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# GENERALIZED LATTICE IDENTITIES IN LATTICE ORDERED GROUPS

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In the paper [4], the notion of a radical class of lattice ordered groups has been introduced. A particular case of this notion, the concept of a torsion class of lattice ordered groups, had been investigated earlier by J. MARTINEZ [7]. Each variety of lattice ordered groups is a torsion class (HOLLAND [1]). Several important classes of lattice ordered groups (e.g., the class of all archimedean $l$-groups, the class of all complete $l$-groups, the class of all projectable $l$-groups, the class of all completely distributive $l$-groups) are radical classes without being torsion classes (cf. [8], [5], [3], [6], [2]).

In this note it will be shown that if $P_1$and $P_2$ are generalized lattice polynomials (which may contain infinitely many variables, cf. the definition below) and if $\mathscr{K}(P_1, P_2)$ is the class of all lattice ordered groups fulfilling the identity $P_1 \equiv P_2$, then $\mathscr{K}(P_1, P_2)$ is a radical class. (In fact, a slightly more general result concerning implications will be proved.)

The method used here is a generalization of the method that has been applied to investigating complete distributivity of lattice ordered groups in [2].

Let us recall the notion of a radical class [4]. A nonempty class $\mathscr{K}$ of lattice ordered groups is said to be a radical class if it fulfils the following conditions:

(i) $\mathscr{K}$ is closed with respect to isomorphisms.

(ii) If $H \in \mathscr{K}$ and $H_1$ is a convex $l$-subgroup of $H$, then $H_1 \in \mathscr{K}$.

(iii) If $G$ is a lattice ordered group and $\{H_i\}$ is a system of convex $l$-subgroups of $G$ such that each $H_i$ belongs to $\mathscr{K}$, then $\bigvee H_i$ belongs to $\mathscr{K}$ as well.

We shall use the following notation: the symbols $x, x_i, x_j, \ldots$ denote variables, while fixed elements of lattices or lattice ordered groups will be denoted by $a, b, c, \ldots$ $\ldots, a_i, b_j, c_k, \ldots$. Instead of $x_i$ and $a_i$ we often write $x(i)$ or $a(i)$, respectively.

An expression of the form

$$S^1_{i_1 \in I_1} S^2_{i_2 \in I_2} \ldots S^n_{i_n \in I_n} \quad x(i_1, \ldots, i_n) \tag{1}$$

(where $S^k \in \{\wedge, \vee\}$ for each $k \in \{1, \ldots, n\}$, and $I_1, \ldots, I_n$ are nonempty sets of indices) is called *a generalized lattice polynomial* with the variables $x(i_1, \ldots, i_n)$. The generalized polynomial (1) will be denoted also by $P\ x(i_1, \ldots, i_n)$ or by $P\ x(i_1, \ldots, i_n)$ $(i_1 \in I_1, \ldots, i_n \in I_n)$.

If $P\ x(i_1, \ldots, i_n)$ is a generalized lattice polynomial and if $L$ is a lattice, then $P$ can be viewed as a partial infinitary operation on $L$. Let us remark that general algebras with infinitary operations have been thoroughly studied by SLOMINSKI [9].

Let us have four generalized polynomials

$$\begin{aligned}
&P_1\ x(i_1, \ldots, i_{n_1})\ (i_1 \in I_1, \ldots, i_{n_1} \in I_{n1}),\\
&P_2\ x(i'_1, \ldots, i'_{n_2})\ (i'_1 \in I'_1, \ldots, i'_{n_2} \in I'_{n_2}),\\
&Q_1\ x(j_1, \ldots, j_{m_1})\ (j_1 \in J_1, \ldots, j_{m_1} \in J_{m_1}),\\
&Q_2\ x(j'_1, \ldots, j'_{m_2})\ (j'_1 \in J'_1, \ldots, j'_{m_2} \in J'_{m_2}).
\end{aligned}$$

