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SCHREIER-ZASSENHAUS THEOREM FOR ALGEBRAS I

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The aim of the present paper is a generalization of Schreier-Zassenhaus theorem for algebras. It is a theme already studied before. GOLDIE [4] discussed a certain kind of generalization of Jordan-Hölder theorem for algebras, which is a special case of the Schreier-Zassenhaus theorem and, moreover, under certain restrictive conditions (the descending and ascending conditions for subalgebras). Another interpretation of this problem was offered in [7] (ORE). The papers of BORŮVKA [1] and CHÂTELET [2] (the latter in the formulation given in [8] Th. 88) served us as a model, namely Theorem 10.8 [1] for the case of partitions of a set (without operations) and Theorems 17.6 [1] and 88 [8] for congruences in algebras. Unfortunately, the theorems mentioned work with too strong assumptions that are not fulfilled in the group case unless invariant series are treated. Our purpose is to give such a theorem for algebras that is applicable in classical structures (in Ω-groups).

We shall call the main attention to the case of sets without operations. In this situation the notion of isomorphism is reduced to that of the set theoretical equivalence. It would be useful (if possible at all) to replace it by another stronger notion. Borůvka (see the book [1] which includes his theory of partitions developed during the World-War II, cf. [2]) has found such a notion, namely that of coupled partitions in a set. Thus, he discovered the set theoretical character of algebraical constructions connected with the Schreier-Zassenhaus theorem.

The notion of coupled partitions is also the central notion used in the present paper. The main result is Theorem 3.5 which is based on Theorems 1.10 and 3.4 (sets) and 2.7 (algebras). Corollary 3.7 gives the group interpretation of Theorem 3.5, the Schreier-Zassenhaus theorem. Another group application of the theory is presented by Theorem 2.5 and by its Corollary 2.6, the general four-group theorem [1] 23.2.

Let us introduce some fundamental notions necessary in the present paper. The reader will find information in greater detail e.g. in [1], [5], [8], [9].

A *partition in* a set $\mathfrak{G}$ is a system of nonempty pairwise disjoint subsets of $\mathfrak{G}$. The system of all partitions in $\mathfrak{G}$ is clearly in a one to one correspondence with the system of all symmetric and transitive binary relations in $\mathfrak{G}$. For this reason we often shall not distinguish between the both notions. The set of all partitions in $\mathfrak{G}$, $P(\mathfrak{G})$, is a complete lattice with regard to the set inclusion; symbols of the operations are $\wedge, \vee, \bigwedge, \bigvee$ (or $\vee = \vee_P$ if necessary). Stable partitions (stable as binary relations) in an algebra $(\mathfrak{G}, \Omega)$ are called *congruences in* $(\mathfrak{G}, \Omega)$. The set of all congruences in $(\mathfrak{G}, \Omega)$, $\mathscr{K}(\mathfrak{G}, \Omega)$, is a complete lattice with regard to the set inclusion; its operations are denoted by $\wedge_{\mathscr{K}}, \vee_{\mathscr{K}}, \bigwedge_{\mathscr{K}}, \bigvee_{\mathscr{K}}$. It holds $\bigwedge_{\mathscr{K}} = \bigwedge_P = \bigcap$ [6] I, 1.1. If A is a binary relation in a set $\mathfrak{G}$, $x \in \mathfrak{G}$ and $\emptyset \neq \mathfrak{B} \subseteq \mathfrak{G}$, we define $A(x) = \{y \in \mathfrak{G} : yAx\}$, $A(\mathfrak{B}) = \bigcup\{A(x) : x \in \mathfrak{B}\}$ and $\bigcup A = \bigcup\{A(x) : x \in \mathfrak{G}\}$. If A is a partition and $A(x) \neq \emptyset$, we call the set $A(x)$ a *block* of the partition A and $\bigcup A$ the *domain* of A [6]. If $\bigcup A = \mathfrak{G}$, we speak about a *partition on* $\mathfrak{G}$ or about a *partition of* $\mathfrak{G}$. If $\emptyset \neq \mathfrak{B} \subseteq \mathfrak{G}$, $\{\mathfrak{B}\}$ means the partition in $\mathfrak{G}$ with the unique block $\mathfrak{B}$, and if A is a binary relation in $\mathfrak{G}$, $\mathfrak{B} \sqcap A$ means the binary relation $(\mathfrak{B} \times \mathfrak{B}) \cap A$. It is called the *intersection* of the relation A with the set $\mathfrak{B}$. Particularly, if A is a partition then $\mathfrak{B} \sqcap A = \{\mathfrak{B}\} \wedge A = \{A^1 \cap \mathfrak{B} : A^1 \in A, A^1 \cap \mathfrak{B} \neq \emptyset\}$ [1] 2.3. Two partitions in $\mathfrak{G}$ are said to be *coupled* if each block of one partition meets exactly one block of the other partition [1] 4.1. The product AB of two partitions A and B in $\mathfrak{G}$ means the product of the corresponding binary relations. Binary relations (or partitions) A and B in $\mathfrak{G}$ are called *commuting* if $AB = BA$. The domain of a congruence in an algebra $(\mathfrak{G}, \Omega)$ is a subalgebra of $(\mathfrak{G}, \Omega)$; if $(\mathfrak{G}, \Omega)$ is an Ω-group, $\emptyset \neq A \in \mathscr{K}(\mathfrak{G}, \Omega)$ then $A(0)$ is an ideal of the Ω-subgroup $\bigcup A$ and $A = \bigcup A/A(0)$ [6] I, 1.4.

1.

We say that a subset $\emptyset \neq \mathfrak{B} \subseteq \mathfrak{G}$ *respects* a partition A in the set $\mathfrak{G}$ if it holds: $A^1 \in A$, $A^1 \cap \mathfrak{B} \neq \emptyset \Rightarrow A^1 \subseteq \mathfrak{B}$ ([6] IV, 4.8).

Lemma 1.1. *Let A, B be partitions in a set $\mathfrak{G}$, $\emptyset \neq \mathfrak{B} \subseteq \mathfrak{G}$. Then $\mathfrak{B} \sqcap (A \vee B) \supseteq (\mathfrak{B} \sqcap A) \vee (\mathfrak{B} \sqcap B)$. The equality follows if $\mathfrak{B}$ respects the partitions A and B or if $\mathfrak{B} \supseteq \bigcup A \cap \bigcup B$. An analogous theorem holds for the product.*

Proof. Let $x, y \in G$. For suitable elements $x_1, \ldots, x_n \in \mathfrak{G}$ $(n \geq 0)$ and for partitions $C_1, \ldots, C_{n+1}$ by turns equal to A or B it holds

$$x\mathfrak{B} \sqcap (A \vee B)\, y \Leftrightarrow x, y \in \mathfrak{B}\,, \quad xC_1x_1C_2x_2 \ldots x_nC_{n+1}y \Leftrightarrow x \in \mathfrak{B} \cap \bigcup C_1\,,$$
$$y \in \mathfrak{B} \cap \bigcup C_{n+1}\,, \quad x_1, \ldots, x_n \in \bigcup A \cap \bigcup B\,, \quad xC_1x_1 \ldots x_nC_{n+1}y \Leftarrow x \in \mathfrak{B} \cap \bigcup C_1\,,$$
$$y \in \mathfrak{B} \cap \bigcup C_{n+1}\,, \quad x_1, \ldots, x_n \in \mathfrak{B} \cap \bigcup A \cap \bigcup B\,, \quad xC_1x_1 \ldots x_nC_{n+1}y \Leftrightarrow$$
$$\Leftrightarrow x(\mathfrak{B} \sqcap C_1)\, x_1(\mathfrak{B} \sqcap C_2)\, x_2 \ldots x_n(\mathfrak{B} \sqcap C_{n+1})\, y \Leftrightarrow x(\mathfrak{B} \sqcap A) \vee (\mathfrak{B} \sqcap B)\, y\,.$$

The middle implication $\Leftarrow$ can be evidently replaced by the both-sided implication $\Leftrightarrow$ if $\mathfrak{B}$ respects both the partitions A and B or if $\mathfrak{B} \supseteq \bigcup A \cap \bigcup B$. The assertion about the product follows from the preceding if we put $n = 1$, $C_1 = A$ and $C_2 = B$.

Remark. The condition in 1.1 for equality is not necessary (even for partitions A and B on $\mathfrak{G}$). Example: $A = B = \{\mathfrak{G}\}$, $\emptyset \neq \mathfrak{B} \subseteq \mathfrak{G}$, $\mathfrak{B} \neq \mathfrak{G}$.

Definition 1.2. Let $B_0 \leqq B$ and $C_0 \leqq C$ be partitions in a set $\mathfrak{G}$, $e \in \bigcup B_0 \cap \bigcup C_0$, $\bigcup B_0 = B(e)$, $\bigcup C_0 = C(e)$. We define $B_{11} = B_0 \vee (B \wedge C)$, $B_{10} = B_0 \vee (B \wedge C_0)$, $\bar{B}_{11} = B_0(B \wedge C)$, $K = B_{11}(e) \sqcap B_{10}$; relations $C_{11}, C_{10}, \bar{C}_{11}$ and L are defined symmetrically with regard to the symbols B and C. Further, let us denote $M = (\mathfrak{A} \sqcap B_0) \vee (\mathfrak{A} \sqcap C_0)$, where $\mathfrak{A} = \bigcup B_0 \cap \bigcup C_0$.

Lemma 1.3. *The relations defined in* 1.2 *have the following properties* (1) *to* (7). *Similar properties can be obtained after the permutation of B and C.*

(1) $\bar{B}_{11}(e) = \bigcup\{B_0(a) : a \in \mathfrak{A}\}$,

(2) $B_{11}(a) = B_{11}(e)$ and $\bar{B}_{11}(a) = \bar{B}_{11}(e)$ for each $a \in \mathfrak{A}$,

(3) $\mathfrak{A} \subseteq \bar{B}_{11}(e) \subseteq B_{11}(e) \subseteq \bigcup B_0$,

(4) $B_{11}(e) \cap C_{11}(e) = \bigcup K \cap \bigcup L = \mathfrak{A}$,

(5) $B_{11}(e) \sqcap (B \wedge C_0) = B_{11}(e) \sqcap C_0$,

(6) $K = (B_{11}(e) \sqcap B_0) \vee (B_{11}(e) \sqcap C_0)$,

(7) $M = \mathfrak{A} \sqcap (B_0 \vee C_0)$, $\bigcup M = \mathfrak{A}$. *If B_0 and C_0 are congruences in an algebra, then so is M.*

Proof. (1) is evident.

(2) The first assertion. It suffices to prove $B_{11}(a) \subseteq B_{11}(b)$ for all $a, b \in \mathfrak{A}$. Let $xB_{11}a$, i.e. $xA_1x_1 \ldots x_{n-1}A_na$, where A_i are equal to B_0 or $B \wedge C$. Since $a, b \in \bigcup B_0 \cap \bigcup C_0$ it holds $b \in B(a) \cap C(a) = (B \wedge C)(a)$, thus $xA_1x_1 \ldots x_{n-1}A_na(B \wedge C)\, b$, i.e. $xB_{11}b$.

Proof of the second assertion follows from the preceding if we put $n = 2$, $A_1 = B_0$ and $A_2 = B \wedge C$.

(3) The first inclusion follows from (1). The second inclusion is evident. The last inclusion: If $x \in B_{11}(e)$ then $xA_1x_1 \ldots x_{n-1}A_ne$, where $A_i \leqq B$ for all i, thus $xBx_1 \ldots \ldots e$, xBe, $x \in B(e) = \bigcup B_0$.

