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ON THE GEOMETRY OF A PARTIAL PRODUCT STRUCTURE

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In his paper [1], A. ŠVEC studied a partial product structure, i.e., a 3-dimensional differentiable manifold with two given tangents at each of its points. In [2], he applied his results to the study of real hypersurfaces of $\mathbb{C}^2$. In what follows, we explain the geometrical meaning of his relative invariants as well as present some other properties of these important structures.

1. Let M be a 3-dimensional differentiable manifold; at each of its points let two distinct tangent lines t_1, t_2 be given so that the field of the tangent planes τ spanned by t_1, t_2 is non-integrable. Let us consider, at a given point $m \in M$, all tangent frames $\{v_1, v_2, v_3\}$ such that $v_1 \in t_1, v_2 \in t_2$; $\{v_1, v_2, v_3\}$ and $\{w_1, w_2, w_3\}$ being two such frames, we have

$$(1) \qquad w_1 = \alpha v_1 , \quad w_2 = \beta v_2 , \quad w_3 = \gamma v_1 + \delta v_2 + \varphi v_3 ; \quad \alpha\beta\varphi \neq 0 .$$

Thus the given structure is a G-structure B_G, G being the group of non-singular matrices of the form

$$(2) \qquad \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ \gamma & \delta & \varphi \end{pmatrix} .$$

Let $\{v_1, v_2, v_3\}$ be a section of B_G. Then we are in the position, see [1], to prove the existence of sections such that

$$(3) \qquad [v_1, v_2] = v_3 , \quad [v_1, v_3] = av_1 + bv_2 , \quad [v_2, v_3] = cv_1 - av_2 ,$$

$$(4) \qquad v_1 c - v_2 a = 0 , \quad v_2 b + v_1 a = 0 ;$$

the sections satisfying equations of this type will be called special sections of our G-structure B_G. Let $\{w_1, w_2, w_3\}$ be another special section satisfying

$$(5) \qquad [w_1, w_2] = w_3 , \quad [w_1, w_3] = Aw_1 + Bw_2 , \quad [w_2, w_3] = Cw_1 - Aw_2 ;$$
$$w_1 C - w_2 A = 0 , \quad w_2 B + w_1 A = 0 .$$

If we have (1) over all of M, we get

$$(6) \qquad \varphi = \alpha\beta ;$$

$$(7) \qquad w_1\alpha = -2\alpha\beta^{-1}\delta , \quad w_2\alpha = -\gamma ; \quad w_1\beta = \delta , \quad w_2\beta = 2\alpha^{-1}\beta\gamma ;$$

$$w_2\gamma = \alpha C - \alpha\beta^2 c , \quad w_1\delta = \beta B - \alpha^2\beta b ,$$

$$w_1\gamma - w_3\alpha = \alpha A - \alpha^2\beta a , \quad w_2\delta - w_3\beta = -\beta A + \alpha\beta^2 a .$$

We then have, see [1], the following

Lemma. *The integrability conditions of the system* (6) + (7) *imply*

$$(8) \qquad w_1w_1A - 2w_3B - 3AB = \alpha^3\beta(v_1v_1a - 2v_3b - 3ab) ,$$

$$w_2w_2A - 2w_3C + 3AC = \alpha\beta^3(v_2v_2a - 2v_3c + 3ac) ;$$

thus the expressions

$$(9) \qquad R = v_1v_1a - 2v_3b - 3ab , \quad S = v_2v_2a - 2v_3c + 3ac$$

are relative invariants of the G-structure B_G.

Our task is to explain their geometrical meaning.