Put

$$I = (I_1 \times \ldots \times I_{n_1}) \cup (I'_1 \times \ldots \times I'_{n_2}) \cup (J_1 \times \ldots \times J_{m_1}) \cup \cup (J'_1 \times \ldots \times J'_{m_2}).$$

Let $L$ be a lattice. $L$ will be said to fulfil the implication

$$P_1 = P_2 \Rightarrow Q_1 = Q_2, \tag{2}$$

if the following condition is valid:

(*) Whenever $\varphi$ is a mapping of the set $I$ into $L$ such that all elements $P_1\ \varphi(i_1, \ldots, i_{n_1})$, $P_2\ \varphi(i'_1, \ldots, i'_{n_2})$, $Q_1\ \varphi(j_1, \ldots, j_{m_1})$ and $Q_2\ \varphi(j'_1, \ldots, j'_{m_2})$ exist in $L$, then we have

$$\begin{aligned}
&P_1\ \varphi(i_1, \ldots, i_{n_1}) = P_2\ \varphi(i'_1, \ldots, i'_{n_2}) \Rightarrow\\
&\Rightarrow Q_1\ \varphi(j_1, \ldots, j_{m_1}) = Q_2\ \varphi(j'_1, \ldots, j'_{m_2}).
\end{aligned}$$

We denote by $\mathscr{L}$ the class of all lattices fulfilling the implication (2). If $L \in \mathscr{L}$, $a, b \in L$, $a \leqq b$, then the interval $[a, b]$ of $L$ belongs to $\mathscr{L}$ as well.

The following result will be proved:

**Theorem 1.** *Let $G$ be a lattice ordered group. There exists a convex l-subgroup $H(G)$ of $G$ such that* (i) *$H(G)$ fulfils the implication* (2); (ii) *if $H_1$ is a convex l-subgroup of $G$ fulfilling the implication* (2), *then $H_1 \subseteq H$;* (iii) *for each automorphism $f$ of the lattice ordered group $G$ we have $f(H(G)) = H(G)$.*

First we prove two lemmas. The symbol $G$ denotes always a lattice ordered group.

**Lemma 1.** *Let $a, b, c \in G$, $a \leqq b \leqq c$. If $[a, b] \in \mathscr{L}$ and $[b, c] \in \mathscr{L}$, then $[a, c]$ belongs to $\mathscr{L}$, too.*

Proof. Let $[a, b], [b, c] \in \mathscr{L}$ and suppose that $[a, c]$ does not belong to $\mathscr{L}$.

Hence there exists a mapping $\varphi : I \to [a, c]$ such that $P_1\, \varphi(i_1, \ldots, i_{n_1})$, $P_2\, \varphi(i'_1, \ldots, i'_{n_2})$, $Q_1\, \varphi(j_1, \ldots, j_{m_1})$, $Q_2\, \varphi(j'_1, \ldots, j'_{m_2})$ exist in $[a, c]$ and

$$P_1\, \varphi(i_1, \ldots, i_{n_1}) = P_2\, \varphi(i'_1, \ldots, i'_{n_2}), \tag{3}$$

$$Q_1\, \varphi(j_1, \ldots, j_{m_1}) \neq Q_2\, \varphi(j'_1, \ldots, j'_{m_2}). \tag{4}$$

Denote

$$u = Q_1\, \varphi(j_1, \ldots, j_{m_1}) \wedge Q_2\, \varphi(j'_1, \ldots, j'_{m_2}), \tag{3'}$$

$$v = Q_1\, \varphi(j_1, \ldots, j_{m_1}) \vee Q_2\, \varphi(j'_1, \ldots, j'_{m_2}). \tag{3''}$$