(4) It holds $\bigcup K \subseteq B_{11}(e) \subseteq \bigcup B_0$, $\bigcup L \subseteq C_{11}(e) \subseteq \bigcup C_0$, hence $\bigcup K \cap \bigcup L \subseteq B_{11}(e) \cap C_{11}(e) \subseteq \bigcup B_0 \cap \bigcup C_0 = \mathfrak{A}$. On the other hand, $\bigcup K = B_{11}(e) \cap \bigcup B_{10} \supseteq \mathfrak{A} \cap [\bigcup B_0 \cup (\bigcup B \cap \bigcup C_0)] = \mathfrak{A}$. Analogously $\bigcup L \supseteq \mathfrak{A}$. Thus (4) is proved.

(5) Since $B_{11}(e) \subseteq \bigcup B_0 = B(e)$ it holds $B_{11}(e) \sqcap (B \wedge C_0) = (B_{11}(e) \sqcap B) \wedge (B_{11}(e) \sqcap C_0) = \{B_{11}(e)\} \wedge (B_{11}(e) \sqcap C_0) = B_{11}(e) \sqcap C_0$.

(6) By 1.1, (3) and (5) it is $K = B_{11}(e) \sqcap B_{10} = B_{11}(e) \sqcap [B_0 \vee (B \wedge C_0)] = [B_{11}(e) \sqcap B_0] \vee [B_{11}(e) \sqcap (B \wedge C_0)] = (B_{11}(e) \sqcap B_0) \vee (B_{11}(e) \sqcap C_0)$.

(7) By 1.1 it holds $\mathfrak{A} \sqcap (B_0 \vee C_0) = (\mathfrak{A} \sqcap B_0) \vee (\mathfrak{A} \sqcap C_0)$ and then $\bigcup M = \bigcup(\mathfrak{A} \sqcap B_0) \cup \bigcup(\mathfrak{A} \sqcap C_0) = \mathfrak{A}$. The assertion concerning congruences follows from the fact that $\vee_P \asymp \vee_{\mathscr{X}}$ for congruences "on" (on the subalgebra $\bigcup B_0 \cap \bigcup C_0 = \mathfrak{A}$).

Lemma 1.4. (Borůvka [1] 4.1.) *Partitions A and D in a set G are coupled if and only if*

(a) $\bigcup D \sqcap A = \bigcup A \sqcap D$,

(b) *every block of the partition A meets $\bigcup D$ (or equivalently $\bigcup A \cap \bigcup D$) and symmetrically.*

Evidently, (a) is equivalent to

(a′) $(\bigcup A \cap \bigcup D) \sqcap A = (\bigcup A \cap \bigcup D) \sqcap D$.

Lemma 1.5. $\mathfrak{A} \sqcap K = \mathfrak{A} \sqcap L = M = K \wedge L$.

Proof. By 1.3, $\bigcup K \cap \bigcup L = \mathfrak{A}$. By 1.3 and 1.1, it follows $\mathfrak{A} \sqcap K = \mathfrak{A} \sqcap [(B_{11}(e) \sqcap B_0) \vee (B_{11}(e) \sqcap C_0)] = (\mathfrak{A} \sqcap B_0) \vee (\mathfrak{A} \sqcap C_0) = M$. A symmetrical statement holds for L, consequently $\mathfrak{A} \sqcap K = M = \mathfrak{A} \sqcap L$. We shall prove $K \wedge L = M$. Since $\bigcup(K \wedge L) = \bigcup K \cap \bigcup L = \mathfrak{A}$ it holds $K \wedge L = \mathfrak{A} \sqcap (K \wedge L) = (\mathfrak{A} \sqcap K) \wedge (\mathfrak{A} \sqcap L) = M$.

Proposition 1.6. *The partitions K and L are coupled if and only if $B_{11}(e) = \bar{B}_{11}(e)$ and $C_{11}(e) = \bar{C}_{11}(e)$.*

Proof. Let K and L be coupled. By 1.4 each block of the partition K intersects the set $\bigcup K \cap \bigcup L$ which is equal to $\mathfrak{A}$ (see 1.3). We shall show that each block of the partition $B_{11}(e) \sqcap B_0$ meets $\mathfrak{A}$. Let $K^1 \in K = (B_{11}(e) \sqcap B_0) \vee (B_{11}(e) \sqcap C_0)$ and let K^1 contain no block of the partition $B_{11}(e) \sqcap C_0$. Then K^1 is equal to a block of the partition $B_{11}(e) \sqcap B_0$ and, as we know, K^1 meets $\mathfrak{A}$. If K^1 contains blocks of the both partitions then every block $D^1 \in B_{11}(e) \sqcap B_0$ with $D^1 \subseteq K^1$ meets a block $E^1 \in B_{11}(e) \sqcap C_0$. It is $E^1 \subseteq B_{11}(e) \cap \bigcup C_0 \subseteq \bigcup B_0 \cap \bigcup C_0 = \mathfrak{A}$, therefore D^1 meets $\mathfrak{A}$. We have proved that each block of $B_{11}(e) \sqcap B_0$ meets $\mathfrak{A}$. Moreover, by 1.3, $B_{11}(e) \subseteq \bigcup B_0$. It follows $B_{11}(e) \subseteq \bigcup\{B_0(a) : a \in \mathfrak{A}\} = \bar{B}_{11}(e)$ (by 1.3). The converse inclusion being evident we conclude $B_{11}(e) = \bar{B}_{11}(e)$. Analogously, we have $C_{11}(e) = \bar{C}_{11}(e)$.

Conversely, let the condition be fulfilled. Then, by 1.3, $B_{11}(e) = \bar{B}_{11}(e) = \bigcup\{B_0(a) : a \in \mathfrak{A}\}$. Consequently, each block of $B_{11}(e) \sqcap B_0$ meets $\mathfrak{A}$ and thus it meets $C_{11}(e) = \bigcup L$ because $C_{11}(e) \supseteq \mathfrak{A}$ (by 1.3). Since each block of $B_{11}(e) \sqcap C_0$ meets $C_{11}(e)$ (is contained in $C_{11}(e)$), each block of the partition $K = (B_{11}(e) \sqcap B_0) \vee (B_{11}(e) \sqcap C_0)$ meets $C_{11}(e)$. Symmetrically, each block of the partition L meets $B_{11}(e)$. Thus the condition 1.4 (b) is verified. The condition 1.4 (a′) is fulfilled in virtue of 1.5. It follows that K and L are coupled.

Lemma 1.7. *Denote* $D = (\bigcup C \sqcap B_0)(\bigcup B_0 \sqcap C)$. *Then* $D(a) = D(e)$ *for each* $a \in \bigcup B_0 \cap \bigcup C_0$.

Proof. It suffices to show $D(a) \subseteq D(b)$ for all $a, b \in \bigcup B_0 \cap \bigcup C_0$. Let $x(\bigcup C \sqcap B_0) . (\bigcup B_0 \sqcap C)\, a$. Since $a, b \in \bigcup B_0 \cap \bigcup C_0 = \bigcup B_0 \cap C(e)$, it is $a \in \bigcup B_0 \cap C(b) = (\bigcup B_0 \sqcap C)(b)$. It follows $x(\bigcup C \sqcap B_0)(\bigcup B_0 \sqcap C)\, a(\bigcup B_0 \sqcap C)\, b$, i.e. $x(\bigcup C \sqcap B_0)(\bigcup B_0 \sqcap C)\, b$.

Proposition 1.8. $B_{11}(e) = \bar{B}_{11}(e)$ *if and only if* $(\bigcup C \sqcap B_0)(\bigcup B_0 \sqcap C)(e) \supseteq (\bigcup B_0 \sqcap C)(\bigcup C \sqcap B_0)(\bigcup B_0 \cap \bigcup C_0)$.

Proof. Let the condition be fulfilled. It suffices to prove $B_{11}(e) \subseteq \bar{B}_{11}(e)$. Let $xB_{11}e$, i.e. $xA_1x_1 \ldots x_{n-1}A_ne$, where A_i are alternately equal to B_0 or $B \wedge C$. Since eB_0e and $e(B \wedge C)\, e$ it can be supposed $n > 4$ and $A_n = B \wedge C$. Moreover, since $x \in \bigcup B_0$ (by 1.3), it can be supposed $A_1 = B_0$. Thus we have $xB_0x_1 \ldots x_{n-3}(B \wedge C)\, x_{n-2}B_0x_{n-1}(B \wedge C)\, e$. Now, there is $x_i \in \bigcup B_0 \cap \bigcup C$ $(i = 1, \ldots, n-1)$ and consequently $xB_0x_1 \ldots x_{n-3}(\bigcup B_0 \sqcap C)\, x_{n-2}(\bigcup C \sqcap B_0)\, x_{n-1}(\bigcup B_0 \sqcap C)\, e$. Evidently $x_{n-1} \in \bigcup B_0 \cap \bigcup C_0$. By supposition, it follows $xB_0x_1 \ldots x_{n-4}(\bigcup C \sqcap B_0)\, x_{n-3}(\bigcup C \sqcap B_0)(\bigcup B_0 \sqcap C)\, e(\bigcup B_0 \sqcap C)\, e$ and therefore $xB_0x_1 \ldots x_{n-4}(\bigcup C \sqcap B_0)\, y_{n-2}(\bigcup B_0 \sqcap C)\, e$ for some $y_{n-2} \in \bigcup B_0 \cap C(e) = \bigcup B_0 \cap \bigcup C_0$. By induction, we obtain $xB_0y(\bigcup B_0 \sqcap C)\, e$ for some $y \in \bigcup B_0 \cap \bigcup C_0$. Then $xB_0(B \wedge C)\, e$, i.e. $x \in \bar{B}_{11}(e)$.

Conversely, let $\bar{B}_{11}(e) = B_{11}(e)$ and $a \in \bigcup B_0 \cap \bigcup C_0$. Then $\bar{B}_{11}(a) = B_{11}(a)$ (by 1.3). It holds $x(\bigcup B_0 \sqcap C)(\bigcup C \sqcap B_0)\, a \Rightarrow x(B \wedge C)\, B_0a \Rightarrow xB_{11}a \Rightarrow x\bar{B}_{11}a \Rightarrow xB_0b(B \wedge C)\, a$ for some $b \in \bigcup B_0 \cap \bigcup C$. Since $x \in \bigcup C$ and $a \in \bigcup B_0$, it follows that $x(\bigcup C \sqcap B_0)(\bigcup B_0 \sqcap C)\, a$. By 1.7, $x(\bigcup C \sqcap B_0)(\bigcup B_0 \sqcap C)\, e$.

Lemma 1.9.

$$(\bigcup C \sqcap B_0)(\bigcup B_0 \sqcap C)(\bigcup B_0 \cap \bigcup C_0) \subseteq (\bigcup B_0 \sqcap C)(\bigcup C \sqcap B_0)(\bigcup B_0 \cap \bigcup C_0).$$

Proof. It holds $\mathfrak{B} := (\bigcup C \sqcap B_0)(\bigcup B_0 \sqcap C)(e) = \bigcup\{(\bigcup C \sqcap B_0)(a) : a \in \bigcup B_0 \cap \bigcup C_0\}$. Hence $\mathfrak{D} := \bigcup\{(\bigcup B_0 \sqcap C)(\bigcup C \sqcap B_0)(a) : a \in \bigcup B_0 \cap \bigcup C_0\} = \bigcup\{(\bigcup B_0 \sqcap C)(b) : b \in \mathfrak{B}\}$. As the domain of the partition $\bigcup B_0 \sqcap C$, $\bigcup B_0 \cap \bigcup C$, contains $\mathfrak{B}$, we have $\mathfrak{D} \supseteq \mathfrak{B}$. Now, we apply 1.7.

Theorem 1.10. *The following conditions are equivalent*:

(1) *The partitions K and L are coupled.*

(2) $B_{11}(e) = \bar{B}_{11}(e)$ *and* $C_{11}(e) = \bar{C}_{11}(e)$.