Theorem 1. *Let $m \in M$ be a fixed point. Then there is (in a neighborhood of m) a special section of B_G such that we have, at m,*

$$(10) \qquad [v_1, [v_1, v_2]] = [v_2, [v_1, v_2]] = 0 ,$$

$$[v_1, [v_1, [v_1, v_2]]] = [v_2, [v_1, [v_1, v_2]]] = [v_2, [v_2, [v_1, v_2]]] = 0 ,$$

$$[v_1, [v_1, [v_1, [v_1, v_2]]]] = [v_2, [v_2, [v_1[v_1, v_2]]]] =$$

$$= [v_1, [v_2, [v_1[v_1, v_2]]]] = [v_2, [v_2, [v_2, [v_1, v_2]]]] = 0 ,$$

$$(11) \qquad [v_2, [v_1, [v_1, [v_1, v_2]]]] = \tfrac{1}{2}Rv_2 , \quad [v_1, [v_2, [v_2, [v_1, v_2]]]] = -\tfrac{1}{2}Sv_1 .$$

Proof. Let $\{v_1, v_2, v_3\}$ be a special section of B_G satisfying (3), and let (1) be another special section. Then we have

$$(12) \qquad [w_1, w_2] = -w_2\alpha v_1 + w_1\beta v_2 + \alpha\beta[v_1, v_2] ,$$

$$(13) \qquad [w_1, [w_1, w_2]] = (-2w_1w_2\alpha + w_2w_1\alpha + \alpha^2\beta a)\, v_1 + (w_1w_1\beta + \alpha^2\beta b)\, v_2 +$$

$$+ (\beta w_1\alpha + 2\alpha w_1\beta)\, [v_1, v_2] ,$$

$$[w_2, [w_1, w_2]] = (-w_2w_2\alpha + \alpha\beta^2 c)\, v_1 + (-w_1w_2\beta + 2w_2w_1\beta - \alpha\beta^2 a)\, v_2 +$$

$$+ (2\beta w_2\alpha + \alpha w_2\beta)\, [v_1, v_2] ,$$

(14) $[w_1, [w_1, [w_1, w_2]]] = \{-3w_1w_1w_2\alpha + 3w_1w_2w_1\alpha - w_2w_1w_1\alpha +$
$+ 3\alpha(\beta w_1 + \alpha w_1\beta)\, a + \alpha^2\beta w_1 a\}\, v_1 + \{w_1w_1w_1\beta + 3\alpha(\beta w_1\alpha + \alpha w_1\beta)\, b +$
$+ \alpha^2\beta w_1 b\}\, v_2 + (\beta w_1w_1\alpha + 3\alpha w_1w_1\beta + 3w_1\alpha w_1\beta + \alpha^3\beta b)\, [v_1, v_2]\,,$

$[w_2, [w_1, [w_1, w_2]]] = \{-2w_2w_1w_2\alpha + w_2w_2w_1\alpha + \alpha(2\beta w_2\alpha + \alpha w_2\beta)\, a +$
$+ \beta(\beta w_1\alpha + 2\alpha w_1\beta)\, c + \alpha^2\beta w_2 a\}\, v_1 + \{2w_1w_2w_1\beta - w_1w_1w_2\beta -$
$- \beta(\beta w_1\alpha + 2\alpha w_1\beta)\, a + \alpha(\beta w_2\alpha + \alpha w_2\beta)\, b + \alpha^2\beta w_2 b\}\, v_2 + (2\beta w_1w_2\alpha +$
$+ 2\alpha w_2w_1\beta + w_1\alpha w_2\beta + 2w_2\alpha w_1\beta - \alpha^2\beta^2 a)\, [v_1, v_2]\,,$

$[w_2, [w_2, [w_1, w_2]]] = \{-w_2w_2w_2\alpha + 3\beta(\beta w_2\alpha + \alpha w_2\beta)\, c + \alpha\beta^2 w_2 c\}\, v_1 +$
$+ \{w_1w_2w_2\beta - 3w_2w_1w_2\beta + 3w_2w_2w_1\beta - 3\beta(\beta w_2\alpha + \alpha w_2\beta)\, a - \alpha\beta^2 w_2 a\}\, v_2 +$
$+ (3\beta w_2w_2\alpha + \alpha w_2w_2\beta + 3w_2\alpha w_2\beta - \alpha\beta^2 c)\, [v_1, v_2]\,;$