From (4) we obtain that

$$u \neq v. \tag{5}$$

Let $\varphi_1$ be a mapping of the set $I$ into $[a, b]$ defined by

$$\varphi_1(i) = \varphi(i) \wedge b \quad \text{for each} \quad i \in I.$$

Analogously we define a mapping $\varphi_2 : I \to [b, c]$ by putting

$$\varphi_2(i) = \varphi(i) \vee b \quad \text{for each} \quad i \in I.$$

Since $G$ is infinitely distributive, we have

$$b \wedge P_1\, \varphi(i_1, \ldots, i_{n_1}) = P_1(b \wedge \varphi(i_1, \ldots, i_{n_1})) = P_1\, \varphi_1(i_1, \ldots, i_{n_1}),$$
$$b \vee P_1\, \varphi(i_1, \ldots, i_{n_1}) = P_1(b \vee \varphi(i_1, \ldots, i_{n_1})) = P_1\, \varphi_2(i_1, \ldots, i_{n_1});$$

analogous relations hold for $P_2$, $Q_1$ and $Q_2$. Thus in view of (3) we obtain

$$P_1\, \varphi_1(i_1, \ldots, i_{n_1}) = P_2\, \varphi_1(i'_1, \ldots, i'_{n_2}), \tag{3.1}$$

$$P_1\, \varphi_2(i_1, \ldots, i_{n_1}) = P_2\, \varphi_2(i'_1, \ldots, i'_{n_2}). \tag{3.2}$$

As both lattices $[a, b]$ and $[b, c]$ fulfil the implication (2), we infer that

$$Q_1\, \varphi_1(j_1, \ldots, j_{m_1}) = Q_2\, \varphi_1(j'_1, \ldots, j'_{m_2}), \tag{4.1}$$

$$Q_1\, \varphi_2(j_1, \ldots, j_{m_1}) = Q_2\, \varphi_2(j'_1, \ldots, j'_{m_2}) \tag{4.2}$$

hold. Denote

$$u_1 = u \wedge b, \quad v_1 = v \wedge b, \quad u_2 = u \vee b, \quad v_2 = v \vee b.$$

From the definition of $u$, $v$, from (4.1) and in view of the infinite distributivity of $G$ we obtain

$$u_1 = v_1. \tag{5.1}$$

Similarly, by using (4.2) we get

$$u_2 = v_2 . \tag{5.2}$$

Since $G$ is a distributive lattice, (5.1) and (5.2) yield $u = v$, which contradicts (5). Hence $[a, c]$ fulfils the implication (2).

**Lemma 2.** *Let $0 \leqq a \in G$, $0 \leqq b \in G$, $[0, a]$, $[0, b] \in \mathscr{L}$. Then $[0, a + b] \in \mathscr{L}$.*

Proof. Since the interval $[a, a + b]$ is isomorphic with $[0, b]$, the assertion follows from Lemma 1.

Proof of Theorem 1:

We denote by $H(G)$ the set of all elements $g \in G$ that fulfil the following condition: there are elements $0 \leqq a_1 \in G$, $0 \leqq a_2 \in G$ with $-a_1 \leqq g \leqq a_2$ such that both $[0, a_1]$ and $[0, a_2]$ belong to $\mathscr{L}$. If $g \in H(G)$, then $-g \in H(G)$. Lemma 2 implies that $H(G)$ is closed with respect to the operation $+$, hence it is a subgroup of the group $G$. Moreover, from the definition of $H(G)$ and from Lemma 2 we infer that $H(G)$ is a sublattice of $G$. From this and from the obvious fact that $g_1 \in G$, $0 \leqq g_1 \leqq$ $\leqq g_2 \in H(G)$ implies $g_1 \in H(G)$ we obtain that $H(G)$ is a convex $l$-subgroup of $G$.

Let $H_1$ be a convex $l$-subgroup of $G$ and suppose that $H_1$ fulfils the implication (2). Let $0 \leqq h_1 \in H_1$. Then the interval $[0, h_1]$ fulfils the implication (2) as well, hence $h_1 \in H(G)$. From this we easily obtain $H_1 \subseteq H(G)$.