(3) $(\bigcup C \sqcap B_0)(\bigcup B_0 \sqcap C)(e) \supseteq (\bigcup B_0 \sqcap C)(\bigcup C \sqcap B_0)(\bigcup B_0 \cap \bigcup C_0)$ *and* $(\bigcup B \sqcap C_0)(\bigcup C_0 \sqcap B)(e) \supseteq (\bigcup C_0 \sqcap B)(\bigcup B \sqcap C_0)(\bigcup B_0 \cap \bigcup C_0)$.

(4) $(\bigcup C \sqcap B_0)(\bigcup B_0 \sqcap C)(e) = (\bigcup B_0 \sqcap C)(\bigcup C \sqcap B_0)(\bigcup B_0 \cap \bigcup C_0)$ *and* $(\bigcup B \sqcap C_0)(\bigcup C_0 \sqcap B)(e) = (\bigcup C_0 \sqcap B)(\bigcup B \sqcap C_0)(\bigcup B_0 \cap \bigcup C_0)$.

(5) $(\bigcup C \sqcap B_0)(\bigcup B_0 \sqcap C)(\bigcup B_0 \cap \bigcup C_0) = (\bigcup B_0 \sqcap C)(\bigcup C \sqcap B_0)(\bigcup B_0 \cap \bigcup C_0)$
and
$(\bigcup B \sqcap C_0)(\bigcup C_0 \sqcap B)(\bigcup B_0 \cap \bigcup C_0) = (\bigcup C_0 \sqcap B)(\bigcup B \sqcap C_0)(\bigcup B_0 \cap \bigcup C_0)$.

(6) *The partitions K and M are coupled.*

(7) *The partitions L and M are coupled.*

Proof. Equivalence of 1, 2 and 3 follows from 1.6 and 1.8.

$3 \Rightarrow 4$: By 1.9 and by a symmetrical assertion.

$4 \Rightarrow 5$: By 1.7 and by a symmetrical assertion.

$5 \Rightarrow 3$: As in the preceding step.

$1 \Rightarrow 6$ and 7: We shall verify the conditions (a) and (b) of 1.4 for $A = K$ and $D = M$. (a) is fulfilled by 1.5 because $\bigcup K \cap \bigcup M = \mathfrak{A}$ and $\mathfrak{A} \sqcap M = M$; in fact, $\bigcup K \sqcap M = (\bigcup K \cap \bigcup M) \sqcap M = \mathfrak{A} \sqcap M = M$ and $\bigcup M \sqcap K = \mathfrak{A} \sqcap K = M$. (b) is fulfilled for K and M, since it is fulfilled for K and L. Indeed, each block of K meets $\bigcup K \cap \bigcup L = \mathfrak{A} = \bigcup M$ and each block of M meets $\bigcup K$, since $\bigcup M = \mathfrak{A}$ and $\bigcup K \supseteq \mathfrak{A}$. Analogously for 7.

$6 \Rightarrow 1$: By 1.4, condition 6 reads that each block of K meets $\bigcup M = \mathfrak{A}$. Since $\bigcup K \cap \bigcup L = \mathfrak{A}$ (by 1.3), the condition 1.4(b) is also true for $A = K$ and $D = L$. Condition 1.4(a) follows from 1.5, since $\bigcup K \sqcap L = (\bigcup K \cap \bigcup L) \sqcap L = \mathfrak{A} \sqcap L = M$ and similarly $\bigcup L \sqcap K = M$. Proof of $7 \Rightarrow 1$ is similar to the preceding one.

2.

Lemma 2.1. *If A and D are congruences in an algebra $(\mathfrak{G}, \Omega)$ and $\bigcup A \supseteq \bigcup D$ then $A \vee_P D = A \vee_{\mathscr{X}} D$.*

Proof. Let $\omega \in \Omega$ be n-ary $(n \geqq 1)$ and $a_i A \vee_P D b_i$, $i = 1, \ldots, n$. Then $a_i E_{i1} e_1^i E_{i2} e_2^i \ldots e_{m_i-1}^i E_{im_i} b_i$, where $E_{i1}, \ldots, E_{im_i}$ are by turns equal to A or D. We can achieve that all m_i's are equal (say equal to m) and that $E_{i1} = A = E_{im}$ $(i = 1, \ldots, n)$, for $\bigcup A \supseteq \bigcup D$ yields $a_i D e_1^i \Rightarrow a_i A a_i D e_1^i$; $e_{m_i-1}^i D b_i \Rightarrow e_{m_i-1}^i D b_i A b_i D b_i \ldots \ldots b_i A b_i$. Hence $a_1 \ldots a_n \omega A e_1^1 \ldots e_1^n \omega D e_2^1 \ldots e_2^n \omega A \ldots A b_1 \ldots b_n \omega$. Then $A \vee_P D$ preserves the operations and is consequently a congruence in $(\mathfrak{G}, \Omega)$.

Let $\mathfrak{A}$ and $\mathfrak{D}$ be Ω-subgroups of an Ω-group $(\mathfrak{G}, \Omega)$.

Definition 2.2. $\mathfrak{A}$ and $\mathfrak{D}$ are called *Ω-commuting* Ω-subgroups if $[\mathfrak{A}, \mathfrak{D}] = \mathfrak{A} + \mathfrak{D}$, where $[\mathfrak{A}, \mathfrak{D}]$ means the Ω-subgroup of $(\mathfrak{G}, \Omega)$ generated by the set $\mathfrak{A} \cup \mathfrak{D}$.

Clearly, Ω-commuting Ω-subgroups are commuting subgroups.

Lemma 2.3. *Let $\mathfrak{U}$ and $\mathfrak{D}$ be Ω-subgroups of an Ω-group $(\mathfrak{G}, \Omega)$. Then the following conditions* (a) *to* (e) *satisfy* a $\Leftrightarrow$ b $\Rightarrow$ c $\Leftrightarrow$ d $\Leftrightarrow$ e.

(a) $\mathfrak{A}$ *and* $\mathfrak{D}$ *are* Ω*-commuting,*

(b) $\mathfrak{G}/_r[\mathfrak{A}, \mathfrak{D}] = \mathfrak{G}/_r \mathfrak{A}\ \mathfrak{G}/_r \mathfrak{D}$,

(c) $\mathfrak{G}/_r \mathfrak{A}$ *and* $\mathfrak{G}/_r \mathfrak{D}$ *commute,*

(d) $\mathfrak{G}/_r \mathfrak{A} \vee_P \mathfrak{G}/_r \mathfrak{D} = \mathfrak{G}/_r \mathfrak{A}\ \mathfrak{G}/_r \mathfrak{D}$,

(e) $\mathfrak{A}$ *and* $\mathfrak{D}$ *commute.*

Analogous assertions hold for the left sided decompositions.

Proof. In the proof let us omit the notation r of the right sided decompositions. Let us remark that $x(\mathfrak{G}/\mathfrak{A}\ \mathfrak{G}/\mathfrak{D})\ y \equiv x\ \mathfrak{G}/\mathfrak{A}\ b\ \mathfrak{G}/\mathfrak{D}\ y$ for some $b \in \mathfrak{G} \equiv b = x + a$, $y = b + d$ for some $a \in \mathfrak{A}$ and some $d \in \mathfrak{D} \equiv -x + y = a + d$ for some $a \in \mathfrak{A}$ and some $d \in \mathfrak{D}$.

Then

(1) $\quad x(\mathfrak{G}/\mathfrak{A}\ \mathfrak{G}/\mathfrak{D})\ y \equiv -x + y = a + d \quad$ for some $\quad a \in \mathfrak{A} \quad$ and $\quad$ some $\quad d \in \mathfrak{D}$.

Thus the condition (b) is equivalent to the condition $[\mathfrak{A}, \mathfrak{D}] = \mathfrak{D} + \mathfrak{A}$, which is (a). Another conclusion from (1) is (c) $\equiv$ (e). The equivalence (c) $\equiv$ (d) follows e.g. from [9] 1.1. The implication (a) $\Rightarrow$ (e) is evident.

Lemma 2.4. *Let* $\mathfrak{G}$ *be a group,* $\mathfrak{Q}$ *a subgroup of* $\mathfrak{G}$*. Then any subsets* $\emptyset = \mathfrak{R} \subseteq \mathfrak{G}$ *and* $\emptyset \neq \mathfrak{S} \subseteq \mathfrak{Q}$ *satisfy* $\mathfrak{Q} \cap (\mathfrak{R} + \mathfrak{S}) = \mathfrak{Q} \cap \mathfrak{R} + \mathfrak{S}$ *and* $\mathfrak{Q} \cap (\mathfrak{S} + \mathfrak{R}) = \mathfrak{S} + \mathfrak{Q} \cap \mathfrak{R}$.

Proof. Evidently, the inclusion $\supseteq$ holds. The converse inclusion (in the first assertion): For an arbitrary element x on the left side of the first equation we have $x = q = r + s$, $s = q'$ for suitable $q, q' \in \mathfrak{Q}$, $r \in \mathfrak{R}$ and $s \in \mathfrak{S}$. Then $r = q - s = q - q' \in \mathfrak{Q}$. Analogously the other equation is proved.

In the following the symbol $/$ means the left sided decomposition of a group by its subgroup. Analogous assertions hold for the right-sided decomposition.

Let $\mathfrak{B}_0 \subseteq \mathfrak{B} \subseteq \mathfrak{B}_2$ and $\mathfrak{C}_0 \subseteq \mathfrak{C} \subseteq \mathfrak{C}_2$ be Ω-subgroups of an Ω-group $(\mathfrak{G}, \Omega)$. Put $B_0 = \mathfrak{B}/\mathfrak{B}_0$, $B = \mathfrak{B}_2/\mathfrak{B}$, $C_0 = \mathfrak{C}/\mathfrak{C}_0$ and $C = \mathfrak{C}_2/\mathfrak{C}$. Then $B_0 \leqq B$, $C_0 \leqq C$ and for $e = 0 =$ the zero element of $\mathfrak{G}$ it holds $\bigcup B_0 = B(0)$ and $\bigcup C_0 = C(0)$. Let K, L and M be as in Definition 1.2.

Theorem 2.5. *Let* $(\mathfrak{G}, \Omega)$ *be an* Ω*-group. Let us keep the preceding notation.*

I. *Then the partitions* K, L *and* M *are pairwise coupled and*

$$K = \{\mathfrak{B}_0 + \mathfrak{E} + a : a \in \mathfrak{B} \cap \mathfrak{C}\},$$

$$L = \{\mathfrak{C}_0 + \mathfrak{E} + a : a \in \mathfrak{B} \cap \mathfrak{C}\}, \quad M = \mathfrak{B} \cap \mathfrak{C}/\mathfrak{E},$$

where $\mathfrak{E}$ *is the subgroup of* $\mathfrak{G}$ *generated by the subgroups* $\mathfrak{B}_0 \cap \mathfrak{C}$ *and* $\mathfrak{C}_0 \cap \mathfrak{B}$*. Further,*

$$(\mathfrak{B} \cap \mathfrak{C}) \sqcap K = (\mathfrak{B} \cap \mathfrak{C}) \sqcap L = K \wedge L = M.$$

II. (i) $\mathfrak{B}_0 + \mathfrak{E}$ *is a subgroup* (*an* Ω-*subgroup*) *if and only if* $\mathfrak{B}_0$ *and* $\mathfrak{B} \cap \mathfrak{C}_0$ *commute* (Ω-*commute*). *In this case* $\mathfrak{B}_0 + \mathfrak{E} = \mathfrak{B}_0 + \mathfrak{B} \cap \mathfrak{C}_0$ *and* $K = (\mathfrak{B}_0 + \mathfrak{B} \cap \cap \mathfrak{C}) \sqcap \mathfrak{B}/\mathfrak{B}_0 + \mathfrak{B} \cap \mathfrak{C}_0$.