(15) $[w_1, [w_1, [w_1, [w_1, w_2]]]] = \{-4w_1w_1w_1w_2\alpha + 6w_1w_1w_2w_1\alpha -$
$- 4w_1w_2w_1w_1\alpha + w_2w_1w_1w_1\alpha + (\cdot)\, a + (\cdot)\, w_1 a + \alpha^2\beta w_1w_1 a)\}\, v_1 +$
$+ \{w_1w_1w_1w_1\beta + (\cdot)\, b + (\cdot)\, w_1 b + \alpha^2\beta w_1w_1 b\}\, v_2 +$
$+ \{\beta w_1w_1w_1\alpha + 4\alpha w_1w_1w_1\beta + 4w_1\beta w_1w_1\alpha + 7w_1\alpha w_1w_1\beta + (\cdot)\, b +$
$+ (\cdot)\, w_1 b\}\, [v_1, v_2]\,,$

$[w_2, [w_1, [w_1[w_1, w_2]]]] = \{-3w_2w_1w_1w_2\alpha + 3w_2w_1w_2w_1\alpha -$
$- w_2w_2w_1w_1\alpha + (\cdot)\, a + (\cdot)\, c + (\cdot)\, w_1 a + (\cdot)\, w_2 a + \alpha^2\beta w_2w_1 a\}\, v_1 +$
$+ \{-w_1w_1w_1w_2\beta + 3w_1w_1w_2w_1\beta - 3w_1w_2w_1w_1\beta + 2w_2w_1w_1w_1\beta +$
$+ (\cdot)\, a + (\cdot)\, b + (\cdot)\, w_1 b + (\cdot)\, w_2 b + \alpha^2\beta w_2w_1 b\}\, v_2 +$
$+ \{3\beta w_1w_1w_2\alpha - 3\beta w_1w_2w_1\alpha + 2\beta w_2w_1w_1\alpha + 3\alpha w_2w_1w_1\beta + w_2\beta w_1w_1\alpha +$
$+ 3w_1\beta w_2w_1\alpha + 3w_2\alpha w_1w_1\beta + 3w_1\alpha w_2w_1\beta +$
$+ (\cdot)\, a + (\cdot)\, b + (\cdot)\, w_1 a + (\cdot)\, w_2 b\}\, [v_1, v_2]\,,$

$[w_1, [w_2, [w_1, [w_1, w_2]]]] = \{w_1w_1w_2w_2\alpha - 4w_1w_2w_1w_2\alpha + w_1w_2w_2w_1\alpha +$
$+ 2w_2w_1w_2w_1\alpha - w_2w_2w_1w_1\alpha + (\cdot)\, a + (\cdot)\, c + (\cdot)\, w_1 a + (\cdot)\, w_2 a +$
$+ (\cdot)\, w_1 c + \alpha^2\beta w_1w_2 a\}\, v_1 +$
$+ \{-w_1w_1w_1w_2\beta + 2w_1w_1w_2w_1\beta + (\cdot)\, a + (\cdot)\, b + (\cdot)\, w_1 a + (\cdot)\, w_1 b +$
$+ (\cdot)\, w_2 b + \alpha^2\beta w_1w_2 b\}\, v_2 + \{2\beta w_1w_1w_2\alpha - \alpha w_1w_1w_2\beta + 4\alpha w_1w_2w_1\beta +$
$+ w_2\beta w_1w_1\alpha + 4w_1\beta w_1w_2\alpha + 2w_2\alpha w_1w_1\beta + w_1\alpha w_1w_2\beta + 2w_1\alpha w_2w_1\beta +$
$+ (\cdot)\, a + (\cdot)\, b + (\cdot)\, w_1 a + (\cdot)\, w_2 b\}\, [v_1, v_2]\,,$