The fact that $H(G)$ is a convex $l$-subgroup of $G$ implies that if $\{a_i\} \subset H(G)$ and if $\sup \{a_i\} = h$ holds in $H(G)$, then $h$ is, at the same time, the least upper bound of the set $\{a_i\}$ in $G$ (and dually). Hence if $P\, x(i_1, \ldots, i_n)$ is a generalized lattice polynomial and the elements $a(i_1, \ldots, i_n)$, $a \in H(G)$ fulfil in $H(G)$ the relation

$$P\, a(i_1, \ldots, i_n) = a ,$$

then this relation holds also with respect to $G$.

Assume that $H(G)$ does not fulfil the implication (2). Then there exists a mapping $\varphi : I \to H(G)$ such that (under the same notation as in Lemma 1) the relations (3), (4) and (5) hold in $H(G)$. In view of the above remark, these relations are valid in $G$ as well. Put $v_1 = v - u$. Then $0 < v_1 \in H(G)$ and thus the interval $[0, v_1]$ fulfils the implication (2). As the intervals $[u, v]$ and $[0, v_1]$ are isomorphic, $[u, v]$ satisfies the implication (2) as well. Consider the mapping $\psi$ of $I$ into $[u, v]$ defined by

$$\psi(i) = (\varphi(i) \vee u) \wedge v .$$

Then we obtain from (3)

$$P_1\, \psi(i_1, \ldots, i_{n_1}) = P_2\, \psi(i'_1, \ldots, i'_{n_2}) . \tag{6}$$

Moreover, from (3′) and (3″) we infer

$$u = Q_1\, \psi(j_1, \ldots, j_{m_1}) \wedge Q_2\, \psi(j'_1, \ldots, j'_{m_2}) ,$$
$$v = Q_1\, \psi(j_1, \ldots, j_{m_1}) \vee Q_2\, \psi(j'_1, \ldots, j'_{m_2}) .$$

Since $u < v$, we must have

$$Q_1\, \psi(j_1, \ldots, j_{m_1}) \neq Q_2\, \psi(j'_1, \ldots, j'_{m_2}) . \tag{7}$$

According to (6) and (7), the interval $[u, v]$ does not fulfil the implication (2), which is a contradiction. Hence $H(G)$ fulfils the implication (2).

Let $f$ be an automorphism of the lattice ordered group $G$. Denote $H_1 = f(H(G))$. Then $H_1$ fulfils the implication (2), hence $H_1 \subseteq H(G)$. Similarly, $H(G) \subseteq H_1$. The proof is complete.

An $l$-subgroup $A$ of $G$ is said to be closed in $G$ if, whenever $\{a_i\} \subseteq A$, $g \in G$ and $\bigvee a_i = g$ holds in $G$, then $g \in A$. If this is valid, then the corresponding dual condition also holds for $A$. It is easy to verify that an $l$-subgroup $B$ of $G$ is closed in $G$ if, whenever $\{b_i\}$ is a set of positive elements of $G$ $g \in G$ and $\bigvee b_i = g$ holds in $G$, then $g \in B$.

**Theorem 2.** *For each lattice ordered group $G$, $H(G)$ is a closed $l$-subgroup of $G$.*

Proof. Let $\{b_k\}_{k \in K} \subseteq H(G)$, $0 \leqq b_k$ for each $k \in K$, $g \in G$. Let $g = \bigvee b_k$ be valid *in $G$. Assume that $g$ does not belong to $H(G)$. Thus the interval $[0, g]$ does not fulfil the implication (2). Hence there is a mapping $\varphi$ of $I$ into $[0, g]$ such that (under the same notation as in Lemma 1) the relations (3), (4) and (5) are valid in $[0, g]$ (since $[0, g]$ is a closed sublattice of $G$, these relations hold also in $G$).* We may assume that $\varphi(i) \in [u, v]$ is valid for each $i \in I$ (namely, the elements $\varphi(i)$ can be replaced by $(\varphi(i) \vee u) \wedge v$ in the same way as we did in the proof of Theorem 1). For each $k \in K$ we denote $u_k = u \wedge b_k$, $v_k = v \wedge b_k$. In view of the infinite distributivity of $G$ we have

$$u = \bigvee_{k \in K} u_k\,, \quad v = \bigvee_{k \in K} v_k\,.$$

From this and from $u < v$ it follows that there exists $k \in K$ with

$$u_k < v_k\,. \tag{8}$$

Let this $k$ be fixed. Since $u_k, v_k \in H(G)$, the interval $[u_k, v_k]$ fulfils the implication (2).

