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STABILIZATION OF SOLUTIONS OF ABSTRACT PARABOLIC EQUATIONS

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In this paper we investigate the stabilization and the rate of stabilization for $t \to \infty$ of the solutions of the equations

$$(1) \qquad u'(t) + A(t)\,u(t) = f(t) \quad (0 < t < \infty)\,, \quad u(0) = u_0\,,$$

where $A(t)$ $(t \geqq 0)$ are monotone, coercive, in general non-linear operators from a real, reflexive B-space V into its dual space V^*. Let H be a real Hilbert space. We assume that the imbedding $V \subset H$ is continuous and that V is dense in H. Under sufficiently general conditions which guarantee the existence and uniqueness of the solution $u(t)$ of (1) (see Remarks 1 and 2) we prove in § 1 that $u(t) \to 0$ in H for $t \to \infty$ provided $f(t)$ decays for $t \to \infty$ in some sense. If $A(t)$ is a strictly or strongly monotone operator (see (13_1), (13_2), (13_3)) then $u(t) \to u_\infty$ in H for $t \to \infty$ provided $f(t)$ tends to f_∞ and $A(t)$ tends to A_∞ for $t \to \infty$ (see (9_2), (12)), where u_∞ is the solution of the stationary equation $A_\infty u_\infty = f_\infty$. (If $A(t) \equiv A$, then $A_\infty = A$). In § 1 we obtain results which are modifications of those in [5], [6], [11]. In § 2 we study the rate of the stabilization of $u(t)$ for $t \to \infty$. For a certain class of stationary operators A we prove that the solution $u(t)$ stabilizes in finite time, i.e., there exists $t_0 = t_0(u_0)$ such that $u(t) = 0$ for $t \geqq t_0$ provided $A(t) \equiv A$ and $f(t) \equiv 0$. If $f : (0, T) \to H$ is continuously differentiable in t and of bounded variation on $\langle 0, \infty)$ then we prove that $u(t) \to u_\infty$ also in the norm of the space V. In § 3 we present some applications of the results from § 1 and § 2 to parabolic initial-boundary value problems.

NOTATION AND DEFINITIONS

Denote by $\|\cdot\|$, $\|\cdot\|_*$ and $|\cdot|$ the norms in V, V^* and H, respectively. If we identify H with its dual H^* then we have

$$V \subset H \subset V^*\,.$$

The duality between $v \in V$ and $f \in V^*$ will be denoted by (f, v). If $f, v \in H$ then (f, v) coincides with the scalar product in H.

Let X be an arbitrary Banach space (X^* its dual space) and $0 < T \leqq \infty$. By $L_p(0, T; X) \equiv Z$ $(1 \leqq p \leqq \infty)$ we denote the Banach space (see, e.g., [15], [7]) of all measurable abstract functions $v : (0, T) \rightarrow X$ satisfying

$$\|v\|_Z^p = \int_0^T \|v(t)\|_X^p \, \mathrm{d}t < \infty \quad \text{for} \quad 1 \leqq p < \infty$$

and

$$\|v\|_Z = \operatorname*{ess\,sup}_{t \in (0,T)} \|v(t)\|_X < \infty \quad \text{for} \quad p = \infty \, .$$

Henceforth, let $p > 1$, $q \geqq 1$ be conjugate numbers $(p^{-1} + q^{-1} = 1)$. The dual space Z^* to Z is $L_q(0, T; X^*)$ (see, e.g., [7]). By $C(0, T; X)$ $(C^1(0, T; X))$ we denote the space of continuous (continuously differentiable) abstract functions $v : \langle 0, T \rangle \rightarrow$ $\rightarrow X$. By $C_w(0, T; X)$ we denote the set of all abstract functions $v : (0, T) \rightarrow X$ satisfying $(x^*, v(t)) \in C(0, T)$ for all $x^* \in X^*$. The abstract function $\mathrm{d}u/\mathrm{d}t : (0, T) \rightarrow X$ is the weak derivative of $u(t)$, iff $(\mathrm{d}/\mathrm{d}t)(x^*, u(t)) = (x^*, \mathrm{d}u(t)/\mathrm{d}t)$ for all $x^* \in X^*$. We denote $C_w^1(0, T; X) = \{v : (0, T) \rightarrow X \text{ for which } \mathrm{d}v/\mathrm{d}t \in C_w(0, T; X)\}$. If $\mathrm{d}v/\mathrm{d}t \in$ $\in L_p(0, T, X)$ then there exists $v'(t)$ (the strong derivative) and $v'(t) = \mathrm{d}v(t)/\mathrm{d}t$ for a.e. $t \in (0, T)$.

We shall assume that $f(t)$ is an abstract function $f : \langle 0, \infty) \rightarrow V^*$ such that $f \in L_q(0, T; V^*)$ (for all $T < \infty$) and u_0 from (1) is an element of H. In some special cases f and u_0 will be supposed to be more regular.

Under the solution of (1) we understand an abstract function $u : (0, \infty) \rightarrow V$ with the following properties: $u \in L_p(0, T; V)$, $u' \in L_q(0, T; V^*)$, $u(0) = u_0$ and $u(t)$ satisfies (1) for a.e. $t \in (0, \infty)$.

In the following remarks we introduce some results concerning existence and uniqueness of the solution of (1).

Remark 1. From [1], [2], [3], the following results follows: If the following assumptions hold:

a_1) $A(t) : V \rightarrow V^*$ (for $t \geqq 0$) is demicontinuous,
b_1) $(A(t)\, v, w)$ is measurable in t for all fixed $v, w \in V$,
c_1) $(A(t)\, v - A(t)\, w, v - w) \geqq 0$ for all $t > 0$ and $v, w \in V$,
d_1) $(A(t)\, v, v) \geqq C_1 \|v\|^p - C_2$, $C_1 > 0$, $1 < p < \infty$,
e_1) $\|A(t)\, v\| \leqq C(1 + \|v\|^{p-1})$ for all $t > 0$,
f_1) $f \in L_q(0, T; V')$ for all $T < \infty$,
g_1) $u_0 \in H$,

then there exists a unique solution of (1).

Remark 2. Existence of a more regular solution of (1) can be guaranteed by stronger assumptions on $f(t)$, A and u_0 as in Remark 1. Let V and H be separable spaces and let $A(t) \equiv A$.