$$[w_2,[w_2,[w_1,[w_1,w_2]]]] = \{-2w_2w_2w_1w_2\alpha + w_2w_2w_2w_1\alpha +$$
$$+ (\cdot)\, a + (\cdot)\, c + (\cdot)\, w_2a + (\cdot)\, w_2c + \alpha^2\beta w_2w_2a\}\, v_1 +$$
$$+ \{w_1w_1w_2w_2\beta - 2w_1w_2w_1w_2\beta - w_2w_1w_1w_2\beta + 4w_2w_1w_2w_1\beta -$$
$$- w_2w_2w_1w_1\beta + (\cdot)\, a + (\cdot)\, b + (\cdot)\, w_2a + (\cdot)\, w_2b + \alpha^2\beta w_2w_2b\}\, v_2 +$$
$$+ \{4\beta w_2w_1w_2\alpha + 2\alpha w_2w_2w_1\beta - \beta w_2w_2w_1\alpha + 2w_2\beta w_1w_2\alpha + w_2\beta w_2w_1\alpha +$$
$$+ 2w_1\beta w_2w_2\alpha + 4w_2\alpha w_2w_1\beta + w_1\alpha w_2w_2\beta + (\cdot)\, a + (\cdot)\, c + (\cdot)\, w_2a\}\, [v_1, v_2]\,,$$

$$[w_1,[w_2,[w_2,[w_1,w_2]]]] = \{-2w_1w_2w_2w_2\alpha + 3w_2w_1w_2w_2\alpha -$$
$$- 3w_2w_2w_1w_2\alpha + w_2w_2w_2w_1\alpha + (\cdot)\, a + (\cdot)\, c + (\cdot)\, w_1c + (\cdot)\, w_2c +$$
$$+ \alpha\beta^2 w_1w_2c\}\, v_1 + \{w_1w_1w_2w_2\beta - 3w_1w_2w_1w_2\beta + 3w_1w_2w_2w_1\beta + (\cdot)\, a +$$
$$+ (\cdot)\, b + (\cdot)\, w_1a + (\cdot)\, w_2a - \alpha\beta^2 w_1w_2a\}\, v_2 +$$
$$+ \{3\beta w_1w_2w_2\alpha + 2\alpha w_1w_2w_2\beta - 3\alpha w_2w_1w_2\beta + 3\alpha w_2w_2w_1\beta +$$
$$+ 3w_2\beta w_1w_2\alpha + 3w_1\beta w_2w_2\alpha + 3w_2\alpha w_1w_2\beta + w_1\alpha w_2w_2\beta + (\cdot)\, a + (\cdot)\, c +$$
$$+ (\cdot)\, w_2a + (\cdot)\, w_1c\}\, [v_1, v_2]\,,$$

$$[w_2,[w_2,[w_2,[w_1,w_2]]]] = \{-w_2w_2w_2w_2\alpha + (\cdot)\, c + (\cdot)\, w_2c +$$
$$+ \alpha\beta^2 w_2w_2c\}\, v_1 + \{-w_1w_2w_2w_2\beta + 4w_2w_1w_2w_2\beta - 6w_2w_2w_1w_2\beta +$$
$$+ 4w_2w_2w_2w_1\beta + (\cdot)\, a + (\cdot)\, w_2a - \alpha\beta^2 w_2w_2a\}\, v_2 + 4\beta w_2w_2w_2\alpha +$$
$$+ \alpha w_2w_2w_2\beta + 6w_2\beta w_2w_2\alpha + 4w_2\alpha w_2w_2\beta + (\cdot)\, c + (\cdot)\, w_2c\}\, [v_1, v_2].$$

From (7_{1-4}), we get

(16) $$\beta w_1\alpha + 2\alpha w_1\beta = 0\,, \quad 2\beta w_2\alpha + \alpha w_2\beta = 0\,;$$

(17) $$\beta w_1w_1\alpha + 2\alpha w_1w_1\beta + 3w_1\alpha w_1\beta = 0\,,$$
$$\beta w_2w_1\alpha + 2\alpha w_2w_1\beta + w_1\alpha w_2\beta + 2w_2\alpha w_1\beta = 0\,,$$
$$2\beta w_1w_2\alpha + \alpha w_1w_2\beta + 2w_2\alpha w_1\beta + w_1\alpha w_2\beta = 0\,,$$
$$2\beta w_2w_2\alpha + \alpha w_2w_2\beta + 3w_2\alpha w_2\beta = 0\,;$$