Consider the mapping $\psi : I \to [u_k, v_k]$ defined by

$$\psi(i) = \varphi(i) \wedge b_k \quad \text{for each} \quad i \in I\,.$$

From (3), (3′) and (3″) we obtain

$$P_1\, \psi(i_1, \ldots, i_{n_1}) = P_2\, \psi(i'_1, \ldots, i'_{n_2})\,,$$
$$u_k = Q_1\, \psi(j_1, \ldots, j_{m_1}) \wedge Q_2\, \psi(j'_1, \ldots, j'_{m_2})\,,$$
$$v_k = Q_1\, \psi(j_1, \ldots, j_{m_1}) \vee Q_2\, \psi(j'_1, \ldots, j'_{m_2})\,.$$

Thus according to (8), the interval $[u_k, v_k]$ does not fulfil the implication (2), which is a contradiction.

An interval $[a, b]$ of $G$ is said to be nontrivial if $a < b$. For each $X \subseteq G$, the polar $X^\delta$ is defined by

$$X^\delta = \{g \in G : |g| \wedge |x| = 0 \text{ for each } x \in X\} .$$

(Cf. ŠIK [10].) Each polar is a closed convex $l$-subgroup of $G$.

**Theorem 3.** *For each lattice ordered group $G$ there exists a convex $l$-subgroup $H'(G)$ of $G$ such that:*

(i) *each nontrivial interval of $H'(G)$ fails to fulfil the implication* (2);

(ii) *if $H_1$ is a convex $l$-subgroup of $G$ such that each nontrivial interval of $H_1$ fails to fulfil the implication* (2), *then $H_1 \subseteq H'(G)$;*

(iii) *$H'(G)$ is a closed $l$-subgroup of $G$;*

(iv) *$f(H'(G)) = H'(G)$ for each automorphism $f$ of the lattice ordered group $G$.*

Proof. Put

$$H'(G) = (H(G))^\delta .$$

Then $H'(G)$ is a convex closed $l$-subgroup of $G$. Let $[a, b]$ be a nontrivial interval of $H'(G)$. Assume that $[a, b]$ fulfils the implication (2). Then $0 < b - a \in H'(G)$ and $[0, b - a]$ fulfils the implication (2). Thus $b - a \in H(G)$ which is a contradiction, since $H(G) \cap H'(G) = \{0\}$.

Let $H_1$ be as in (ii), $0 \leqq h_1 \in H_1$. Further let $0 \leqq h \in H(G)$. Denote $h \wedge h_1 = c$. If $c > 0$, then $[0, c] \subseteq H(G)$, hence $[0, c]$ fulfils (2); at the same time, $[0, c] \subseteq H_1$, hence $[0, c]$ fails to fulfil (2), which is a contradiction. Therefore $c = 0$ and this yields $h_1 \in H'(G)$.

Let $f$ be an automorphism of the lattice ordered group $G$. According to Theorem 1 we have $f(H(G)) = H(G)$, whence $f((H(G))^\delta) = (H(G))^\delta$.

**Remark 1.** From (i) it follows that the $l$-subgroup $H'(G)$ is uniquely determined. Moreover, $H'(G)$ can be defined as the set of all elements $g \in G$ that fulfil the following condition:

($\alpha$) Each nontrivial subinterval of the interval $[0, |g|]$ fails to satisfy the implication (2).

In fact, if $g \in H'(G)$, then ($\alpha$) is valid according to (i). Let $g \in G$ and suppose that $g$ fulfils ($\alpha$). Then clearly $|g| \wedge |h| = 0$ for each $h \in H(G)$, hence $g \in (H(G))^\delta = H'(G)$.