If the following assumptions are satisfied:

a_2) $A : V \to V^*$ is demicontinuous and bounded,
b_2) $(Av - Aw, v - w) \geqq 0$ for all $v, w \in V$,
c_2) $\|v\|^{-1} (Av, v) \to \infty$ for $\|v\| \to \infty$,
d_2) $u_0 \in V$ and $Au_0 \in H$,
e_2) $f : \langle 0, T \rangle \to H$ is Lipschitz continuous on each compact subset of $(0, \infty)$,

then there exists a unique solution $u(t)$ of (1) (see, e.g., [5], [6]) with the following properties:

$u : \langle 0, \infty) \to H$ is Lipschitz continuous on each compact subset of $\langle 0, \infty)$, $u \in L_\infty(0, T; V)$, $u' \in L_\infty(0, T; H)$ and $Au \in L_\infty(0, T; H)$.

Moreover, if $f \in C^1(0, T; H)$ then $u \in C^1_w(0, T; H)$, $Au \in C_w(0, T; H)$ and if we replace $u'(t)$ by $\mathrm{d}u(t)/\mathrm{d}t$ then (1) is valid for all $t > 0$ (see [8], [9]). The estimate

$$\left|\frac{\mathrm{d}u(t)}{\mathrm{d}t}\right| \leqq |f(0)| + |Au|_0 + \int_0^T |f'(t)| \,\mathrm{d}t$$

holds (see [8], Remark 2 and Lemma 5). A similar result (but under some additional assumptions) is proved also for the nonstationary case $A(t) \not\equiv A$ in [10].

Positive constants will be denoted by C and the dependence of C on the parameter ε by $C(\varepsilon)$. Constants C and $C(\varepsilon)$ may denote also various constants in the same discussion.

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In this paper we assume that there exists a unique solution (in the previously defined sense) $u(t)$ of (1). Since $u \in L_p(0, T; V)$ and $u' \in L_q(0, T; V^*)$, we have $u \in C(0, T; H)$ for all $T < \infty$ and

$$|u(r)|^2 - |u(s)|^2 = 2\int_s^r (u'(t), u(t)) \,\mathrm{d}t$$

for all $0 \leqq r, s < \infty$ (see [1], [7]).

Let $\gamma(t)$ be a continuous function satisfying: $\gamma(0) = 0$, $\gamma(t) > 0$ for $t > 0$ and there exists $\delta > 0$ and $t_0 > 0$ such that $\gamma(t) > \delta$ for $t \geqq t_0$.

Coerciveness of $A(t)$ will be assumed in some of the following forms:

(3_1) $(A(t)\, v, v) \geqq 0$,
(3_2) $(A(t)\, v, v) \geqq \gamma(\|v\|)$,
(3_3) $(A(t)\, v, v) \geqq C\|v\|^p$ $(1 < p < \infty)$.

Clearly, (3_2) implies (3_1). We shall assume $f(t)$ to have the following properties:

(4_1) $f \in L_1(0, T; H)$,
(4_2) $f \in L_q(0, \infty; V^*)$,
(4_3) $f \in L_q(0, T; V^*)$ for all $T < \infty$.

Lemma 1. *Let one of the assumptions* i) *or* ii) *be satisfied, where*

i) (3_1), (4_1),

ii) (3_3), (4_2).

Then $u \in L_\infty(0, \infty; H)$.

Proof. i) From (1) we deduce

$$(u'(t), u(t)) + (A(t)\, u(t), u(t)) = (f(t), u(t)) . \tag{5}$$

Integrating (5) over $\langle 0, t \rangle$ and using (3_1) we have

$$|u(t)|^2 - |u(0)|^2 \leqq 2 \int_0^t |f(s)|\, |u(s)|\, \mathrm{d}s ,$$

which implies ($u \in C(\langle 0, t \rangle, H)$)

$$\max_{0 \leq \xi \leq t} |u(\xi)|^2 \leqq |u(0)|^2 + 2 \max_{0 \leq \xi \leq t} |u(\xi)| \int_0^t |f(s)|\, \mathrm{d}s .$$

From this inequality we easily obtain

$$|u(t)| \leqq |u(0)| + 2 \int_0^\infty |f(s)|\, \mathrm{d}s$$

for all $t \geqq 0$ which proves the assertion.

ii) In this case (5) and (3_3) imply

$$(u'(t), u(t)) + C\|u(t)\|^p \leqq \|f(t)\|_* \, \|u(t)\| \leqq \leqq \frac{\varepsilon^{-q}}{q} \|f(t)\|_*^q + \frac{\varepsilon^p}{p} \|u(t)\|^p , \tag{6}$$

where Young's inequality has been used ($\varepsilon > 0$). Integrating (6) over $\langle s, r \rangle$ for a suitable ε we obtain

$$|u(r)|^2 - |u(s)|^2 + C \int_s^r \|u(t)\|^p \, \mathrm{d}t \leqq C_1 \int_s^r \|f(t)\|_*^q \, \mathrm{d}t \tag{7}$$

where $C_1 = C_1(\varepsilon)$. From (7) (for $s = 0$) and (4_2) we deduce the required result.

Theorem 1. *Let one of the assumptions* i) *or* ii) *be satisfied, where*

i) (3_2), (4_1),

ii) (3_3), (4_2).

Then $u(t) \to 0$ *in* H *for* $t \to \infty$.

Proof. i) Integrating (5) over the interval $\langle s, r \rangle$ and using (3_2) we obtain

$$|u(r)|^2 - |u(s)|^2 + 2 \int_s^r \gamma(\|u(t)\|)\, \mathrm{d}t \leqq 2 \int_s^{r'} |f(t)|\, |u(t)|\, \mathrm{d}t . \tag{8}$$

Using Lemma 1 and (4_1), we deduce from (8) that

$$\int_0^\infty \gamma(\|u(t)\|)\,dt < \infty$$

which implies: There exists a subsequence $\{t_n\}$, $t_n \to \infty$ for $n \to \infty$, such that $\|u(t_n)\| \to 0$ for $n \to \infty$. Thus, $|u(t_n)| \to 0$ for $n \to \infty$ since $V \subset H$. From this fact and from Lemma 1, (4_1) and (8) we obtain the required result.

ii) From (7) (for $s = 0$), (4_2) and Lemma 1 we deduce

$$\int_0^\infty \|u(t)\|^p\,dt < \infty\,.$$

Hence, using (7) and (4_2), by the same argument as in Assertion i) we deduce the required result.