(18) $$\beta w_1w_1w_1\alpha + 2\alpha w_1w_1w_1\beta + 4w_1\beta w_1w_1\alpha + 5w_1\alpha w_1w_1\beta = 0\,,$$
$$\beta w_2w_1w_1\alpha + 2\alpha w_2w_1w_1\beta + w_2\beta w_1w_1\alpha + 3w_1\beta w_2w_1\alpha +$$
$$+ 2w_2\alpha w_1w_1\beta + 3w_1\alpha w_2w_1\beta = 0\,,$$
$$\beta w_1w_2w_1\alpha + 2\alpha w_1w_2w_1\beta + w_2\beta w_1w_1\alpha + 2w_1\beta w_1w_2\alpha + w_1\beta w_2w_1\alpha +$$
$$+ 2w_2\alpha w_1w_1\beta + w_1\alpha w_1w_2\beta + 2w_1\alpha w_2w_1\mathrm{b} = 0\,,$$
$$\beta w_2w_2w_1\alpha + 2\alpha w_2w_2w_1\beta + 2w_2\beta w_2w_1\alpha + 2w_1\beta w_2w_2\alpha +$$
$$+ 4w_2\alpha w_2w_1\beta + w_1\alpha w_2w_2\beta = 0\,,$$

$$2\beta w_1 w_1 w_2\alpha + \alpha w_1 w_1 w_2\beta + w_2\beta w_1 w_1\alpha + 4w_1\beta w_1 w_2\alpha +$$
$$+ 2w_2\alpha w_1 w_1\beta + 2w_1\alpha w_1 w_2\beta = 0\,,$$
$$2\beta w_2 w_1 w_2\alpha + \alpha w_2 w_1 w_2\beta + 2w_2\beta w_1 w_2\alpha + w_2\beta w_2 w_1\alpha + 2w_1\beta w_2 w_2\alpha +$$
$$+ w_2\alpha w_1 w_2\beta + w_1\alpha w_2 w_2\beta + 2w_2\alpha w_2 w_1\beta = 0\,,$$
$$2\beta w_1 w_2 w_2\alpha + \alpha w_1 w_2 w_2\beta + 3w_2\beta w_1 w_2\alpha + 2w_1\beta w_2 w_2\alpha +$$
$$+ 3w_2\alpha w_1 w_2\beta + w_1\alpha w_2 w_2\beta = 0\,,$$
$$2\beta w_2 w_2 w_2\alpha + \alpha w_2 w_2 w_2\beta + 5w_2\beta w_2 w_2\alpha + 4w_2\alpha w_2 w_2\beta = 0\,;$$

