)(n - 4) - 2$ edges so that $L \in \mathcal{G}_3$ and is thus embeddable. The map τ defined by $\tau(v) = v_1$, $\tau(w_1) = v$, $\tau(v_1) = w$, and $\tau(w) = w_1$, is an embedding of H. Since $\tau\{v_1, w_1\} = \{v, w\}$, Lemma 9 applies and gives an embedding of G.

Lemma 12. *If G has no vertices of degree* 1, *then G is embeddable.*

Proof. By Lemma 10 it is assumed that each vertex has positive degree. If G contains t vertices of degree 2, no pair of which are adjacent, then $2t \leqq 6n/5-2$ and $2t + 3(n - t) \leqq 2(6n/5-2)$. Since these inequalities are incompatible, there exists at least one pair of adjacent vertices of degree 2. Hence this result follows from Lemma 11.

Lemma 13. *If G has two vertices of degree* 1 *which have no common adjacency, then G is embeddable.*

Proof. Let v and w be vertices of degree 1 with v adjacent to v_1 and w adjacent to w_1, $v_1 \neq w_1$. Since $\mathcal{G}_2 \subseteq \mathcal{G}_3$ and elements of $\mathcal{G}_2$ are embeddable, it will be assumed throughout the proof that G has more than n edges.

If v_1 is of degree 1, then let $u \in G - v - v_1$ of maximal degree in G. Since G has more than n edges, the degree of u is at least 3. Let H be the subgraph $\langle u, v, v_1\rangle$ and let $L = G - H$. Embed H by the map τ where $\tau(u) = v$, $\tau(v) = u$, and $\tau(v_1) = v_1$. The graph L has $n - 3$ vertices, at most $(6n/5) - 6 \leqq (6/5)(n - 3) - 2$ edges, is not a star, and contains no cycles of length 3 or 4. Hence L is embeddable, so that this embedding and the embedding τ of H can be extended to an embedding for G by Lemma 9.

Next consider the case where the degree of v_1 is 2. Let v and v_2 be the neighbors of v_1. Select a vertex u in $G - v - v_1 - v_2$ such that u is not adjacent to v_2 and such that the degree of u is as large as possible. This choice of u is possible since G has no cycles of length 3 and G is assumed to have at least $n + 1$ edges. Let H be the graph $\langle v, v_1, v_2, u\rangle$ and let $L = G - H$. It is easy to see that L has $n - 4$ vertices and at most $6n/5 - 7 \leqq (6/5)(n - 4) - 2$ edges. Also L is not a star and contains no cycles of length 3 or 4 so that L is embeddable. The graph H can be embedded via the map τ defined as follows: $\tau(u) = v$, $\tau(v_2) = v_1$, $\tau(v_1) = u$, $\tau(u) = v_2$. By applying Lemma 9 this embedding of H together with the embedding of L can be extended to an embedding of G.

Finally the remaining case when the degree of v_1 is at least 3 is considered. From the cases considered there is no loss of generality in assuming that w_1 is of degree at least 3. Let H be the subgraph $\langle v, v_1, w, w_1\rangle$ and let $L = G - H$. As before L is not a star, contains no cycles of length 3 or 4, has $n - 4$ vertices, and no more than $(6/5)(n - 4) - 2$ edges so that it is embeddable. The mapping $\tau(v) = w_1$, $\tau(v_1) = v$, $\tau(w_1) = w$, $\tau(w) = v_1$ is an embedding of H. Applying Lemma 9 again gives an embedding of G.

Lemma 14. *If G has at most two vertices of degree* 1, *then G is embeddable.*

Proof. From Lemmas 12 and 13 it will be assumed that G has either one or two vertices of degree 1 and these vertices are adjacent to a common vertex. Let v be the vertex of G adjacent to the vertices of degree 1. The case when v is of degree 2 was considered in the body of the proof of Lemma 13. Hence it will be assumed that the degree of v is at least 3. Further, by Lemma 11, if G contains two adjacent vertices of degree 2, G is embeddable. But if there are t vertices in G of degree 2, no two of which are adjacent, both $2t \leqq 6n/5 - 2$ and

$$\max\{1 + 2t + 3(n - t - 1),\ 2 + 2t + 3(n - t - 2)\} \leqq 2(6n/5 - 2).$$

Since these inequalities are incompatible, G is always embeddable.

Lemma 15. *If G has at least three vertices of degree* 1, *then G is embeddable.*

Proof. By Lemma 13 it can be assumed that there is a vertex v of G adjacent to the vertices of degree 1.

First consider the case where G has two vertices u and w of degree 2 neither of which is adajcent to v. Because of Lemma 11, it is assumed that u and w are not adjacent. Let the neighbors of u be u_1 and u_2 and of w be w_1 and w_2 with $u_2 = w_2$ if the neighborhoods of u and w intersect. Also let v_1, v_2, v_3 be three distinct neighbors of v of degree 1. Let H be the subgraph $\langle v, v_1, v_2, v_3, u, w, u_1, u_2, w_1, w_2\rangle$ and $L = G - H$. Again by Lemma 11 it can be assumed that each of u_1, u_2, w_1, w_2 are vertices of degree 3 or more. Hence since G has no cycles of length 3 or 4, L has at most $(6/5)(n - t) - 2$ edges with t the number of vertices in H. Also L is not a star. To see this suppose the contrary. Then the nonexistence of cycles of length 3 or 4 implies that there are at most five edges joining vertices of H to vertices of L. Thus, since all the vertices of degree 1 in G are adjacent to v, the star L has at most five end vertices. Each of the possibilities force G to have more than $(6/5)n - 2$ edges, a contradiction. Hence L is not a star and therefore is an embeddable graph.