Let f_∞ be an element of the space H or V^*. We shall assume that $f(t)$ tends to f_∞ for $t \to \infty$ in the following sense:

$$(9_1) \qquad \int_0^\infty |f(t) - f_\infty|\,dt < \infty\,,$$

$$(9_2) \qquad \int_0^\infty \|f(t) - f_\infty\|_*^q\,dt < \infty\,.$$

Let A_∞ be an operator from V into V^* and let $u_\infty \in V$ be a solution of the equation

$$(10) \qquad A_\infty u_\infty = f_\infty\,.$$

We shall assume that $A(t)$ tends to A_∞ for $t \to \infty$ in the following sense:

$$(11) \qquad \int_0^\infty \|A(t)\,u_\infty - A_\infty u_\infty\|_*^q\,dt < \infty\,.$$

Assumption (11) is clearly satisfied, if

$$(12) \qquad \int_0^\infty \|A(t)\,v - A_\infty v\|_*^q\,dt < \infty$$

holds for all $v \in V$. In particular, if $A(t) \equiv A$ for $t > 0$, then $A \equiv A_\infty$.

Monotonicity of $A(t)$ will be considered in the form

$$(13_1) \qquad (A(t)\,v - A(t)\,w,\, v - w) > 0 \quad \text{for all} \quad v, w \in V,\quad v \neq w\,,$$

$$(13_2) \qquad (A(t)\,v - A(t)\,w,\, v - w) \geqq \gamma(\|v - w\|) \quad \text{for all} \quad v, w \in V,$$

$$(13_3) \qquad (A(t)\,v - A(t)\,w,\, v - w) \geqq C\|v - w\|^p \ (1 < p < \infty)$$

for all $v, w \in V$. Clearly, (13_2) implies (13_1).

Theorem 2. *Suppose* (10). *Let one of the assumptions* i) *or* ii) *be satisfied, where*

i) $(9_1), (13_2), A(t) \equiv A$,

ii) $(9_2), (13_3), (11)$.

Then $u(t) \to u_\infty$ *in* H *for* $t \to \infty$.

Proof. i) From (10) and (1) we obtain

$$(u'(t), u(t) - u_\infty) + (A\,u(t) - Au_\infty, u(t) - u_\infty) = \\ = (f(t) - f_\infty, u(t) - u_\infty)\,. \tag{14}$$

Integrating (14) over $\langle s\ \ r\rangle$ and using (13_2) we deduce

$$|u(r) - u_\infty|^2 - |u(s) - u_\infty|^2 + 2\int_s^r \gamma(\|u(t) - u_\infty\|)\,\mathrm{d}t \leqq \\ \leqq 2\int_s^r |f(t) - f_\infty|\,|u(t) - u_\infty|\,\mathrm{d}t\,. \tag{15}$$

From (15) and (9_1) similarly as in Lemma 1, we deduce $u \in L_\infty(0, \infty; H)$. Hence, from (15) we conclude

$$\int_0^\infty \gamma(\|u(t) - u_\infty\|)\,\mathrm{d}t < \infty\,.$$

From this fact, analogously as in Theorem 1, the required result follows.

ii) From (1) and (10) we have

$$(u'(t), u(t) - u_\infty) + (A(t)\,u(t) - A(t)\,u_\infty,\ u(t) - u_\infty) = \\ = (f(t) - f_\infty,\ u(t) - u_\infty) - (A(t)\,u_\infty - A_\infty u_\infty,\ u(t) - u_\infty)\,. \tag{16}$$

Using (13_3), (9_2), Hölder's and Young's inequalities in (16) we obtain

$$(u'(t), u(t) - u_\infty) + C\|u(t) - u_\infty\|^p \leqq \frac{2\varepsilon^p}{p}\,\|u(t) - u_\infty\|^p + \\ + \frac{\varepsilon^{-q}}{q}\,(\|f(t) - f_\infty\|_*^q + \|A(t)\,u_\infty - A_\infty u_\infty\|_*^q)\,. \tag{17}$$

Integrating (17) over the interval $\langle s, r\rangle$ for a suitable $\varepsilon > 0$ we deduce

$$|u(r) - u_\infty|^2 - |u(s) - u_\infty|^2 + C_1\int_s^r \|u(t) - u_0\|^p\,\mathrm{d}t \leqq \\ \leqq C_2\int_s^r (\|f(t) - f_\infty\|_*^q + \|A(t)\,u_\infty - A_\infty u_\infty\|_*^q)\,\mathrm{d}t$$

$(C_1 = C_1(\varepsilon))$. Hence, analogously as in the previous part we successively deduce

$$u \in L_\infty(0, \infty; H), \quad \int_0^\infty \|u(t) - u_\infty\|^p \, dt < \infty$$

and then the required result.

Consequence. Theorem 2 implies that the solution u_∞ of (10) is unique in V.

Remark 3. If in (3_2), (13_2) $\|v - w\|$ is replaced by $|v - w|$, then Theorems 1 and 2 remain true. Moreover, in this case the assumption $V \subset H$ can be weakened to the assumption that $V \cap H$ is dense in V and H.

Theorem 3. *Suppose* $A(t) \equiv A$, (9_1), (10) *and* $a_2)$, $c_2)$, $d_2)$ (*from Remark* 2). *Assume that* $f : \langle 0, \infty) \to H$ *is continuously differentiable and satisfies*

$$\int_0^\infty |f'(t)| \, dt < \infty . \tag{18}$$

i) *If the imbedding* $V \subset H$ *is compact and* (13_1) *holds then* $u(t) \to u_\infty$ *in* H *for* $t \to \infty$.

ii) *If* (13_2) *holds then* $u(t) \to u_\infty$ *in* V *for* $t \to \infty$.

Proof. i) From the estimates (2), (18) and the equation

$$\frac{du(t)}{dt} + A\,u(t) - Au_\infty = f(t) - f_\infty \quad \text{for all} \quad t > 0 \tag{19}$$

(see Remark 2) we deduce that there exist C_1, C_2 such that

$$\left|\frac{du(t)}{dt}\right| \leqq C_1 \quad \text{for all} \quad t > 0 \tag{20}$$

and

$$|A\,u(t)| \leqq C_2 \quad \text{for all} \quad t > 0 . \tag{21}$$

From (21) and $c_2)$ we conclude

$$\|u(t)\| \leqq C_3 , \quad |u(t)| \leqq C_4 \quad \text{for all} \quad t > 0 \tag{22}$$

$(C_3, C_4$ are suitable constants$)$

since $|A\,u(t)| \geqq \|A\,u(t)\|_* \geqq \|u(t)\|^{-1} (A\,u(t), u(t))$ and $V \subset H$. Hence, integrating (14) over $(0, \infty)$ we obtain the estimate

$$\int_0^\infty (A\,u(t) - Au_\infty, u(t) - u_\infty) \, dt \leqq C_5 \left(\int_0^\infty |f(t) - f_\infty| \, dt + 1 \right).$$