(ii) *The domain of* K, $\bigcup K = \mathfrak{B}_0 + \mathfrak{B} \cap \mathfrak{C}$, *is a subgroup* (*an* Ω-*subgroup*) *if and only if* $\mathfrak{B}_0$ *and* $\mathfrak{B} \cap \mathfrak{C}$ *commute* (Ω-*commute*).

(iii) *If the conditions of* (i) *and* (ii) *are fulfilled then* $K = \mathfrak{B}_0 + \mathfrak{B} \cap \mathfrak{C}/\mathfrak{B}_0 + + \mathfrak{B} \cap \mathfrak{C}_0$, *both* $\mathfrak{B}_0 + \mathfrak{B} \cap \mathfrak{C}$ *and* $\mathfrak{B}_0 + \mathfrak{B} \cap \mathfrak{C}_0$ *being subgroups* (Ω-*subgroups*). *Similar results hold for* L.

(iv) *If* $\mathfrak{B}_0$ Ω-*commutes with* $\mathfrak{B} \cap \mathfrak{C}_0$ *and* $\mathfrak{C}_0$ Ω-*commutes with* $\mathfrak{C} \cap \mathfrak{B}_0$ *then* $\mathfrak{E}$ *is an* Ω-*subgroup*, $\mathfrak{E} = \mathfrak{B}_0 \cap \mathfrak{C} + \mathfrak{C}_0 \cap \mathfrak{B}$ *and thus*, $\mathfrak{B}_0 \cap \mathfrak{C}$ *and* $\mathfrak{C}_0 \cap \mathfrak{B}$ Ω-*commute. In this case* $M = \mathfrak{B} \cap \mathfrak{C}/\mathfrak{B}_0 \cap \mathfrak{C} + \mathfrak{C}_0 \cap \mathfrak{B}$.

III. (Zassenhaus Lemma.) *If in particular* $\mathfrak{B}_0$ *or* $\mathfrak{C}_0$ *is an ideal of* $\mathfrak{B}$ *or* $\mathfrak{C}$, *respectivelly, then*

$$K = \mathfrak{B}_0 + \mathfrak{B} \cap \mathfrak{C}/\mathfrak{B}_0 + \mathfrak{B} \cap \mathfrak{C}_0 \,,$$

$$L = \mathfrak{C}_0 + \mathfrak{C} \cap \mathfrak{B}/\mathfrak{C}_0 + \mathfrak{C} \cap \mathfrak{B}_0 \,, \quad M = \mathfrak{B} \cap \mathfrak{C}/\mathfrak{B}_0 \cap \mathfrak{C} + \mathfrak{C}_0 \cap \mathfrak{B} \,,$$

all the decompositions K, L *and* M *are congruences in* $(\mathfrak{G}, \Omega)$ *and the corresponding* (*factor*) Ω-*groups are isomorphic.*

Proof. I. Since the condition (3) of Theorem 1.10 is evidently fulfilled, K and L are coupled and $B_{11}(0) = \bar{B}_{11}(0)$. By 1.4(b), each block of the partition K meets the set $\bigcup K \cap \bigcup L$, which is equal to $\bigcup B_0 \cap \bigcup C_0 = \mathfrak{B} \cap \mathfrak{C}$ (1.3). An arbitrary block of K, $K(a)$ for $a \in \mathfrak{B} \cap \mathfrak{C}$, is given by $K(a) = \{\bar{B}_{11}(0) \sqcap [B_0 \vee (B \wedge C_0)]\}\,(a) = \{(\mathfrak{B}_0 + \mathfrak{B} \cap \mathfrak{C}) \sqcap (\mathfrak{B}/\mathfrak{B}_0 \vee_P \mathfrak{B}_2 \cap \mathfrak{C}/\mathfrak{B} \cap \mathfrak{C}_0)\}\,(a)$ or shorter by $K(a) = (\mathfrak{B}_0 + \mathfrak{D}_1) \cap [(\mathfrak{B}/\mathfrak{B}_0 \vee_P \mathfrak{D}/\mathfrak{D}_0)]\,(a)\;(a \in \mathfrak{D}_1)$, where $\mathfrak{D} = \mathfrak{B}_2 \cap \mathfrak{C}$, $\mathfrak{D}_1 = \mathfrak{B} \cap \mathfrak{C}$ and $\mathfrak{D}_0 = \mathfrak{B} \cap \mathfrak{C}_0$. Thus, $K(a)$ is the set of all $x \in \mathfrak{B}_0 + \mathfrak{D}_1$ with the property

$$x = x_n \ldots x_3 \mathfrak{D}/\mathfrak{D}_0 x_2 \mathfrak{B}/\mathfrak{B}_0 x_1 \mathfrak{D}/\mathfrak{D}_0 a \tag{1}$$

or

$$x = y_n \ldots y_3 \mathfrak{B}/\mathfrak{B}_0 y_2 \mathfrak{D}/\mathfrak{D}_0 y_1 \mathfrak{B}/\mathfrak{B}_0 a \,, \tag{2}$$

where x_i and y_j are elements of $\mathfrak{G}$. But (2) can be converted into a sequence of type (1), since $a \mathfrak{D}/\mathfrak{D}_0 a$. Denote by X_n the set of all x for which there exists a chain $a, x_1, x_2, \ldots, x_n$ that fulfils (1). Then 2.4 yields

$$X_1 = \mathfrak{D} \cap (\mathfrak{D}_0 + a) = \mathfrak{D}_0 + a \,, \quad X_2 = \mathfrak{B} \cap (\mathfrak{B}_0 + X_1) = \mathfrak{B}_0 + \mathfrak{D}_0 + a \,,$$
$$X_3 = \mathfrak{D} \cap (\mathfrak{D}_0 + X_2) = \mathfrak{D}_0 + \mathfrak{D} \cap X_2 = \mathfrak{D}_0 + \mathfrak{D} \cap (\mathfrak{B}_0 + \mathfrak{D}_0 + a) =$$
$$= \mathfrak{D}_0 + (\mathfrak{D} \cap \mathfrak{B}_0) + \mathfrak{D}_0 + a \,.$$

By induction, we obtain

$$X_{2n-1} = \mathfrak{D}_0 + (\mathfrak{D} \cap \mathfrak{B}_0) + \mathfrak{D}_0 + \ldots + \mathfrak{D}_0 + (\mathfrak{D} \cap \mathfrak{B}_0) + \mathfrak{D}_0 + a$$
$$(2n \text{ summands}) \,, \quad X_{2n} = \mathfrak{B}_0 + X_{2n-1} \quad (n = 1, 2, \ldots) \,.$$

Evidently, it holds for $n = 1, 2, \cdots$

$$X_{2n} = \mathfrak{B}_0 + X_{2n-1} \supseteq X_{2n-1} \,.$$

Consequently, for $a \in \mathfrak{Q}_1$ we have

$$K(a) = (\mathfrak{B}_0 + \mathfrak{Q}_1) \cap \bigcup_{n=1}^{\infty} X_{2n} \,.$$

It holds

$$\text{(3)} \qquad \bigcup_{n=1}^{\infty} X_{2n} = \mathfrak{B}_0 + \bigcup_{n=1}^{\infty} X_{2n-1} = \mathfrak{B}_0 + \bigcup_{n=1}^{\infty} [\mathfrak{Q}_0 + (\mathfrak{Q} \cap \mathfrak{B}_0) + \dots$$
$$\dots + (\mathfrak{Q} \cap \mathfrak{B}_0) + \mathfrak{Q}_0] + a = \mathfrak{B}_0 + \mathfrak{E} + a \,,$$

where $\mathfrak{E}$ denotes the subgroup of $\mathfrak{G}$ generated by the subgroups $\mathfrak{Q} \cap \mathfrak{B}_0$ and $\mathfrak{Q}_0$ or, returning to the original notation, $\mathfrak{E}$ is the subgroup of $\mathfrak{G}$ generated by the subgroups $\mathfrak{B}_0 \cap \mathfrak{C}$ and $\mathfrak{C}_0 \cap \mathfrak{B}$. It follows

$$K(a) = (\mathfrak{B}_0 + \mathfrak{B} \cap \mathfrak{C}) \cap (\mathfrak{B}_0 + \mathfrak{E} + a) \ (a \in \mathfrak{B} \cap \mathfrak{C})$$

and by 2.4

$$K(a) = \mathfrak{B}_0 + \mathfrak{E} + a \,, \quad (a \in \mathfrak{B} \cap \mathfrak{C}) \,.$$

A similar consideration leads to the result on L.

We shall express blocks of the partition $M = (\mathfrak{A} \sqcap B_0) \vee (\mathfrak{A} \sqcap C_0) = (\mathfrak{B} \cap \mathfrak{C}) \sqcap \sqcap \mathfrak{B}/\mathfrak{B}_0 \vee_P (\mathfrak{B} \cap \mathfrak{C}) \sqcap \mathfrak{C}/\mathfrak{C}_0 = \mathfrak{B} \cap \mathfrak{C}/\mathfrak{B}_0 \cap \mathfrak{C} \vee_P \mathfrak{B} \cap \mathfrak{C}/\mathfrak{C}_0 \cap \mathfrak{B}$. Since each block of M, $M(a)$ where $a \in \mathfrak{B} \cap \mathfrak{C}$, meets the set $\mathfrak{B} \cap \mathfrak{C}$, we obtain by the preceding method $M(a) = \mathfrak{E} + a$ and then $M = \mathfrak{B} \cap \mathfrak{C}/\mathfrak{E}$.

The partitions K, L and M are pairwise coupled by Theorem 1.10 as claimed. The final assertion of Part I follows from 1.5.

II. In the following proof we use Lemma 2.3 several times.

a) Let $\mathfrak{B}_0 + \mathfrak{E}$ be a subgroup. Then $\mathfrak{B}_0$ and $\mathfrak{E}$ commute. Let b_i, $\bar{b}_i$ or $\bar{c}_i$ be elements of $\mathfrak{B}_0, \mathfrak{C} \cap \mathfrak{B}_0$ or $\mathfrak{B} \cap \mathfrak{C}_0$, respectively. There exists $e \in \mathfrak{E}$ such that $b_1 + \bar{c}_1 = \bar{c}_1 + e$, where $e = \bar{b}_2 + \bar{c}_2 + \bar{b}_3 + \bar{c}_3 + \dots$ for suitable $\bar{b}_i, \bar{c}_i$. Since $\bar{b}_2 \in \mathfrak{B}_0$ and $\bar{c}_2 + \bar{b}_3 + \bar{c}_3 + \dots \in \mathfrak{E}$, we have $e = \bar{c}_2 + \bar{b}_3 + \bar{c}_3 + \dots + b_2$ for some $b_2 \in B_0$ and so $b_1 + \bar{c}_1 = (\bar{c}_1 + \bar{c}_2) + \bar{b}_3 + \bar{c}_3 + \dots + b_2$. By induction, we obtain $b_1 + \bar{c}_1 = (\bar{c}_1 + \bar{c}_2 + \dots) + (\dots + b_3 + b_2)$, where $\bar{c}_1 + \bar{c}_2 + \dots \in \mathfrak{B} \cap \mathfrak{C}_0$ and $\dots + b_3 + b_2 \in \mathfrak{B}_0$. Thus $\mathfrak{B}_0$ and $\mathfrak{B} \cap \mathfrak{C}_0$ commute.