To apply Lemma 9 an appropriate embedding τ of H is needed. This embedding will be given only when H has 10 vertices, i.e. when $u_2 \neq w_2$, since the embedding when $u_2 = w_2$ and H has 9 vertices is similar. Define τ on H as follows: $\tau(u_1) = v_1$, $\tau(u_2) = v_2$, $\tau(w_1) = v_3$, $\tau(w_2) = u$, $\tau(v) = w$, $\tau(v_1) = v$, $\tau(v_2) = u_1$, $\tau(v_3) = u_2$, $\tau(u) = w_1$, $\tau(w) = w_2$. Lemma 9 now applies and G is embeddable.

It remains to consider the case where all except possibly one of the vertices of degree 2 in G are adjacent to v. Thus if r denotes the number of vertices of degree 2 in G and t denotes the number of vertices of degree 1, vertex v is at least of degree $r + t - 1$. Furthermore, a lower bound on the sum of degrees of all vertices of G is $t + 2r + r + t - 1 + 3(n - r - t - 1) \leqq 2(6n/5 - 2)$ implying that $t \geqq 3n/5$. Thus G contains at least $3n/5$ vertices of degree 1. Let $T = \{v_1, v_2, \ldots, v_t\}$, $t \geqq 3n/5$, denote this set of vertices and let L be the subgraph of G induced by the vertices $\{x \mid x \notin T \cup \{v\}\}$. Since G is not a star, there exists a vertex w in L which is not adjacent to v. Further let $M = \{v'_1, v'_2, \ldots, v'_m\}$ be the set of vertices of L adjacent to v and $S = \{w_1, w_2, \ldots, w_s\}$ be the vertices of L adjacent to w. Since G contains no cycles of length 4 it is clear that M and S have at most one element in common. Also since $t \geqq 3n/5$, if L has l vertices, then $l \leqq 2n/5 - 1$. Embeddings of G will be given separately when M and S are disjoint and when they have an element in common.

Consider the case when $M \cap S = \phi$. Since G has no cycles of length 3, the set $T \cup S$ is an independent set of vertices of G. Also since $s + 1 \leqq l \leqq 2n/5 - 1$ and $t \geqq 3n/5$, select a subset T_1 of T of cardinality $l - s$. Let σ be any bijection of the set of vertices of L not belonging to S onto T_1. Extend σ to any bijection for G such that $\sigma(v) = w$, and $\sigma(y) = y$ for each $y \in S$. It is easy to see that σ is an embedding of G.

Finally, the case where M and S have an element in common will be considered. Let $M \cap S = \{y\}$. In this case it is first shown that there exists a vertex u in L, $u \notin M \cup S \cup \{w\}$. Suppose the contrary. Since G contains no cycles of length 3 or 4 each vertex in S different from y is of degree 2 and not adjacent to v. But by assumption all but one vertex of degree 2 is adjacent to v. Hence, since w is not adjacent to v and therefore not of degree 1, S must contain exactly two elements one of which is degree 2 and not adjacent to v. This vertex and w are both of degree 2 and non-adjacent to v, a contradiction. Thus the assumption is false and there exists a u in L, $u \notin M \cup S \cup \{w\}$.

It remains to describe an embedding for G. Select a subset T_1 of T of cardinality $l - s$. Let σ be any bijection of those vertices of L disjoint from $(S \setminus \{y\}) \cup \{u\}$ onto T_1. Extend σ to any bijection of G such that $\sigma(v) = w$, $\sigma(u) = y$, and $\sigma(z) = z$ for each $z \in S \setminus \{y\}$. Again using the independence of $T \cup S$ in G it is easy to check that σ is an embedding of G which completes the proof of the lemma and consequently the proof of Theorem 8.

Conjecture. *Each graph which is not a star and contains no cycles of length* 3 *or* 4 *as subgraphs is embeddable.*

The reason for proving Theorem 8 is to provide some evidence that the above conjecture might hold. This conjecture, if true, would fit nicely with many other characterization theorems which specify that all but a family of forbidden graphs satisfy a given property or are of a given type.

References

[1] *M. Behzad* and *G. Chartrand:* Introduction to the Theory of Graphs, Allyn and Bacon, Boston, 1971.

[2] *B. Bollobas* and *S. E. Eldridge:* Packings of Graphs and Applications to Computational Complexity, J. of Comb. Theory (B), *25* (1978), 105—124.

[3] *D. Burns* and *S. Schuster:* Every $(p, p - 2)$ Graph is Contained in its Complement, J. of Graph Theory, *1* (1977), 277—299.

[4] *P. A. Catlin:* Embedding Subgraphs under Extremal Degree Conditions, Proc. 8th S-E Conf. Combinatorics, Graph Theory, and Computing, (1977), 139—145.
[5] *P. A. Catlin:* Subgraphs of Graph, I, Discrete Math., *10* (1974), 225—233.
[6] *F. Harary:* Graph Theory, Addison-Wesley, Reading, Mass., 1969.
[7] *N. Sauer* and *J. Spencer:* Edge Disjoint Placement of Graphs, J. of Comb. Theory (B), 25 (1978), 295—302.
[8] *S. Schuster:* Embedding $(p, p - 1)$ Graphs Their Complements, Israel J. of Math., *30* (1978), 313—320.

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