Thus, there exists a sequence $\{t_n\}$, $t_n \to \infty$ with $n \to \infty$ such that

(23) $$(A\,u(t_n) - Au_\infty, u(t_n) - u_\infty) \to 0 \quad \text{with} \quad n \to \infty\,.$$

From (21), (22), from the reflexivity of the space V and from the compactness of the imbedding $V \subset H$ we conclude that there exists $y \in H$ and $v \in V \cap H$ such that $A\,u(t_{n_k}) \rightharpoonup y$ in H (weak convergence in H) and $u(t_{n_k}) \to v$ in H for $k \to \infty$ ($\{t_{n_k}\}$ is a suitable subsequence of $\{t_n\}$). From these facts and the monotonicity of A we deduce easily $y = Av$. Then (23) implies $(Av - Au_\infty, v - u_\infty) = 0$ and hence (13_1) yields $v = u_\infty$. From $u(t_{n_k}) \to u_\infty$ in H, (9_1) and the formula

(24) $$|u(r) - u_\infty|^2 - |u(s) - u_\infty|^2 \leqq C_6 \int_s^r |f(t) - f_\infty|\,\mathrm{d}t$$

we obtain the required result.

ii) From (23) we deduce $u(t_n) \to u_\infty$ in V for $n \to \infty$ and hence $u(t_n) \to u_\infty$ in H for $t \to \infty$. Thus, from (24) we conclude $u(t) \to u_\infty$ in H for $t \to \infty$. On the other hand, from (19), (13_2) and from the estimates (22) we obtain the estimate

$$\gamma(\|u(t) - u_\infty\|) \leqq C_7|u(t) - u_\infty| \quad \text{for} \quad t > 0$$

which yields the required result.

2

Estimating the rate of stabilization of the solution $u(t)$ of (1) (for $t \to \infty$) we use the following assertion on the asymptotical behaviour for the solution $y(t)$ of the equation

(25) $$y'(t) = -C_0\,y(t)^\alpha + \varphi(t) \quad (0 < t < \infty,\ C_0 > 0)$$

where $y(0) \geqq 0$, $0 < \alpha$, and $\varphi(t)$ is a measurable nonnegative function.

Assertion 1. a) *If $\varphi(t) \to 0$ for $t \to \infty$, then $y(t) \to 0$ for $t \to \infty$.*

b) *Let $0 < \alpha < 1$.*

i) *If $\varphi(t) \equiv 0$, then $y(t) = 0$ for $t \geqq y(0)^{1-\alpha}/C_0(1 - \alpha)$ (C_0 is from (25)).*

ii) *If $\varphi(t) = O(t^{-\beta})$ ($\beta > 1$), then $y(t) = O(t^{-\beta+1})$.*

c) *Let $\alpha = 1$.*

i) *If $\varphi(t) = O(t^{-\beta})$ ($\beta > 1$), then $y(t) = O(t^{-\beta})$.*

ii) *If $\varphi(t) = O(\mathrm{e}^{-\lambda t})$ ($\lambda > 0$), then $y(t) = O(\mathrm{e}^{-\delta t})$ where $\delta = \min(C_0, \lambda)$.*

d) *Let $1 < \alpha < \infty$. If $\varphi(t) = O(t^{-\beta})$ ($\beta > 1$), then $y(t) = O(t^{-\delta})$, where $\delta = \min(1/(\alpha - 1), \beta/\alpha, \beta - 1)$.*

Remark 4. Assertion 1, d) and Assertion 1, b) (ii)) can be deduced from a more general result due to Hardy (see [13], Chap. V, Theorem 3, where α, β are integers) via the transformation $u^s = y$ if $\alpha = r/s$ and $z^m = t$ if $\beta = n/m$.

Theorem 4. *Suppose* (3_3), (4_3) *and* $f(t) \to 0$ *in* V^* *for* $t \to \infty$. *Then* $u(t) \to 0$ *in* H *for* $t \to \infty$. *Moreover, if* $f(t) \equiv 0$ *and* $1 < p < 2$ *then* $u(t) = 0$ *for* $t \geqq \geqq 2C_1|u_0|^{2-p}/C(2-p)$ *(*C *is from* (3_3) *and* C_1 *is from* (27)*).*

Consequence of Theorem 4. If (3_3) (for $1 < p < 2$) holds then the converse problem

$$u'(t) + A(t)\,u(t) = 0 \qquad 0 < t < T,$$
$$u(T) = 0$$

has many different solutions for sufficiently big T.

In the following theorems we assume that (10) is satisfied and $u(t)$ is a solution of (1).

Theorem 5. *Suppose* (4_3), (13_3) *and* $f(t) \to f_\infty$, $A(t)\,u_\infty \to A_\infty u_\infty$ *in* V^* *for* $t \to \infty$. *Then* $u(t) \to u_\infty$ *in* H *for* $t \to \infty$.

Theorem 6. *Let* $p = 2$ *and let* (13_3) *hold.*

i) *If* $\|f(t) - f_\infty\|_* = O(t^{-\beta})$ *and* $\|A(t)\,u_\infty - A_\infty u_\infty\|_* = O(t^{-\beta})$, *then* $|u(t) - u_\infty|^2 = O(t^{-q\beta})$.

ii) *If* $\|f(t) - f_\infty\|_* = O(e^{-\lambda t})$ *and* $\|A(t)\,u_\infty - A_\infty u_\infty\|_* = O(e^{-\lambda t})$ $(\lambda > 0)$, *then* $|u(t) - u_\infty|^2 = O(e^{-\delta t})$, *where* $\delta = \min(C_2, \lambda)$ *and* C_2 *is from* (28).