(19)
$$\beta w_1 w_1 w_1 w_1\alpha + 2\alpha w_1 w_1 w_1 w_1\beta + 5w_1\beta w_1 w_1 w_1\alpha + 7w_1\alpha w_1 w_1 w_1\beta +$$
$$+ 9w_1 w_1\alpha w_1 w_1\beta = 0\,,$$
$$\beta w_2 w_1 w_1 w_1\alpha + 2\alpha w_2 w_1 w_1 w_1\beta + w_2\beta w_1 w_1 w_1\alpha + 4w_1\beta w_2 w_1 w_1\alpha +$$
$$+ 2w_2\alpha w_1 w_1 w_1\beta + 5w_1\alpha w_2 w_1 w_1\beta + 4w_1 w_1\alpha w_2 w_1\beta + 5w_2 w_1\alpha w_1 w_1\beta = 0\,,$$
$$\beta w_1 w_2 w_1 w_1\alpha + 2\alpha w_1 w_2 w_1 w_1\beta + w_2\beta w_1 w_1 w_1\alpha + 3w_1\beta w_1 w_2 w_1\alpha +$$
$$+ w_1\beta w_2 w_1 w_1\alpha + 2w_2\alpha w_1 w_1 w_1\beta + 3w_1\alpha w_1 w_2 w_1\beta + 2w_1\alpha w_2 w_1 w_1\beta +$$
$$+ w_1 w_1\alpha w_1 w_2\beta + 3w_1 w_1\alpha w_2 w_1\beta + 2w_1 w_2\alpha w_1 w_1\beta + 3w_2 w_1\alpha w_1 w_1\beta = 0\,,$$
$$\beta w_2 w_2 w_1 w_1\alpha + 2\alpha w_2 w_2 w_1 w_1\beta + 2w_2\beta w_2 w_1 w_1\alpha + 3w_1\beta w_2 w_2 w_1\alpha +$$
$$+ 4w_2\alpha w_2 w_1 w_1\beta + 3w_1\alpha w_2 w_2 w_1\beta + w_1 w_1\alpha w_2 w_2\beta + 6w_2 w_1\alpha w_2 w_1\beta +$$
$$+ 2w_2 w_2\alpha w_1 w_1\beta = 0\,,$$
$$\beta w_1 w_1 w_2 w_1\alpha + 2\alpha w_1 w_1 w_2 w_1\beta + w_2\beta w_1 w_1 w_1\alpha + 2w_1\beta w_1 w_1 w_2\alpha +$$
$$+ 2w_1\beta w_1 w_2 w_1\alpha + 2w_2\alpha w_1 w_1 w_1\beta + w_1\alpha w_1 w_1 w_2\beta + 4w_1\alpha w_1 w_2 w_1\beta +$$
$$+ 2w_1 w_1\alpha w_1 w_2\beta + 2w_1 w_1\alpha w_2 w_1\beta + 4w_1 w_2\alpha w_1 w_1\beta + w_2 w_1\alpha w_1 w_1\beta = 0\,,$$
$$\beta w_2 w_1 w_2 w_1\alpha + 2\alpha w_2 w_1 w_2 w_1\beta + w_2\beta w_1 w_2 w_1\alpha + w_2\beta w_2 w_1 w_1\alpha +$$
$$+ 2w_1\beta w_2 w_1 w_2\alpha + w_1\beta w_2 w_2 w_1\alpha + 2w_2\alpha w_1 w_2 w_1\beta + 2w_2\alpha w_2 w_1 w_1\beta +$$
$$+ w_1\alpha w_2 w_1 w_2\beta + 2w_1\alpha w_2 w_2 w_1\beta + w_1 w_1\alpha w_2 w_2\beta + 2w_1 w_2\alpha w_2 w_1\beta +$$
$$+ w_2 w_1\alpha w_1 w_2\beta + 3w_2 w_1\alpha w_2 w_1\beta + 2w_2 w_2\alpha w_1 w_1\beta = 0\,,$$
$$\beta w_1 w_2 w_2 w_1\alpha + 2\alpha w_1 w_2 w_2 w_1\beta + 2w_2\beta w_1 w_2 w_1\alpha + w_1\beta w_2 w_2 w_1\alpha +$$
$$+ 2w_1\beta w_1 w_2 w_2\alpha + 4w_2\alpha w_1 w_2 w_1\beta + w_1\alpha w_1 w_2 w_2\beta + 2w_1\alpha w_2 w_2 w_1\beta +$$
$$+ w_1 w_1\alpha w_2 w_2\beta + 4w_1 w_2\alpha w_2 w_1\beta + 2w_2 w_1\alpha w_1 w_2\beta + 2w_2 w_2\alpha w_1 w_1\beta = 0\,,$$

$$\beta w_2w_2w_2w_1\alpha + 2\alpha w_2w_2w_2w_1\beta + 3w_2\beta w_2w_2w_1\alpha + 2w_1\beta w_2w_2w_2\alpha +$$
$$+ 6w_2\alpha w_2w_2w_1\beta + w_1\alpha w_2w_2w_2\beta + 3w_2w_1\alpha w_2w_2\beta + 6w_2w_2\alpha w_2w_1\beta = 0\,,$$

$$2\beta w_1w_1w_1w_2\alpha + \alpha w_1w_1w_1w_2\beta + w_2\beta w_1w_1w_1\alpha + 6w_1\beta w_1w_1w_2\alpha +$$
$$+ 2w_2\alpha w_1w_1w_1\beta + 3w_1\alpha w_1w_1w_2\beta + 6w_1w_2\alpha w_1w_1\beta + 3w_1w_1\alpha w_1w_2\beta = 0\,,$$