b) If $\mathfrak{B}_0$ and $\mathfrak{B} \cap \mathfrak{C}_0$ commute (Ω-commute) then $\mathfrak{B}_0 + \mathfrak{B} \cap \mathfrak{C}_0$ is a subgroup (an Ω-subgroup). This set is equal to $\mathfrak{B}_0 + \mathfrak{E}$. Namely, the latter set contains the former one and the converse inclusion is verified as follows: Given $b \in \mathfrak{B}_0$, $\bar{b}_i \in \mathfrak{B}_0 \cap \cap \mathfrak{C}(\subseteq \mathfrak{B}_0)$, $\bar{c}_i \in \mathfrak{B} \cap \mathfrak{C}_0$ and $e = \dots + \bar{b}_i + \bar{c}_i + \dots \in \mathfrak{E}$, then $b + e = b + \dots \dots + \bar{b}_i + \bar{c}_i + \dots$ is an element of the subgroup generated by $\mathfrak{B}_0$ and $\mathfrak{E}$, i.e. by $\mathfrak{B}_0$, $\mathfrak{B}_0 \cap \mathfrak{C}$ and $\mathfrak{C}_0 \cap \mathfrak{B}$ and thus by $\mathfrak{B}_0$ and $\mathfrak{B} \cap \mathfrak{C}_0$ and this is equal to $\mathfrak{B}_0 + \mathfrak{B} \cap \mathfrak{C}_0$. Consequently, $\mathfrak{B}_0 + \mathfrak{E} = \mathfrak{B}_0 + \mathfrak{B} \cap \mathfrak{C}_0$ is a subgroup (an Ω-subgroup).

c) Let $\mathfrak{B}_0 + \mathfrak{C}$ be an Ω-subgroup. Then by a), $\mathfrak{B}_0$ and $\mathfrak{B} \cap \mathfrak{C}_0$ commute. By b) $\mathfrak{B}_0 + \mathfrak{B} \cap \mathfrak{C}_0 = \mathfrak{B}_0 + \mathfrak{C}$, hence $\mathfrak{B}_0 + \mathfrak{B} \cap \mathfrak{C}_0$ is an Ω-subgroup. Therefore $\mathfrak{B}_0$ and $\mathfrak{B} \cap \mathfrak{C}_0$ Ω-commute.

d) The assertion concerning the domain of K, $\bigcup K$, is evident.

e) If $\mathfrak{B}_0$ Ω-commutes with $\mathfrak{B} \cap \mathfrak{C}_0$ and $\mathfrak{C}_0$ Ω-commutes with $\mathfrak{B}_0 \cap \mathfrak{C}$, then $\mathfrak{B}_0 + \mathfrak{B} \cap \mathfrak{C}_0$ and $\mathfrak{C}_0 + \mathfrak{C} \cap \mathfrak{B}_0$ are Ω-subgroups. By I and II(i) $M = [(\mathfrak{B} \cap \mathfrak{C}) \sqcap \sqcap K] \wedge [(\mathfrak{B} \cap \mathfrak{C}) \sqcap L] = [(\mathfrak{B} \cap \mathfrak{C}) \sqcap \mathfrak{B}/\mathfrak{B}_0 + \mathfrak{B} \cap \mathfrak{C}_0] \wedge [(\mathfrak{B} \cap \mathfrak{C}) \sqcap \mathfrak{C}/\mathfrak{C}_0 + + \mathfrak{C} \cap \mathfrak{B}_0] = \mathfrak{B} \cap \mathfrak{C}/(\mathfrak{B}_0 + \mathfrak{B} \cap \mathfrak{C}_0) \cap (\mathfrak{C}_0 + \mathfrak{C} \cap \mathfrak{B}_0) = \mathfrak{B} \cap \mathfrak{C}/\mathfrak{E}$. Thus $\mathfrak{E} = = (\mathfrak{B}_0 + \mathfrak{B} \cap \mathfrak{C}_0) \cap (\mathfrak{C}_0 + \mathfrak{C} \cap \mathfrak{B}_0)$ is an Ω-subgroup and by 2.4, it is $\mathfrak{E} = = (\mathfrak{B}_0 + \mathfrak{B} \cap \mathfrak{C}_0) \cap (\mathfrak{C}_0 + \mathfrak{C} \cap \mathfrak{B}_0) = (\mathfrak{B}_0 + \mathfrak{B} \cap \mathfrak{C}_0) \cap \mathfrak{C}_0 + \mathfrak{C} \cap \mathfrak{B}_0 = \mathfrak{B}_0 \cap \cap \mathfrak{C}_0 + \mathfrak{B} \cap \mathfrak{C}_0 + \mathfrak{C} \cap \mathfrak{B}_0 = \mathfrak{B} \cap \mathfrak{C}_0 + \mathfrak{C} \cap \mathfrak{B}_0$. Then $\mathfrak{B} \cap \mathfrak{C}_0$ and $\mathfrak{C} \cap \mathfrak{B}_0$ Ω-commute and $\mathfrak{E} = \mathfrak{B}_0 \cap \mathfrak{C} + \mathfrak{C}_0 \cap \mathfrak{B}$.

III. If $\mathfrak{B}_0$ or $\mathfrak{C}_0$ is an ideal of $\mathfrak{B}$ or $\mathfrak{C}$, respectively, then $\mathfrak{B}_0$ Ω-commutes with $\mathfrak{B} \cap \mathfrak{C}$ and therefore $\mathfrak{B}_0 + \mathfrak{B} \cap \mathfrak{C}$ ($= \bigcup\{B_0(a) : a \in \mathfrak{A}\} = \bar{B}_{11}(0)$) is an Ω-subgroup. The partition K is P-join of congruences in $(\mathfrak{G}, \Omega)$, $K = (\bar{B}_{11}(0) \sqcap B_0) \vee_P (\bar{B}_{11}(0) \sqcap \sqcap C_0) = [(\mathfrak{B}_0 + \mathfrak{B} \cap \mathfrak{C}) \sqcap \mathfrak{B}/\mathfrak{B}_0] \vee_P [(\mathfrak{B}_0 + \mathfrak{B} \cap \mathfrak{C}) \sqcap \mathfrak{C}/\mathfrak{C}_0]$ (see 1.3), the domain of the first congruence contains that of the second one, thus K is a congruence (2.1). Similarly, L is a congruence. The partition M as an intersection of two congruences, $M = K \wedge L$ (see Part I), is a congruence, too. This completes the proof of Theorem 2.5.

Theorem 2.5 implies the following "General Four-Group Theorem" [1] 23.2:

Corollary 2.6. (Borůvka [1] 23.2.) *Let $\mathfrak{B}_0 \subseteq \mathfrak{B}$ and $\mathfrak{C}_0 \subseteq \mathfrak{C}$ be Ω-subgroups of an Ω-group $(\mathfrak{G}, \Omega)$. Let the following Ω-subgroups be Ω-commuting*:

$$\mathfrak{B} \cap \mathfrak{C} \;\; \textit{and} \;\; \mathfrak{B} \cap \mathfrak{C}_0 \;\; \textit{with} \;\; \mathfrak{B}_0 ; \qquad \mathfrak{C} \cap \mathfrak{B} \;\; \textit{and} \;\; \mathfrak{C} \cap \mathfrak{B}_0 \;\; \textit{with} \;\; \mathfrak{C}_0 .$$

Then the decompositions K, L and M,

$$K = \mathfrak{B}_0 + \mathfrak{B} \cap \mathfrak{C}/\mathfrak{B}_0 + \mathfrak{B} \cap \mathfrak{C}_0 ,$$

$$L = \mathfrak{C}_0 + \mathfrak{C} \cap \mathfrak{B}/\mathfrak{C}_0 + \mathfrak{C} \cap \mathfrak{B}_0 \quad \textit{and} \quad M = \mathfrak{B} \cap \mathfrak{C}/\mathfrak{B}_0 \cap \mathfrak{C} + \mathfrak{C}_0 \cap \mathfrak{B}$$

are pairwise coupled. All given sums of Ω-subgroups are Ω-subgroups.

Theorem 2.7. *Let $(\mathfrak{G}, \Omega)$ be an algebra, $B_0 \leqq B$ and $C_0 \leqq C$ congruences in $(\mathfrak{G}, \Omega)$, $a \in \bigcup B_0 \cap \bigcup C_0$, $\bigcup B_0 = B(e)$, $\bigcup C_0 = C(e)$. Then the relations K, L and M are congruences on the subalgebras $B_{11}(e)$, $C_{11}(e)$ and $\mathfrak{A}$, respectively.*

Proof. First, we prove that $B_{11}(e)$ is a subalgebra. Let $xB_{11}e$. Then there exist $x_1, \ldots, x_{n-1} \in \mathfrak{G}$ such that

$$(1) \qquad xA_1x_1 \ldots x_{n-1}A_na ,$$

where A_i are alternately equal to B_0 or $B \wedge C$. The sequence (1) can be arbitrarily extended to the right because of eB_0e, $e(B \wedge C)\,e$. Thus, $n \geqq 2$ can be supposed. Let $A_1 = B \wedge C$. Then $A_2 = B_0$, therefore $x_1 \in \bigcup B_0 = B(e)$ and hence $x \in B(x_1) = B(e) = \bigcup B_0$. In this case, we can rewrite (1) as

$$xB_0x(B \wedge C)\,x_1 \ldots x_{n-1}A_ne\,.$$

Now, let $\omega \in \Omega$ be m-ary, $m \geqq 1$, $x^k \in B_{11}(e)$ $(k = 1, \ldots, m)$,

$$x^kA_1^{(k)}x_1^k \ldots x_{n_k-1}^kA_{n_k}^{(k)}e\,,$$

where $A_i^{(k)}$ (for a given $k = 1, \ldots, m$) are by turns equal to B_0 or $B \wedge C$. As above, we can suppose $A_1^{(k)} = B_0$ $(k = 1, \ldots, m)$ and the existence of n with $n = n_k$ $(k = 1, \ldots, m)$. It follows that for a given i $(i = 1, \ldots, n)$, $A_i^{(k)}$ is the same congruence (say A_i) for all $k = 1, \ldots, m$. Since B_0 and $B \wedge C$ are congruences and $\mathfrak{A}$ is a subalgebra, we have $a = e \ldots e\omega \in \mathfrak{A}$ and $x^1 \ldots x^m\omega A_1x_1^1 \ldots x_1^m\omega \ldots x_{n-1}^1 \ldots \ldots x_{n-1}^m\omega A_ne \ldots e\omega$, hence $x^1 \ldots x^m\omega \in B_{11}(a) = B_{11}(e)$ (see 1.3). Finally, $B_{11}(e)$ contains all nullary operations, since $B_{11}(e) \supseteq \mathfrak{A}$ (1.3), and $\mathfrak{A}$ is a subalgebra. Then $B_{11}(e)$ is a subalgebra. Similarly, $C_{11}(e)$ is a subalgebra.

K is a congruence: By 1.3, $K = (B_{11}(e) \sqcap B_0) \vee (B_{11}(e) \sqcap C_0)$, $\bigcup(B_{11}(e) \sqcap B_0) = B_{11}(e) \cap \bigcup B_0 = B_{11}(e)$, $\bigcup(B_{11}(e) \sqcap C_0) = B_{11}(e) \cap \bigcup C_0 \subseteq \bigcup B_0 \cap \bigcup C_0 = \mathfrak{A} \subseteq B_{11}(e)$. By 2.1, P-join of congruences whose domains are comparable is their $\mathscr{K}$-join, hence a congruence. Similarly, L and M are congruences.