Theorem 7. *Let* $p > 2$ *and let* (13_3) *hold. If* $\|f(t) - f_\infty\|_* = O(t^{-\beta})$ and $\|A(t)\,u_\infty - A_\infty u_\infty\|_* = O(t^{-\beta})$, *then* $|u(t) - u_\infty|^2 = O(t^{-\delta})$, *where*

$$\delta = \min\left(\frac{2}{p-2}, \frac{2q}{p}, q\beta - 1\right).$$

Proof of Theorems 4–7. From (17) we deduce the estimate

$$\frac{d}{dt}|u(t) - u_\infty|^2 + \left(C - \frac{2\varepsilon^p}{p}\right)\|u(t) - u_\infty\|^p \leqq \tag{26}$$
$$\leqq \frac{\varepsilon^{-q}}{q}\left(\|f(t) - f_\infty\|_*^q + \|A(t)\,u_\infty - A_\infty u_\infty\|_*^q\right)$$

for a.e. $t > 0$, since $|u(t)|^2$ is an absolutely continuous function in t and

$$\frac{d}{dt}|u(t)|^2 = 2(u'(t), u(t))$$

holds for a.e. $t > 0$. Due to the imbedding $V \subset H$ we have

$$|v| \leqq C_1 \|v\| \quad \text{for all} \quad v \in V \tag{27}$$

and hence from (20) for a suitable $\varepsilon > 0$ we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t} |u(t) - u_\infty|^2 \leqq -C_2(|u(t) - u_\infty|^2)^{p/2} + \tag{28}$$
$$+ C_3(\|f(t) - f_\infty\|_*^q + \|A(t)\, u_\infty - A_\infty u_\infty\|_*^q),$$

where $C_2 = C_2(C, C_1, \varepsilon)$. In the case of Theorem 4 we obtain the estimate

$$\frac{\mathrm{d}}{\mathrm{d}t} |u(t)|^2 \leqq -C_2(|u(t)|^2)^{p/2} + C_3\|f(t)\|_*^q .$$

Thus, putting $z(t) = |u(t) - u_\infty|^2$, $\alpha = \frac{1}{2}p$ and

$$\varphi(t) = C_3(\|f(t) - f_\infty\|_*^q + \|A(t)\, u_\infty - A_\infty u_\infty\|_*^q)$$

we obtain the differential inequality

$$z'(t) \leqq -C_2\, z(t)^\alpha + \varphi(t) \tag{29}$$

where $z(t) \geqq 0$, $\varphi(t) \geqq 0$ for $t > 0$. Comparing any two solutions $y(t)$ of (25) and $z(t)$ of (29) with $y(0) = z(0) \geqq 0$ we conclude that $z(t) \leqq y(t)$ for all $t > 0$. From this fact and Assertion 1 we successively obtain Theorems 4–7.

Theorem 8. *Let $A(t) \equiv A$ and let the assumptions of Remark 2 be satisfied. If (9_1), (18) and (13_3) hold then the estimate*

$$\|u(t) - u_\infty\| = O(|u(t) - u_\infty|^{1/p} + \|f(t) - f_\infty\|_*^{q/p})$$

takes place.

Proof. From (19) and (13_3) we deduce

$$C\|u(t) - u_\infty\|^p \leqq \left|\frac{\mathrm{d}u(t)}{\mathrm{d}t}\right| |u(t) - u_\infty| + \|f(t) - f_\infty\|_* \|u(t) - u_\infty\| .$$

Hence, using (20) and Young's inequality, we obtain the required result.

Remark 5. In many applications it is more suitable to replace the assumptions $\|A(t)\, u_\infty - A_\infty u_\infty\|_* \to 0$ for $t \to \infty$ and $\|A(t)\, u_\infty - A_\infty u_\infty\|_* = O(\cdot)$ in Theorems 5, 6 and 7 by stronger assumptions

$$\|A(t)\, v - A_\infty v\|_* \to 0 \quad \text{for} \quad t \to \infty \quad \text{for all} \quad v \in V \tag{30}$$

and

$$\|A(t)\, v - A_\infty v\|_* = O(\cdot) \quad \text{for an arbitrary} \quad v \in V, \tag{31}$$

which can be directly verified. Then, in Theorems 5, 6 and 7 it suffices to assume the existence of the solution u_∞ of (10), which is guaranteed by certain properties of A_∞.

3

Let us consider nonlinear parabolic equations of the form

$$\frac{\partial u}{\partial t} + \sum_{|i| \leqq k} (-1)^{|i|} D^i a_i(t, x, Du) = f(t, x) \tag{32}$$

in the domain $Q = \Omega \times (0, \infty)$, where Ω is a bounded domain in E^N (N-dimensional Euclidean space) with a Lipschitzian boundary $\partial\Omega$, $x \in \Omega$, $t > 0$, i is a multiindex and Du is the vector function $Du = (D^i u, |i| \leqq k)$.

The functions $a_i(t, x\ \xi)$ $\xi \in E^d$ ($d = \text{card}\{i, |i| \leqq k\}$) for $|i| \leqq k$ are supposed to be real, defined for $0 \leqq t < \infty$, $x \in \Omega$ and $|\xi| < \infty$, continuous in all the variables (it suffices to assume Caratheodory's conditions).

Let us consider the first initial – boundary value problem

$$u(x, 0) = u_0(x)\,, \quad D_\nu^l u(x, t)|_{\partial\Omega \times (0,T)} = 0 \quad \text{for} \quad l = 0, 1, \ldots, k-1\,, \tag{33}$$

where D_ν^l is the outward normal derivative of order l with respect to $\partial\Omega$.

The functions $a_i(t, x, \xi)$ are supposed to satisfy the growth condition

$$|a_i(t, x, \xi)| \leqq C(1 + |\xi|^{p-1}) \quad \text{for} \quad |i| \leqq k\,, \tag{34}$$

where $1 < p < \infty$. Let W_p^k be the Sobolev space ($W_p^k \equiv \{u \in L_p(\Omega);\ D^i u \in L_p(\Omega)$ for $|i| \leqq k\}$ with the norm $\|\cdot\|_W = \sum_{|i| \leqq k} \|D^i u\|_{L_p}$). By the duality form

$$(A(t)\, v, w) = \sum_{|i| \leqq k} \int_\Omega D^i w\, a_i(t, x, Dv)\, \mathrm{d}x \quad \text{for} \quad v, w \in W_p^k$$

we define an (in general nonlinear) operator

$$A(t) : W_p^k \to W_q^{-k} \quad (W_q^{-k} \text{ is the dual space to } W_p^k)\,,$$

which is continuous and bounded because of Nemyckij's theorem;

$$a_{ij}(t, x, \xi) = \frac{\partial a_i(t, x, \xi)}{\partial \xi_j} \quad (|i|, |j| \leqq k)\,.$$

Remark 6. Monotonicity and coerciveness of $A(t)$ is guaranteed by

$$\sum_{|i| \leqq k} [a_i(t, x, \xi) - a_i(t, x, \eta)]\,(\xi_i - \eta_i) \geqq 0\,, \tag{35}$$

$$\sum_{|i| \leqq k} a_i(t, x, \xi)\, \xi_i \geqq C_1 |\xi|^p - C_2\,. \tag{36}$$

Remark 7. Let $p \geqq 2$. If the estimate

$$(37) \qquad \sum_{|i|,|j| \leqq k} a_{ij}(t, x, \xi)\, \eta_i \eta_j \geqq C \sum_{|i|=k} |\xi_i|^{p-2}\, \eta_i^2$$

holds for all $\xi, \eta \in E^d$ and $t > 0$, then $A(t)$ satisfies (13_3) – see [12].