We denote by $\mathscr{K}_1$ the class of all lattice ordered groups $G$ such that the lattice $(G; \leqq)$ fulfils the implication (2). Further, let $\mathscr{K}_2$ be the class of all lattice ordered groups $G_1$ having the property that for each $g \in G_1$ the condition ($\alpha$) is valid. Both the classes $\mathscr{K}_1$ and $\mathscr{K}_2$ fulfil the conditions (i) and (ii) from the definition of a radical class (cf. Introduction); in view of Theorem 1 and Theorem 3, the condition (iii) of this definition also holds. Hence we obtain the following

**Corollary 1.** *$\mathscr{K}_1$ and $\mathscr{K}_2$ are radical classes.*

**Theorem 4.** *For each lattice ordered group $G$, $H(G)$ is a polar of $G$ (i.e., $(H(G))^{\delta\delta} = H(G)$).*

Proof. Clearly $H(G) \subseteq (H(G))^{\delta\delta}$. Let $0 < g \in (H(G))^{\delta\delta}$ and assume that $g$ does not belong to $H(G)$. Then the interval $[0, g]$ does not fulfil the implication (2). Hence there exists a mapping $\varphi : I \to [0, g]$ such that (under the same notation as in the proof of Lemma 1) the relation (5) is valid. Let $[u_1, v_1]$ be a nontrivial subinterval of $[u, v]$. Consider the mapping $\psi : I \to [u_1, v_1]$ defined by

$$\psi(i) = (\varphi(i) \wedge v_1) \vee u_1 \quad \text{for each} \quad i \in I .$$

Now by an analogous reasoning as in the proof of Theorem 2 we obtain that the interval $[u_1, v_1]$ does not satisfy the implication (2). Hence each nontrivial subinterval of $[u, v]$ fails to satisfy the implication (2); because $[0, v - u]$ is isomorphic with $[u, v]$, the same holds for the interval $[0, v - u]$. Thus according to Remark 1, $v - u$ belongs to $H'(G)$. On the other hand, $v - u \in H(G)$; since $v - u > 0$, we have a contradiction. Therefore $g \in H(G)$ and hence $(H(G))^{\delta\delta} \subseteq H(G)$.

Since each polar of a complete lattice ordered group is a direct factor, we obtain

**Corollary 2.** *For each complete lattice ordered group $G$ we have $G = H(G) \times H'(G)$.*

It can be shown by examples that the assertion of Corollary 2 need not hold for noncomplete lattice ordered groups.

**Remark 2.** If $P_1\, x(i_1, \ldots, i_{n_1}) = x_1$, $P_2\, x(i'_1, \ldots, i'_{n_2}) = x_1$, then Theorems 1, 2 and 3 turn out to be assertions concerning lattice ordered groups that fulfil the identity $Q_1\, x(j_1, \ldots, j_{m_1}) \equiv Q_2\, x(j'_1, \ldots, j'_{m_2})$.

**Remark 3.** Let $K_1$ and $K_2$ be sets of indices and let $P_1^{k_1}, P_2^{k_1} (k_1 \in K_1)$, $Q_1^{k_2}, Q_2^{k_2} (k_2 \in K_2)$ be generalized lattice polynomials. The above methods and results remain valid (under more complicated notation) also for the implication (2′) that we obtain from (2) by replacing the left hand side of (2) by the conjunction of all relations $P_1^{k_1} = P_2^{k_1} (k_1 \in K_1)$ and the right hand side of (2) by the conjunction of all relations $Q_1^{k_2} = Q_2^{k_2} (k_2 \in K_2)$.

**Remark 4.** The above investigations remain valid also in the case when we define the notion of a generalized polynomial by the expression

$$S_{i_1 \in I_1} S_{i_2 \in I_2(i_1)}, \ldots S_{i_n \in I_n(i_1, i_2, \ldots, i_{n-1})}\, x(i_1, \ldots, i_n)$$

(where $I_1, I_2(i_1), \ldots, I_n(i_1, i_2, \ldots, i_{n-1})$ are nonempty sets of indices) instead of the expression (1).

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