Lemma 2.8. *Let $B_0 \leqq B$ and $C_0 \leqq C$ be congruences in an Ω-group $(\mathfrak{G}, \Omega)$, $\bigcup B_0 = B(0)$ and $\bigcup C_0 = C(0)$. Then for $e = 0 =$ the zero element of the group $\mathfrak{G}$, K, L and M are congruences in $(\mathfrak{G}, \Omega)$. They are pairwise coupled (as partitions) and thus isomorphic (as factor Ω-groups).*

Proof. K, L and M are congruences by 2.7. We prove that the condition (4), Theorem 1.10, is fulfilled. Indeed, by 2.4, $(\bigcup C \sqcap B_0)(\bigcup B_0 \sqcap C)(0) = \bigcup C \cap \bigcup B_0 \cap [\bigcup C \cap B_0(0) + \bigcup B_0 \cap C(0)] = \bigcup C \cap B_0(0) + \bigcup B_0 \cap C(0)$, $(\bigcup B_0 \sqcap C) \,.\, (\bigcup C \sqcap B_0)(\bigcup B_0 \cap \bigcup C_0) = \bigcup B_0 \cap \bigcup C \cap \{\bigcup B_0 \cap C(0) + \bigcup C \cap \bigcup B_0 \cap [\bigcup C \cap B_0(0) + \bigcup B_0 \cap \bigcup C_0]\} = \bigcup B_0 \cap C(0) + \bigcup C \cap B_0(0) + \bigcup B_0 \cap C(0) = \bigcup C \cap B_0(0) + \bigcup B_0 \cap C(0)$.

The second part of (4) can be verified analogously.

3.

Definition 3.1. ([1] I, 10.1.) A finite chain

$$(1) \qquad A_1 \leqq A_2 \leqq \ldots \leqq A_n$$

of partitions in a set $\mathfrak{G}$ is called a *partition series* (from A_1 to A_n) in the set $\mathfrak{G}$.

([1] I, 10.2.) Let $e \in \bigcup A_1$. A *local chain* of a partition series (1) is the partition chain $\{A_1(e)\} \leqq A_2(e) \sqcap A_1 \leqq A_3(e) \sqcap A_2 \leqq \ldots \leqq A_n(e) \sqcap A_{n-1}$. We also speak about an *e-chain*.

([1] I, 10.6.) We say that two partition series are *e-joint*, if there exists a bijection of the e-chain of one series on the e-chain of the other one such that the corresponding partitions are coupled.

Two partition series in $\mathfrak{G}$

$$(2)\qquad A_0(=A_{10}) \leqq A_{11} \leqq \ldots \leqq A_{1s}(=A_{20}) \leqq A_{21} \leqq \ldots \\ \ldots \leqq A_{r-1,s}(=A_{r0}) \leqq A_{r1} \leqq \ldots \leqq A_{rs},$$

$$(3)\qquad D_0(=D_{10}) \leqq D_{11} \leqq \ldots \leqq D_{1r}(=D_{20}) \leqq D_{21} \leqq \ldots \\ \ldots \leqq D_{s-1,r}(=D_{s0}) \leqq D_{s1} \leqq \ldots \leqq D_{sr}$$

are called *regularly e-joint* (for $e \in \bigcup A_{10} \cap \bigcup D_{10}$) if the e-chains of these series, i.e. the partition series in $\mathfrak{G}$

$$(4)\qquad \{A_0(e)\} \leqq H_{11} \leqq \ldots \leqq H_{1s} \leqq H_{21} \leqq \ldots \leqq H_{r-1,s} \leqq H_{r1} \leqq \ldots \leqq H_{rs},$$

$$(5)\qquad \{D_0(e)\} \leqq N_{11} \leqq \ldots \leqq N_{1r} \leqq N_{21} \leqq \ldots \leqq N_{s-1,r} \leqq N_{s1} \leqq \ldots \leqq N_{sr},$$

where

$$(6)\qquad H_{ij} = A_{ij}(e) \sqcap A_{i,j-1}, \quad N_{ji} = D_{ji}(e) \sqcap D_{j,i-1},$$

fulfil the following condition:

(7) the partitions H_{ij} and N_{ji} are coupled, $1 \leqq i \leqq r$; $1 \leqq j \leqq s$.

Indeed, regularly e-joint series are e-joint.

Definition 3.2. Let $B_{i-1} \leqq B_i$ and $C_{j-1} \leqq C_j$ be partitions in a set $\mathfrak{G}$, $e \in \bigcup B_{i-1} \cap \cap \bigcup C_{j-1}$ and $\bigcup B_{i-1} = B_i(e)$, $\bigcup C_{j-1} = C_j(e)$. Then we define

$$(1)\qquad \begin{aligned} B_{ij} &= B_{i-1} \vee (B_i \wedge C_j), \quad \bar{B}_{ij} = B_{i-1}(B_i \wedge C_j), \\ C_{ji} &= C_{j-1} \vee (C_j \wedge B_i), \quad \bar{C}_{ji} = C_{j-1}(C_j \wedge B_i), \\ K_{ij} &= B_{ij}(e) \sqcap B_{i,j-1}, \quad L_{ji} = C_{ji}(e) \sqcap C_{j,i-1}, \\ M_{ij} &= (C_j(e) \sqcap B_{i-1}) \vee (B_i(e) \sqcap C_{j-1}). \end{aligned}$$

A. W. Goldie [9] calls two congruences B and C in an algebra $(\mathfrak{G}, \Omega)$ *weakly permutable* (with respect to a subalgebra $\mathfrak{G}_0$ of $\mathfrak{G}$) if $(\bigcup C \sqcap B)(\bigcup B \sqcap C)(\mathfrak{G}_0) = (\bigcup B \sqcap C)(\bigcup C \sqcap B)(\mathfrak{G}_0)$. We introduce a similar concept as follows.

Definition 3.3. Let B and C be partitions in a set $\mathfrak{G}$ and $e \in \bigcup B \cap \bigcup C$. The partitions B and C are said to *weakly commute on the element e* (shortly *weakly e-commute*) if $(\bigcup C \sqcap B)(\bigcup B \sqcap C)(\mathfrak{Q}) = (\bigcup B \sqcap C)(\bigcup C \sqcap B)(\mathfrak{Q})$, where $\mathfrak{Q} = (\bigcup B \cap \cap C(e)) \cup (\bigcup C \cap B(e))$.

Theorem 3.4. *Under the suppositions and the notation of Definition 3.2 it holds*

$$B_{i+1,j}(e) = \bar{B}_{i+1,j}(e) \quad \text{and} \quad C_{j+1,i}(e) = \bar{C}_{j+1,i}(e)$$

if and only if B_i and C_j weakly e-commute.

Proof. By 1.8 and 1.7, the left-hand side condition is equivalent to the following conditions (1) and (2):

$$(1) \qquad (\bigcup C_j \sqcap B_i)(\bigcup B_i \sqcap C_j)(\bigcup B_i \cap \bigcup C_{j-1}) \supseteq \supseteq (\bigcup B_i \sqcap C_j)(\bigcup C_j \sqcap B_i)(\bigcup B_i \cap \bigcup C_{j-1})$$

and

$$(2) \qquad (\bigcup B_i \sqcap C_j)(\bigcup C_j \sqcap B_i)(\bigcup C_j \cap \bigcup B_{i-1}) \supseteq \supseteq (\bigcup C_j \sqcap B_i)(\bigcup B_i \sqcap C_j)(\bigcup C_j \cap \bigcup B_{i-1}).$$

Putting in 1.9 C_j, B_{i-1} or B_i instead of B_0, C_0 or C, respectively, we obtain

$$(3) \qquad (\bigcup C_j \sqcap B_i)(\bigcup B_i \sqcap C_j)(\bigcup C_j \cap \bigcup B_{i-1}) \supseteq \supseteq (\bigcup B_i \sqcap C_j)(\bigcup C_j \sqcap B_i)(\bigcup C_j \cap \bigcup B_{i-1}).$$

Analogously,

$$(4) \qquad (\bigcup B_i \sqcap C_j)(\bigcup C_j \sqcap B_i)(\bigcup B_i \cap \bigcup C_{j-1}) \supseteq \supseteq (\bigcup C_j \sqcap B_i)(\bigcup B_i \sqcap C_j)(\bigcup B_i \cap \bigcup C_{j-1}).$$

Now, the union of the left-hand sides of (1) and (3) is equal to the union of the left-hand sides of (2) and (4). This is the desired equality.

The following theorem is a generalization for sets and algebras of the Schreier-Zassenhaus Theorem for groups — see Corollary 3.7. For the sake of simplicity we separate the set theoretical and algebraical version; after all, the former (set case) need not be a special case of the latter (algebra case) — see Remark 2 below.

Theorem 3.5. I. *Let*

$$(a) \qquad B_0 \leqq B_1 \leqq \ldots \leqq B_r, \quad C_0 \leqq C_1 \leqq \ldots \leqq C_s$$

be two partition series in a set $\mathfrak{G}$ satisfying for some $e \in \bigcup B_0 \cap \bigcup C_0$ the following conditions (b) *and* (b′):

$$(b) \quad \bigcup B_i = B_{i+1}(e), \quad 0 \leqq i \leqq r - 1 \quad \text{and} \quad \bigcup C_j = C_{j+1}(e), \quad 0 \leqq j \leqq s - 1;$$

$$(b') \qquad B_0 = C_0 \quad \text{and} \quad B_r = C_s.$$

Then (1) *and* (2) (see proof) *are refinements of the series* (a). *The partitions* (1) *and* (2) *are regularly e-joint if and only if the following equivalent conditions* (c) *and* (d) *are fulfilled*:

(c) $\quad B_{ij}(e) = \bar{B}_{ij}(e)$ *and* $C_{ji}(e) = \bar{C}_{ji}(e)\,,\quad 1 \leqq i \leqq r\,,\quad 1 \leqq j \leqq s\,;$

(d) $\quad B_i$ *and* C_j *weakly e-commute*, $\quad 1 \leqq i \leqq r - 1\,,\quad 1 \leqq j \leqq s - 1\,.$

The refinements (1) *and* (2) *do not depend on the element* $e \in \bigcup B_0 \cap \bigcup C_0$. *Each of the partitions* K_{ij} *and* L_{ji} *is coupled with the partition* M_{ij}.

II. *If, moreover,* $(\mathfrak{G}, \Omega)$ *is an algebra and, for some* (i, j) $(1 \leqq i \leqq r, 1 \leqq j \leqq s)$, B_{i-1}, B_i, C_{j-1} *and* C_j *are congruences in* $(\mathfrak{G}, \Omega)$, *then* K_{ij}, L_{ji} *and* M_{ij} *are congruences in* $(\mathfrak{G}, \Omega)$. *They are isomorphic as (factor) algebras.*

Remark 1. We shall prove Part I of Theorem 3.5 assuming the condition

(b′) $$B_0 = C_0 \quad\text{and}\quad B_r = C_s\,.$$

If this condition is not true we shall extend the series (a) so that the extensions already fulfil (b′) (and, moreover also (b)).

Let $B_r \neq C_s$. If $\bigcup C_s \nsubseteq \bigcup B_r$ we continue the first series on the right by $B_{r+1} = \{\bigcup B_r, \bigcup C_s \setminus \bigcup B_r\}$, $B_{r+2} = \{\bigcup B_r \cup \bigcup C_s\}$. If $\bigcup C_s \subseteq \bigcup B_r$, we extend it by $B_{r+1} = \{\bigcup B_r\}$ only.

Let $B_0 \neq C_0$. If $C_0(e) \nsupseteq B_0(e)$ we continue the first series on the left by $B_{-1} = \{B_0(e) \cap C_0(e),\ B_0(e) \setminus C_0(e)\}$, $B_{-2} = \{B_0(e) \cap C_0(e)\}$. If $C_0(e) \supseteq B_0(e)$, we continue it by $B_{-1} = \{B_0(e)\}$ only. Similarly the second series will be extended.

Proof. I. In the following $\vee$ and $\wedge$ denote the operations in the lattice of partitions in $\mathfrak{G}$ (let us point out that this supposition also holds for Part II of congruences).