$$2\beta w_2w_1w_1w_2\alpha + \alpha w_2w_1w_1w_2\beta + 2w_2\beta w_1w_1w_2\alpha + w_2\beta w_2w_1w_1\alpha +$$
$$4w_1\beta w_2w_1w_2\alpha + 2w_2\alpha w_2w_1w_1\beta + w_2\alpha w_1w_1w_2\beta + 2w_1\alpha w_2w_1w_2\beta +$$
$$+ 2w_2w_2\alpha w_1w_1\beta + 2w_2w_1\alpha w_1w_2\beta + 4w_1w_2\alpha w_2w_1\beta + w_1w_1\alpha w_2w_2\beta = 0\,,$$

$$2\beta w_1w_2w_1w_2\alpha + \beta w_1w_2w_1w_2\alpha + 2w_2\beta w_1w_1w_2\alpha + w_2\beta w_1w_2w_1\alpha +$$
$$+ 2w_1\beta w_1w_2w_2\alpha + 2w_1\beta w_2w_1w_2\alpha + w_2\alpha w_1w_1w_2\beta + 2w_2\alpha w_1w_2w_1\beta +$$
$$+ w_1\alpha w_1w_2w_2\beta + w_1\alpha w_2w_1w_2\beta + 2w_2w_2\alpha w_1w_1\beta + 3w_1w_2\alpha w_1w_2\beta +$$
$$+ w_2w_1\alpha w_1w_2\beta + 2w_1w_2\alpha w_2w_1\beta + w_1w_1\alpha w_2w_2\beta = 0\,,$$

$$2\beta w_2w_2w_1w_2\alpha + \alpha w_2w_2w_1w_2\beta + 4w_2\beta w_2w_1w_2\alpha + w_2\beta w_2w_2w_1\alpha +$$
$$+ 2w_1\beta w_2w_2w_2\alpha + 2w_2\alpha w_2w_1w_2\beta + 2w_2\alpha w_2w_2w_1\beta + w_1\alpha w_2w_2w_2\beta +$$
$$+ w_2w_2\alpha w_1w_2\beta + 4w_2w_2\alpha w_2w_1\beta + 2w_1w_2\alpha w_2w_2\beta + 2w_2w_1\alpha w_2w_2\beta = 0\,,$$

$$2\beta w_1w_1w_2w_2\alpha + \alpha w_1w_1w_2w_2\beta + 3w_2\beta w_1w_1w_2\alpha + 4w_1\beta w_1w_2w_2\alpha +$$
$$+ 3w_2\alpha w_1w_1w_2\beta + 2w_1\alpha w_1w_2w_2\beta + 2w_2w_2\alpha w_1w_1\beta + 6w_1w_2\alpha w_1w_2\beta +$$
$$+ w_1w_1\alpha w_2w_2\beta = 0\,,$$

$$2\beta w_2w_1w_2w_2\alpha + \alpha w_2w_1w_2w_2\beta + 2w_2\beta w_1w_2w_2\alpha + 3w_2\beta w_2w_1w_2\alpha +$$
$$+ 2w_1\beta w_2w_2w_2\alpha + w_2\alpha w_1w_2w_2\beta + 3w_2\alpha w_2w_1w_2\beta + w_1\alpha w_2w_2w_2\beta +$$
$$+ 3w_2w_2\alpha w_1w_2\beta + 2w_2w_2\alpha w_2w_1\beta + 3w_1w_2\alpha w_2w_2\beta + w_2w_1\alpha w_2w_2\beta = 0\,,$$

$$2\beta w_1w_2w_2w_2\alpha + \alpha w_1w_2w_2w_2\beta + 5w_2\beta w_1w_2w_2\alpha + 2w_1\beta w_2w_2w_2\alpha +$$
$$+ 4w_2\alpha w_1w_2w_2\beta + w_1\alpha w_2w_2w_2\beta + 5w_2w_2\alpha w_1w_2\beta + 4w_1w_2\alpha w_2w_2\beta = 0\,,$$

$$2\beta w_2w_2w_2w_2\alpha + \alpha w_2w_2w_2w_2\beta + 7w_2\beta w_2w_2w_2\alpha + 5w_2\alpha w_2w_2w_2\beta +$$
$$+ 9w_2w_2\alpha w_2w_2\beta = 0\,.$$