Remark 8. Let $p \geqq 2$ and $a_i(t, x, \xi) = g_i(t, x)\, |\xi_i|^{p-2}\, \xi_i$ ($|i| \leqq k$), where $g_i(t, x) \in C(Q) \cap L_\infty(Q)$ ($|i| \leqq k$). If

$$(38) \qquad g_i(t, x) \geqq C > 0 \quad \text{for all} \quad |i| = k\,, \quad g_i(t, x) \geqq 0 \quad \text{for all} \quad |i| < k$$

then we can verify by elementary computation that the operator $A(t)$ generated by $a_i(t, x, \xi)$ ($|i| \leqq k$) satisfies (13_3).

Now, let $A(t)$, A be generated by $a_i(t, x, \xi)$, $a_i(x, \xi)$ ($|i| \leqq k$), respectively.

Assertion 2. *Let $a_i(t, x, \xi)$, $a_i(x, \xi)$ satisfy (34). If $a_i(t, x, \xi) \to a_i(x, \xi)$ with $t \to \infty$ for all fixed $|i| \leqq k$, $x \in \Omega$ and $|\xi| < \infty$, then (30) holds with $A = A_\infty$.*

Proof. We have

$$(39) \qquad \|A(t)\, v - Av\|_* = \sup_{\|z\|_W \leqq 1} |(A(t)\, v - Av, z)| \leqq \sum_{|i| \leqq k} \|a_i(t, x, Dv) - a_i(x, Dv)\|_{L_q}\,,$$

where $\|\cdot\|_*$ is the norm in W_q^{-k}. From (34) we deduce the estimate

$$|a_i(t, x, \xi) - a_i(x, \xi)| \leqq C(1 + |\xi|^{p-1}) \quad \text{for} \quad |i| \leqq k\,.$$

Since

$$|a_i(t, x, Dv) - a_i(x, Dv)|^q \leqq C(1 + \sum_{|j| \leqq k} |D^j v|^p)$$

and $a_i(t, x, Dv) \to a_i(x, Dv)$ with $t \to \infty$ for all $x \in \Omega$, Lebesgue's convergence theorem and (39) yield the required result.

Using the estimate (39) we can estimate also the rate of convergence

$$\|A(t)\, v - Av\|_* \to 0 \quad \text{for} \quad t \to \infty\,.$$

If, e.g., $a_i(t, x, \xi) = g_i(t, x)\, a_i(x, \xi)$ ($|i| \leqq k$), where $g_i(t, x)$ are continuous functions for $x \in \bar{\Omega}$, $t \geqq 0$, then we easily deduce

$$\|A(t)\, v - Av\|_* = O(\max_{|i| \leqq k, x \in \bar{\Omega}} |g_i(t, x) - 1|)\,.$$

Now let us consider a nonhomogeneous problem (32), (33'),

$$(33') \qquad u(x, 0) = u_0(x, 0)\,, \quad D_\nu^l\, u(x, t)|_{\partial\Omega \times (0,\infty)} = D_\nu^l\, u_0(x, t)|_{\partial\Omega \times (0,\infty)}\,, \quad l = 0, 1, \ldots, k - 1\,,$$

where $u_0(x, t)$ is a sufficiently smooth function in $\Omega \times (0, \infty)$. Considering u in the form $u = u_0 + z$ we can transform (32) (33′) into a homogeneous problem (32*) (33*):

$$\frac{\partial z}{\partial t} + \sum_{|i| \leqq k} (-1)^{|i|} D^i a_i^*(t, x, Dz) = f^*(x, t), \tag{32*}$$

$$z(x, 0) = 0, \quad D_\nu^l z(x, t)|_{\partial\Omega \times (0,\infty)} = 0, \quad l = 0, 1, \ldots, k-1, \tag{33*}$$

where $a_i^*(t, x, Dz) = a_i(t, x, Du_0 + Dz)$ $(|i| \leqq k)$, $f^*(x, t) = f(x, t) - \partial u_0/\partial t$.

By means of $a_i(t, x, \xi)$ $(|i| \leqq k)$ we define the operator $A^*(t)$. If $a_i(t, x, \xi)$ satisfy (34), (35), (36), (37), respectively, then $A^*(t)$ has the corresponding properties as $A(t)$ – see Remarks 6, 7 and 8.

Let $u_0(x)$, $u_0(x, t)$ $W_\infty^k(\Omega)$ (for all $t > 0$). We shall assume

$$u_0(x, t) \to u_0(x) \quad \text{in} \quad W_p^k(\Omega) \quad \text{for} \quad t \to \infty \tag{40}$$

and

$$\|u_0(x, t)\|_{W_\infty{}^k(\Omega)} \leqq C \quad \text{for all} \quad t > 0. \tag{41}$$

By means of $a_i^*(x, \xi)$ $(a_i^*(x, Dz) = a_i(x, Du_0 + Dz))$ $(|i| \leqq k)$ let us define a stationary operator A^*.

Assertion 3. *Suppose $a_i(t, x, \xi)$ and $a_i(x, \xi)$ $(|i| \leqq k)$ satisfy (34) and*

$$a_i(t, x, \xi) \to a_i(x, \xi) \quad \textit{with} \quad t \to \infty \tag{42}$$

for all fixed $x \in \Omega$ uniformly for ξ from a bounded set in E^d.

If (34), (40) and (41) are satisfied then (30) holds with $A^*(t)$ and A^*.

Proof. Analogously as in the proof of Assertion 2 we have

$$\|A^*(t)\, v - A^* v\|_* \leqq \tag{43}$$
$$\leqq \sum_{|i| \leqq k} \|a_i(t, x, Du_0(x, t) + Dv) - a_i(x, Du_0(x) + Dv)\|_{L_q}$$

and

$$|a_i(t, x, Du_0(x, t) + Dv) - a_i(x, Du_0(x, t) + Dv)|^q \leqq C(1 + \sum_{|i| \leqq k} |D^i v|^p)$$

because of (34) and (41). Thus, from (41), (42) and Lebesgue's convergence theorem we conclude

$$(a_i(t, x, Du_0(x, t) + Dv) - a_i(x, Du_0(x, t) + Dv)) \to 0 \tag{44}$$

with $t \to \infty$ in $L_q(\Omega)$ for all $v \in W_p^k$.