Refinements will be chosen as follows

(1) $$B_0(=B_{10}) \leqq B_{11} \leqq \ldots \leqq B_{1s}(=B_1 = B_{20}) \leqq B_{21} \leqq \ldots$$
$$\ldots \leqq B_{r-1,s}(=B_{r-1} = B_{r0}) \leqq B_{r1} \ldots \leqq B_{r,s-1} \leqq (B_{rs}=)\,B_r\,;$$

(2) $$C_0(=C_{10}) \leqq C_{11} \leqq \ldots \leqq C_{1r}(=C_1 = C_{20}) \leqq C_{21} \leqq \ldots$$
$$\ldots \leqq C_{s-1,r}(=C_{s-1} = C_{s0}) \leqq C_{s1} \leqq \ldots \leqq C_{s,r-1} \leqq (C_{sr}=)\,C_s\,.$$

The inequalities are clear. The equalities follow from (b′):

$$B_{is} = B_{i-1} \vee (B_i \wedge C_s) = B_{i-1} \vee (B_i \wedge B_r) = B_i\,,$$
$$B_{i0} = B_{i-1} \vee (B_i \wedge C_0) = B_{i-1} \vee (B_i \wedge B_0) = B_{i-1}\,.$$

Similarly for (2).

The e-chains of the series (1) and (2) are

(3) $$\{B_0(e)\} \leqq K_{11} \leqq \ldots \leqq K_{1s} \leqq K_{21} \leqq \ldots \leqq K_{r-1,s} \leqq K_{r1} \leqq \ldots$$
$$\ldots \leqq K_{r,s-1} \leqq K_{rs}(=B_{r-1} \vee C_{s-1})\,,$$

(4) $$\{C_0(e)\} \leqq L_{11} \leqq \ldots \leqq L_{1r} \leqq L_{21} \leqq \ldots \leqq L_{s-1,r} \leqq L_{s1} \leqq \ldots$$
$$\ldots \leqq L_{s,r-1} \leqq L_{sr}(=C_{s-1} \vee B_{r-1})\,.$$

To prove this it suffices to show that $K_{i+1,1} = B_{i+1,1}(e) \sqcap B_{is}$ $(0 \leqq i \leqq r - 1)$. This is, however, evident from the fact that $B_{i+1,0} = B_{is}$ (see (1)). Similarly for (4).

Now, the assertion of Part I follows from Theorems 1.10 and 3.4. In fact, by 1.10 one necessary and sufficient condition for the regular e-jointness of the series (1) and (2) is $B_{ij}(e) = \bar{B}_{ij}(e)$ and $C_{ji}(e) = \bar{C}_{ji}(e)$ for $1 \leqq i \leqq r$, $1 \leqq j \leqq s$. By 3.4, this is equivalent to the weak e-commutativity of B_i and C_j for $1 \leqq i \leqq r - 1$, $1 \leqq j \leqq s - 1$, together with the conditions

$$B_{1j}(e) = \bar{B}_{1j}(e)\,, \quad 1 \leqq j \leqq s - 1\,, \qquad B_{is}(e) = \bar{B}_{is}(e)\,, \quad 1 \leqq i \leqq r\,,$$
$$C_{1i}(e) = \bar{C}_{1i}(e)\,, \quad 1 \leqq i \leqq r - 1\,, \qquad C_{jr}(e) = \bar{C}_{jr}(e)\,, \quad 1 \leqq j \leqq s\,.$$

By 1.10, these conditions are equivalent to the following conditions (i) to (iv):

(i) $$(\bigcup C_j \sqcap B_0)(\bigcup B_0 \sqcap C_j)(e) \supseteq$$
$$\supseteq (\bigcup B_0 \sqcap C_j)(\bigcup C_j \sqcap B_0)(\bigcup B_0 \cap \bigcup C_{j-1})\,, \quad 1 \leqq j \leqq s - 1\,;$$

(ii) $$(\bigcup C_s \sqcap B_{i-1})(\bigcup B_{i-1} \sqcap C_s)(e) \supseteq$$
$$\supseteq (\bigcup B_{i-1} \sqcap C_s)(\bigcup C_s \sqcap B_{i-1})(\bigcup B_{i-1} \cap \bigcup C_{s-1})\,, \quad 1 \leqq i \leqq r\,.$$

The remaining two conditions (iii) and (iv) are expressed symmetrically (with respect to the symbols B and C). All these conditions are fulfilled, for the left-hand side set of (i) or (ii) is equal to $\bigcup B_0$ or $\bigcup B_{i-1}$ and evidently contains the right-hand side set (which is a subset of the domain of the partition $\bigcup B_0 \sqcap C_j$ or $\bigcup B_{i-1} \sqcap C_s$, respectively). It suffices to recall the condition (b′). The proof of Part I is complete.

II. follows from Theorem 2.7.

Remark 2. a) Part I of Theorem 3.5 is proved under the assumption (b′) $B_0 = C_0$ and $B_r = C_s$. Provided this assumption was not fulfilled we extended both the series (a) as described in Remark 1. It is useful to point out that the added partitions need not be congruences even if all the B_i's and C_j's given in (a) are.

If all B_i's and C_j's in (a) are congruences and (b′) is valid, all members of the e-chains (3) and (4) of the refinements (1) and (2) are congruences. On the other hand, the members of the refinements (1) and (2) themselves need not be congruences, since the symbol $\vee$ denotes $\vee_P$ in (1), Definition 3.2, even for the case II.

b) If $(\mathfrak{G}, \Omega)$ is an Ω-group, all B_i's and C_j's are congruences and (b′) is not fulfilled, then we can extend the series (a) so that all the partitions of the extended series are congruences (and its terminal members are the same) provided that the following condition is true (for $e = 0 =$ the zero element of the group $\mathfrak{G}$)

$$\bigcup B_r \text{ and } \bigcup C_s \text{ are ideals of an } \Omega\text{-subgroup } \mathfrak{G}^* \text{ of } (\mathfrak{G}, \Omega).$$

If this is the case we extend the first series by $B_{r+1} = \mathfrak{G}^*/\bigcup B_r$, $B_{r+2} = \{\mathfrak{G}^*\}$, $B_{-1} = B_0(0)/\{0\}$, $B_{-2} = \{\{0\}\}$, and similarly for the second series. Then the extended series have the same terminal members as required in (b′) and fulfil (b), too. The regular e-jointness of refinements of the type (1) and (2), Theorem 3.5 (of refinements of the congruence series $B_{-2} \leqq B_{-1} \leqq \ldots \leqq B_{r+1} \leqq B_{r+2}$, $C_{-2} \leqq C_{-1} \leqq \ldots \leqq C_{s+1} \leqq C_{s+2}$) is guaranteed by Lemma 2.8.

The following definition presents an analogue of the concept of "the isomorphism of two normal series of Ω-subgroups of an Ω-group".

Definition 3.6. Two series of Ω-subgroups of an Ω-group $(\mathfrak{G}, \Omega)$

$$(1) \qquad \mathfrak{B}_1 \subseteq \mathfrak{B}_2 \subseteq \ldots \subseteq \mathfrak{B}_r \quad \text{and} \quad \mathfrak{C}_1 \subseteq \mathfrak{C}_2 \subseteq \ldots \subseteq \mathfrak{C}_s$$

are said to be *joint* if there exists a bijection of the sequence of the (left-sided) factors of the first series (1), $K_i = \mathfrak{B}_i/\mathfrak{B}_{i-1}$ $(2 \leqq i \leqq r)$, on the sequence of the (left-sided) factors of the second series (1), $L_j = \mathfrak{L}_j/\mathfrak{L}_{j-1}$ $(2 \leqq j \leqq s)$, such that the corresponding factors are coupled partitions.

An analogous definition with right-sided factors.

Corollary 3.7. *Let*

$$(1) \qquad \mathfrak{B}_0 \subseteq \mathfrak{B}_1 \subseteq \ldots \subseteq \mathfrak{B}_r, \quad \mathfrak{C}_0 \subseteq \mathfrak{C}_1 \subseteq \ldots \subseteq \mathfrak{C}_s$$

be two series of Ω-subgroups of an Ω-group $(\mathfrak{G}, \Omega)$. Let

$$(1') \qquad \mathfrak{B}_0 = \mathfrak{C}_0 \quad and \quad \mathfrak{B}_r = \mathfrak{C}_s$$

be fulfilled.

I. *Then the following partitions K_{ij}, L_{ji} and M_{ij} $(1 \leqq i \leqq r,\ 1 \leqq j \leqq s)$ are pairwise coupled*:

$$(2) \qquad K_{ij} = \{\mathfrak{B}_{i-1} + \mathfrak{E}_{ij} + a : a \in \mathfrak{B}_i \cap \mathfrak{C}_j\},$$
$$L_{ji} = \{\mathfrak{C}_{j-1} + \mathfrak{E}_{ji} + a : a \in \mathfrak{C}_j \cap \mathfrak{B}_i\},$$
$$(3) \qquad M_{ij} = \mathfrak{B}_i \cap \mathfrak{C}_j /_l \mathfrak{E}_{ij},$$

where $\mathfrak{E}_{ij} = \mathfrak{E}_{ji}$ is the subgroup of the group $\mathfrak{G}$ generated by the subgroups $\mathfrak{B}_{i-1} \cap \cap \mathfrak{C}_j$ and $\mathfrak{C}_{j-1} \cap \mathfrak{B}_i$.

Further,

$$(\mathfrak{B}_i \cap \mathfrak{C}_j) \sqcap K_{ij} = (\mathfrak{B}_i \cap \mathfrak{C}_j) \sqcap L_{ji} = K_{ij} \wedge L_{ji} = M_{ij}.$$

II. *Let the following Ω-subgroups Ω-commute $(1 \leqq i \leqq r,\ 1 \leqq j \leqq s)$: $\mathfrak{B}_i \cap \mathfrak{C}_j$ and $\mathfrak{B}_i \cap \mathfrak{C}_{j-1}$ with $\mathfrak{B}_{i-1}$, $\mathfrak{C}_j \cap \mathfrak{B}_i$ and $\mathfrak{C}_j \cap \mathfrak{B}_{i-1}$ with $\mathfrak{C}_{j-1}$. Then the following series of Ω-subgroups* (4) *and* (5)

$$(4) \qquad \mathfrak{B}_0(=\mathfrak{B}_{10}) \subseteq \mathfrak{B}_{11} \subseteq \ldots \subseteq \mathfrak{B}_{1s}(=\mathfrak{B}_1 = \mathfrak{B}_{20}) \subseteq \mathfrak{B}_{21} \subseteq \ldots$$
$$\ldots \subseteq \mathfrak{B}_{r-1,s}(=\mathfrak{B}_{r-1} = \mathfrak{B}_{r0}) \subseteq \mathfrak{B}_{r1} \subseteq \ldots \subseteq \mathfrak{B}_{r,s-1} \subseteq (\mathfrak{B}_{rs}=)\, \mathfrak{B}_r,$$

$$(5) \qquad \mathfrak{C}_0(=\mathfrak{C}_{10}) \subseteq \mathfrak{C}_{11} \subseteq \ldots \subseteq \mathfrak{C}_{1r}(=\mathfrak{C}_1 = \mathfrak{C}_{20}) \subseteq \mathfrak{C}_{21} \subseteq \ldots$$
$$\ldots \subseteq \mathfrak{C}_{s-1,r}(=\mathfrak{C}_{s-1} = \mathfrak{C}_{s0}) \subseteq \mathfrak{C}_{s1} \subseteq \ldots \subseteq \mathfrak{C}_{s,r-1} \subseteq (\mathfrak{C}_{sr}=)\, \mathfrak{C}_s,$$

where $\mathfrak{B}_{ij} = \mathfrak{B}_{i-1} + \mathfrak{B}_i \cap \mathfrak{C}_j$ and $\mathfrak{C}_{ji} = \mathfrak{C}_{j-1} + \mathfrak{C}_j \cap \mathfrak{B}_i$, are joint refinements of the series (1). *The coupled left-sided factors of the refinements* (4) *and* (5) *are K_{ij}*

and L_{ji}. *Now, these partitions have the form*

$$K_{ij} = \mathfrak{B}_{ij} /_l \mathfrak{B}_{i,j-1} \quad \textit{and} \quad L_{ji} = \mathfrak{C}_{ji} /_l \mathfrak{C}_{j,i-1} \, .$$

Both these factors are coupled with the partition (3), *which has the form*

$$M_{ij} = \mathfrak{B}_i \cap \mathfrak{C}_j /_l \mathfrak{B}_{i-1} \cap \mathfrak{C}_j + \mathfrak{C}_j \cap \mathfrak{B}_{i-1} \, .$$

III. (Schreier-Zassenhaus Theorem.) *If* (1) *are normal series of* Ω*-subgroups of* $(\mathfrak{G}, \Omega)$, *then the refinements* (4) *and* (5) *are normal series as well, and* K_{ij}, L_{ji} *and* M_{ij} *are isomorphic factor* Ω*-groups.*

Remark. 1) An analogous theorem is true for the right-sided decompositions.