Due to the theorem of Nemyckij (see [14]) and (40) we have

$$a_i(x, Du_0(x, t) + Dv) \to a_i(x, Du_0(x) + Dv) \quad \text{with} \quad t \to \infty \quad \text{in} \quad L_q(\Omega) \tag{45}$$

for all $v \in W_p^k$. The inequality

$$\left|a_i(t, x, Du_0(x, t) + Dv) - a_i(x, Du_0(x) + Dv)\right| \leqq$$
$$\leqq \left|a_i(t, x, Du_0(x, t) + Dv) - a_i(x, Du_0(x, t) + Dv)\right| +$$
$$+ \left|a_i(x, Du_0(x, t) + Dv) - a_i(x, Du_0(x) + Dv)\right|$$

together with (44), (45) and (43) implies the required result.

Remark 9. If $a_i(t, x, \xi) \equiv a_i(x, \xi)$ ($|i| \leqq k$) then Assertion 3 holds true if we assume $u_0(x), u_0(x, t) \in W_p^k(\Omega)$ (for all $t > 0$) and $u_0(x, t) \to u_0(x)$ with $t \to \infty$ in W_p^k instead of (40), (41).

Investigating (31) we can easily prove

Assertion 4. *Let $a_i(t, x, \xi) = g_i(x, t)\, a_i(x, \xi)$ $|i| \leqq k$), where $g_i(x, t)$ ($|i| \leqq k$) are continuous functions in $\bar{\Omega} \times (0, \infty)$. Suppose $u_0(x), u_0(x, t) \in W_p^k(\Omega)$ and $\|u_0(x, t)\|_W \leqq C$ for all $t > 0$. If $a_i(x, \xi)$ satisfy (34) and $a_{ij}(x, \xi)$ satisfy*

$$(46) \qquad \left|a_{ij}(x, \xi)\right| \leqq C(1 + |\xi|^{p-2}) \quad \text{where} \quad p \geqq 2\,,$$

then the estimate

$$\|A^*(t)\, v - A^* v\|_* = O\big(\max_{|i| \leqq k, x \in \Omega} |g_i(x, t) - 1| +$$
$$+ \|v\|_W^{p-2}\, \|u_0(x, t) - u_0(x)\|_W + \|u_0(x, t) - u_0(x)\|_W^{p-1}\big)$$

takes place.

For the proof we use (43), the formula

$$a_i(x, D\, u_0(x, t) + Dv) - a_i(x, D\, u_0(x) + Dv) =$$
$$= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\, a_i(x, D\, u_0(x) + s\, D(u_0(x, t) - u_0(x)))\, \mathrm{d}s\,,$$

the stimate (46) and Hölder's inequality.

Remark 10. Let $p \geqq 2$. If $a_i(t, x, \xi)$ satisfy (34), (37), $u_0(x, t) \in W_p^k$ (for all $t > 0$) and $\partial u_0(x, t)/\partial t$, $f(x, t) \in L_q(0, T; W_q^{-k})$ (for all $T < \infty$) then there exists a unique solution $u(x, t)$ of (32), (33′) – see Remark 1 ((37) implies c_1) and d_1)). If $u_0(x) \in W_p^k$, $f(x) \in W_q^{-k}$ and $a_i(x, \xi)$ ($|i| \leqq k$) satisfy (34)–(36) then there exists a solution $u(x)$ of the stationary problem

$$(47) \qquad \sum_{|i| = k} (-1)^{|i|}\, D^i\, a_i(x, Du) = f(x)\,,$$

$$(48) \qquad D_\nu^l\, u(x)|_{\partial\Omega} = D_\nu^l\, u_0(x)|_{\partial\Omega}\,, \quad l = 0, 1, \ldots, k - 1\,.$$

If in (35) the sign $>$ holds for $\xi \neq \eta$, then the solution $u(x)$ is unique.

Applying certain results of this section and § 2 we obtain

Theorem 10. *Let $u(x, t)$ be a solution of* (32), (33') *and let $u(x)$ be a solution of* (47), (48). *Let us assume* (40), $p \geqq 2$ *and let $a_i(t, x, \xi)$ $(|i| \leqq k)$ satisfy* (34), (37).

i) *Suppose that the assumptions of Assertion* 3 *or Assertion* 4 *are satisfied. If*

$$\frac{\partial u_0(x, t)}{\partial t}, \quad f(x, t) \in W_q^{-k}(\Omega) \quad (\textit{for all } t > 0)$$

and

$$\frac{\partial u_0(x, t)}{\partial t} \to 0, \quad f(x, t) \to f(x) \quad \textit{in} \quad W_q^{-k} \quad \textit{for} \quad t \to \infty,$$

then $u(x, t) \to u(x)$ in $L_2(\Omega)$ for $t \to \infty$.

ii) *Suppose $a_i(t, x, \xi) \equiv a_i(x, \xi)$ $(|i| \leqq k)$, $u_0(x, t) \equiv u_0(x)$ and* d_2) (form Remark 2). *We assume $f \in C^1(\langle 0, \infty), L_2(\Omega))$,*

$$\int_0^\infty \left\| \frac{\partial f(x, t)}{\partial t} \right\|_{L_2} \mathrm{d}t < \infty \quad \textit{and} \quad \int_0^\infty \|f(x, t) - f(x)\|_{L_2} \mathrm{d}t < \infty.$$

If in (35) *the sign $>$ holds for $\xi \neq \eta$ and if $p > pN/(N - kp)$, then $u(x, t) \to u(x)$ in $L_2(\Omega)$ for $t \to \infty$.*

iii) *Suppose that the assumptions of* ii) *are satisfied. If $p \geqq 2$ and if* (37) *holds, then $u(x, t) \to u(x)$ in $W_p^k(\Omega)$ for $t \to \infty$.*

Assertion i) is a consequence of Theorem 5. Theorem 3 implies Assertions ii) and iii).

Applying other results of §§ 1, 2 we can deduce the corresponding results on stabilization of the solution of the initial-boundary value problems (32), (33') and (32), (33), respectively.

The above results can be applied to the following examples.

Example 1. Let $u(x, t)$, $u(x)$ be the solutions of the problems

$$\frac{\partial u}{\partial t} + \sum_{|i|=k} (-1)^{|i|} D^i(g_i(x) |D^i u|^{p-2} D^i u) = 0,$$

$$u(x, 0) = u_0(x), \quad D_\nu^l u(x, t)|_{\partial\Omega} = D_\nu^l u_0(x)|_{\partial\Omega} \quad \text{for} \quad t > 0, \quad l = 0. 1, \ldots, k-1.$$

We assume that $u_0(x) \in W_p^k$ $(p > 1)$ and that $g_i(x) \in C(\bar{\Omega})$ $(|i| \leqq k)$ satisfy (38). If $2 > p \geqq p_0$ then the identity $u(x, t) \equiv 0$ holds for $t \geqq 2 C_1 (C(2 - p) C_2)^{-1}$. $\|u_0(x)\|_{L_2}^{2-p}$. The constants C, C_1 are obtained from (38), (27), respectively, and C_2 is obtained from the inequality $\sum_{|i|=k} \|D^i u\|_{L_p} \geqq C_2 \|u\|_W$ for all $u \in \mathring{W}_p^k(\Omega)$ (equivalence of norms in $\mathring{W}_p^k$).

Example 2. Let $u(x, t)$, $u(x)$ be the solutions of the problems

$$\frac{\partial u}{\partial t} + \sum_{|i| \leqq k} (-1)^{|i|} D^i(g_i(x, t) |D^i u|^{p-2} D^i u) = r(t) f(x),$$

$$u(x, 0) = s(0)\, u_0(x)\,, \quad D_\nu^l\, u(x, t)|_{\partial\Omega \times (0,\infty)} = s(t)\, D\, u_0(x)|_{\partial\Omega}$$
$$\text{for} \quad t > 0\,, \quad l = 0, 1, \ldots, k - 1$$

and

$$\sum_{|i| \leqq k} (-1)^{|i|}\, D^i(g_i(x)\, |D^i u|^{p-2}\, D^i u) = f(x)\,,$$
$$D_\nu^l\, u(x)|_{\partial\Omega} = D_\nu^l\, u_0(x)|_{\partial\Omega} \quad \text{for} \quad l = 0, 1, \ldots, k - 1\,,$$

respectively. Suppose $p \geqq 2$, (38), $f(x) \in W_q^{-k}$, $u_0(x) \in W_p^k$, $g_i(x, t) \in C(Q) \cap L_\infty(Q)$ and $g_i(x) \in C(\bar{\Omega})$ ($|i| \leqq k$).

i) Let $s'(t), r(t) \in L_q(\langle 0, T \rangle)$ for all $T < \infty$.

If $s(t) \to 1$, $s'(t) \to 0$, $r(t) \to 1$ for $t \to \infty$, and if $g_i(x, t) \to g_i(x)$ ($|i| \leqq k$) for $x \in \Omega$ and $t \to \infty$ then $u(x, t) \to u(x)$ in $L_2(\Omega)$ for $t \to \infty$.

ii) Suppose $g_i(x, t) \equiv g_i(x)$ ($|i| \leqq k$), $s(t) \equiv 1$ (stationary case). If

$$\int_0^\infty |r'(t)|\, \mathrm{d}t < \infty \quad \text{and} \quad \int_0^\infty |r(t) - 1|\, \mathrm{d}t < \infty$$

then $u(x, t) \to u(x)$ in the norm of the space W_p^k for $t \to \infty$.

Example 3. Let $u(x, t)$ be the solution of the problem

$$\frac{\partial u}{\partial t} - \Delta u + f(x, u) = 0\,,$$

$$u(x, 0) = u_0(x)\,,$$

a) $u(x, t)|_{\partial\Omega} = 0$ ($t > 0$); b) $(\partial u(x, t)/\partial \nu)|_{\partial\Omega} = 0$ ($t > 0$) and let $u(x)$ be the solution of the stationary problem

$$-\Delta u + f(x, u) = 0$$

a) $u|_{\partial\Omega} = 0$; b) $\partial u/\partial \nu|_{\partial\Omega} = 0$.

Let $f(x, s)$ be a continuous function in all its variables. Assume

$$(f(x, \xi) - f(x, \eta))\,(\xi - \eta) > 0 \quad \text{for all} \quad \xi, \eta \in E^1\,, \quad \xi \neq \eta$$

and

$$C_1 |s| \leqq s\, f(x, s) \leqq C_2(1 + |s|^r) \quad \text{for} \quad |s| < \infty\,,$$

where $r \leqq 2N(N - 1)^{-1}$ for $N > 1$ and r is arbitrary for $N = 1$. Then in the case of the boundary conditions a) or b) we have $u(x, t) \to u(x)$ in $W_2^1(\Omega)$ for $t \to \infty$.

References

[1] *J. L. Lions:* Quelques méthodes de résolution des problémes aux limites non linéaires, Dunod Gauthier-Villars, Paris, 1969.

[2] *F. E. Browder:* Non-linear initial value problems, Ann. of Math., *82*, (1965), 51—87.

[3] *F. E. Browder:* Existence theorems for nonlinear partial differential equations, Global analysis, Proc. Symp. Pure Math. Vol. *16*, Amer. Math. Soc. (1970), 1—60.

[4] *H. Brézis:* Monotonicity methods in Hilbert spaces and some applications to nonlinear partial differential equations, Contributions to Nonlinear Functional Analysis, E. Zarantonello ed. Acad. Press (1971).

[5] *V. Barbu:* Semi-groups of nonlinear contractions in Banach spaces, (Rumanian), Academia, Bucharest, 1974.

[6] *H. Brézis:* Opérateurs maximaux monotones et semigroups de contractions dans les espaces de Hilbert, Math. Studies, *5*, North Holland, (1973).

[7] *H. Gajewski, K. Gröger, K. Zacharias:* Theorie der nichtlinearen Operatoren und Differentialgleichungen, Academia-Verlag, Berlin, 1974.

[8] *J. Kačur:* Method of Rothe and nonlinear parabolic boundary value problems of arbitrary order I, II, Czechoslovak Math. J., to appear.

[9] *J. Nečas:* Application of Rothe's method to abstract parabolic equations, Czechoslovak Math. J.,

[10] *J. Kačur:* Applications of Rothe's method to nonlinear evolution equations, Mat. Časopis Sloven. Akad. Vied, *25*, (1975), N-1, 63—81.

[11] *C. V. Pao:* On the stability of nonlinear operator differential equations and applications, Arch. Rational Mech. Anal. *35*, (1969), 30—46.

[12] *J. A. Dubinskij:* Quasilinear elliptic and parabolic equations of arbitrary order (Russian), Uspehi Mat. Nauk, *23*, 1 (1968), 45—90.

[13] *R. Bellman:* Stability theory of differential equations, New York—Toronto—London, 1953.

[14] *M.A. Krasnoselskij:* Topological methods in the theory of nonlinear integral equations, Pergamon Press, New York, 1964.

[15] *K. Yosida:* Functional Analysis, Springer, Berlin—Heidelberg—New York, 1969.

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