2) When the condition (1′) is not true, the series (1) can be extended in such a way that (1′) is again fulfilled; namely, we continue both the series on the right-hand side by an Ω-subgroup $\mathfrak{B}_{r+1} = \mathfrak{C}_{s+1}$ between $[\mathfrak{B}_r, \mathfrak{C}_s]$ and $\mathfrak{G}$, and on the left-hand side by an Ω-subgroup $\mathfrak{B}_{-1} = \mathfrak{C}_{-1}$ between $\{0\}$ and $\mathfrak{B}_0 \cap \mathfrak{C}_0$. If the conditions of Part II are true for $1 \leqq i \leqq r$, $1 \leqq j \leqq s$, they are true for $0 \leqq i \leqq r + 1$, $0 \leqq j \leqq s + 1$, too. If the series (1) are normal, then the extended series become normal under evident supplementary suppositions.

Proof. Define $B_i = \mathfrak{B}_{i+1}/\mathfrak{B}_i$ $(0 \leqq i \leqq r - 1)$, $C_j = \mathfrak{C}_{j+1}/\mathfrak{C}_j$ $(0 \leqq j \leqq s - 1)$ (we put shortly $/$ instead of $/_l$) and $B_r = \{\mathfrak{B}_r\} = \{\mathfrak{C}_s\} = C_s$. Then

$$B_0 \leqq B_1 \leqq \ldots \leqq B_r \, , \quad C_0 \leqq C_1 \leqq \ldots \leqq C_s \, ,$$
$$\bigcup B_{i-1} = B_i(0) \, , \quad \bigcup C_{j-1} = C_j(0) \, , \quad 1 \leqq i \leqq r \, , \quad 1 \leqq j \leqq s \, .$$

Define K_{ij}, L_{ji} and M_{ij} as in Definition 3.2.

I. follows from Theorem 2.5, I.

II. Inclusions in (4) and (5) follow directly from the definition and from (1′). By Theorem 2.5, II (iii),

$$\left.\begin{aligned} K_{ij} &= \mathfrak{B}_{i-1} + \mathfrak{B}_i \cap \mathfrak{C}_j/\mathfrak{B}_{i-1} + \mathfrak{B}_i \cap \mathfrak{C}_{j-1} = \mathfrak{B}_{ij}/\mathfrak{B}_{i,j-1} \\ L_{ji} &= \mathfrak{C}_{j-1} + \mathfrak{C}_j \cap \mathfrak{B}_i/\mathfrak{C}_{j-1} + \mathfrak{C}_j \cap \mathfrak{B}_{i-1} = \mathfrak{C}_{ji}/\mathfrak{C}_{j,i-1} \\ M_{ij} &= \mathfrak{B}_i \cap \mathfrak{C}_j/\mathfrak{B}_{i-1} \cap \mathfrak{C}_j + \mathfrak{C}_{j-1} \cap \mathfrak{B}_r \, , \end{aligned}\right\} \begin{aligned} 1 &\leqq i \leqq r \, , \\ 1 &\leqq j \leqq s \end{aligned}$$

all given sums being Ω-subgroups. These partitions are pairwise coupled by I.

III. Normality of the series (4) means that $\mathfrak{B}_{i,j-1}$ is an ideal of $\mathfrak{B}_{i,j}$. By [6] III, 3.5.7, it is

$$\begin{aligned} (6) \qquad & [B_{i-1} \vee_P (B_i \wedge C_{j-1})] \, (0) = \\ & = [B_{i-1}(0) + \bigcup B_{i-1} \cap (B_i \wedge C_{j-1}) \, (0)] \cup \\ & \cup [\bigcup (B_i \wedge C_{j-1}) \cap B_{i-1}(0) + (B_i \wedge C_{j-1}) \, (0)] = \\ & = [\mathfrak{B}_{i-1} + \mathfrak{B}_i \cap \mathfrak{B}_i \cap \mathfrak{C}_{j-1}] \cup [\mathfrak{B}_{i+1} \cap \mathfrak{C}_j \cap \mathfrak{B}_{i-1} + \mathfrak{B}_i \cap \mathfrak{C}_{j-1}] = \\ & = (\mathfrak{B}_{i-1} + \mathfrak{B}_i \cap \mathfrak{C}_{j-1}) \cup (\mathfrak{B}_{i-1} \cap \mathfrak{C}_j + \mathfrak{B}_i \cap \mathfrak{C}_{j-1}) = \\ & = \mathfrak{B}_{i-1} + \mathfrak{B}_i \cap \mathfrak{C}_{j-1} = \mathfrak{B}_{i,j-1} \end{aligned}$$

and by the same theorem, this is an ideal of the Ω-subgroup

$$B_{i-1}(0) + \bigcup B_{i-1} \cap \bigcup(B_i \wedge C_{j-1}) = \mathfrak{B}_{i-1} + \mathfrak{B}_i \cap \mathfrak{C}_j = \mathfrak{B}_{ij}\,.$$

Since K_{ij}, L_{ji} and M_{ij} are pairwise coupled (as partitions), they are isomorphic (as factor Ω-groups). The proof is complete.

Remark. The proof of the Schreier-Zassenhaus Theorem (Corollary 3.7, III) is based on Theorem 2.5. We can obtain another proof by using Theorem 3.5. If we make use of the notation given at the beginning of the proof to Corollary 3.7, then

$$\text{(a)} \qquad B_0 \leqq B_1 \leqq \ldots \leqq B_r\,, \quad C_0 \leqq C_1 \leqq \ldots \leqq C_s$$

are two congruence series in $(\mathfrak{G}, \Omega)$ which fulfil the conditions (b) and (b′) of Theorem 3.5 for $e = 0 =$ the zero element of the group $\mathfrak{G}$. The condition (d), Theorem 3.5, is fulfilled by Lemma 2.8.

We shall show that factors of the Schreier-Zassenhaus refinements of the first series (1) are formed by the congruences

$$K_{ij} = [B_{i-1} \vee (B_i \wedge C_j)]\,(0) \sqcap [B_{i-1} \vee (B_i \wedge C_{j-1})]\,.$$

By [6] III, 3.5.7 it is

$$[B_{i-1} \vee (B_i \wedge C_{j-1})]\,(0) = \mathfrak{B}_{i-1} + \mathfrak{B}_i \cap \mathfrak{C}_{j-1}$$

(see (6) of the preceding proof). By the same theorem, this set is an ideal of the Ω-subgroup

$$B_{i-1}(0) + \bigcup B_{i-1} \cap \bigcup(B_i \wedge C_{j-1}) = \mathfrak{B}_{i-1} + \mathfrak{B}_i \cap \mathfrak{C}_j\,.$$

Analogously $[B_{i-1} \vee (B_i \wedge C_j)]\,(0) = \mathfrak{B}_{i-1} + \mathfrak{B}_i \cap \mathfrak{C}_j$.

From the preceding it follows that

$$K_{ij}(0) = (\mathfrak{B}_{i-1} + \mathfrak{B}_i \cap \mathfrak{C}_j) \cap (\mathfrak{B}_{i-1} + \mathfrak{B}_i \cap \mathfrak{C}_{j-1}) = \mathfrak{B}_{i-1} + \mathfrak{B}_i \cap \mathfrak{C}_{j-1}$$

is an ideal of the Ω-subgroup

$$\bigcup K_{ij} = (\mathfrak{B}_{i-1} + \mathfrak{B}_i \cap \mathfrak{C}_j) \cap (\mathfrak{B}_i + (\mathfrak{B}_{i+1} \cap \mathfrak{C}_j)) = \mathfrak{B}_{i-1} + \mathfrak{B}_i \cap \mathfrak{C}_j\,.$$

Since K_{ij} is a congruence in $(\mathfrak{G}, \Omega)$ (see Theorem 3.5), [6] I, 1.6 implies

$$\text{(7)} \qquad K_{ij} = \bigcup K_{ij}/K_{ij}(0) = \mathfrak{B}_{i-1} + \mathfrak{B}_i \cap \mathfrak{C}_j/\mathfrak{B}_{i-1} + \mathfrak{B}_i \cap \mathfrak{C}_{j-1}\,.$$

Similarly

$$\text{(8)} \qquad L_{ji} = \bigcup L_{ji}/L_{ji}(0) = \mathfrak{C}_{j-1} + \mathfrak{C}_j \cap \mathfrak{B}_i/\mathfrak{C}_{j-1} + \mathfrak{C}_j \cap \mathfrak{B}_{i-1}\,.$$

Thus, the Schreier-Zassenhaus refinement of the first or second series in (1) is formed by Ω-subgroups

$$\text{(9)} \qquad \mathfrak{B}_{ij} = \mathfrak{B}_{i-1} + \mathfrak{B}_i \cap \mathfrak{C}_j \quad \text{or} \quad \mathfrak{C}_{ji} = \mathfrak{C}_{j-1} + \mathfrak{C}_j \cap \mathfrak{B}_i\,, \quad \text{respectively}\,.$$

By Theorem 3.5, congruences K_{ij} (see (7)) and L_{ji} (see (8)) are coupled (as partitions), thus isomorphic (as Ω-groups). Moreover, with each of them the following congruence M_{ij} is coupled:

$$\begin{aligned} M_{ij} &= (B_i(0) \sqcap C_{j-1}) \vee (C_j(0) \sqcap B_{i-1}) = (\mathfrak{B}_i \sqcap \mathfrak{C}_j/\mathfrak{C}_{j-1}) \vee (\mathfrak{C}_j \sqcap \mathfrak{B}_i/\mathfrak{B}_{i-1}) = \\ &= (\mathfrak{B}_i \cap \mathfrak{C}_j/\mathfrak{B}_i \cap \mathfrak{C}_{j-1}) \vee_P (\mathfrak{C}_j \cap \mathfrak{B}_i/\mathfrak{C}_j \cap \mathfrak{B}_{i-1}) = \\ &= (\mathfrak{B}_i \cap \mathfrak{C}_j/\mathfrak{B}_i \cap \mathfrak{C}_{j-1}) \vee_{\mathcal{K}} (\mathfrak{C}_j \cap \mathfrak{B}_i/\mathfrak{C}_j \cap \mathfrak{B}_{i-1}) = \\ &= \mathfrak{B}_i \cap \mathfrak{C}_j/(\mathfrak{B}_i \cap \mathfrak{C}_{j-1} + \mathfrak{C}_j \cap \mathfrak{B}_{i-1}) . \end{aligned}$$

Thus it is shown that the refinements of the series (1) formed by the Ω-subgroups (9) are isomorphic.

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