Let us recall the obvious identities

$$\text{(20)}\qquad \begin{aligned} [w_1, [w_1, w_2]] &= w_1w_1w_2 - 2w_1w_2w_1 + w_2w_1w_1\,,\\ [w_2, [w_1, w_2]] &= -w_1w_2w_2 + 2w_2w_1w_2 - w_2w_2w_1\,; \end{aligned}$$

$$(21)\qquad \begin{aligned} [w_1,[w_1,[w_1,w_2]]] &= \ \ w_1w_1w_1w_2 - 3w_1w_1w_2w_1 + \\ &\quad + 3w_1w_2w_1w_1 - w_2w_1w_1w_1\,, \\ [w_2,[w_1,[w_1,w_2]]] &= -2w_2w_1w_2w_1 + w_2w_2w_1w_1 - \\ &\quad - w_1w_1w_2w_2 + 2w_1w_2w_1w_2\,, \\ [w_2,[w_2,[w_1,w_2]]] &= \ \ w_1w_2w_2w_2 - 3w_2w_1w_2w_2 + \\ &\quad + 3w_2w_2w_1w_2 - w_2w_2w_2w_1\,. \end{aligned}$$

At the given point $m \in M$, (16) implies the existence of numbers $P_1, P_2 \in \mathbb{R}$ such that

$$(22)\qquad w_1\alpha = 2\alpha P_1\,,\quad w_2\alpha = -\alpha P_2\,,\quad w_1\beta = -\beta P_1\,,\quad w_2\beta = 2\beta P_2\,;$$

from (17) we infer the existence of $Q'_1, \ldots, Q'_4 \in \mathbb{R}$ satisfying

$$(23)\qquad \begin{aligned} w_1w_1\alpha &= \ \ \alpha(2Q'_1 + 3P_1^2)\,, & w_1w_1\beta &= -\tfrac{1}{2}\beta(2Q'_1 - 3P_1^2)\,, \\ w_1w_2\alpha &= \tfrac{1}{2}\alpha(2Q'_2 - 3P_1P_2)\,, & w_1w_2\beta &= -\beta(2Q'_2 + 3P_1P_2)\,, \\ w_2w_1\alpha &= \ \ \alpha(2Q'_3 - 3P_1P_2)\,, & w_2w_1\beta &= -\tfrac{1}{2}\beta(2Q'_3 + 3P_1P_2)\,, \\ w_2w_2\alpha &= \tfrac{1}{2}\alpha(2Q'_4 + 3P_2^2)\,, & w_2w_2\beta &= -\beta(2Q'_4 - 3P_2^2)\,. \end{aligned}$$

Inserting these expressions into (13), we get, at $m \in M$,

$$(24)\qquad \begin{aligned} [w_1,[w_1,w_2]] &= \ \ 2(Q'_3 - Q'_2 + \tfrac{1}{2}\alpha\beta a)\,w_1 - \tfrac{1}{2}(2Q'_1 - 3P_1^2 - 2\alpha^2 b)\,w_2\,, \\ [w_2,[w_1,w_2]] &= -\tfrac{1}{2}(2Q'_4 + 3P_2^2 - 2\beta^2 c)\,w_1 - 2(Q'_3 - Q'_2 + \tfrac{1}{2}\alpha\beta a)\,w_2\,. \end{aligned}$$

Choosing suitably $Q'_1, \ldots, Q'_4$ we can achieve, at $m \in M$,

$$(25)\qquad [w_1,[w_1,w_2]] = [w_2,[w_1,w_2]] = 0\,.$$

Let us restrict ourselves to special sections satisfying (10_1) at $m \in M$. Then

$$(26)\qquad Q'_1 = \tfrac{3}{2}P_1^2\,,\quad Q = Q'_2 = Q'_3\,,\quad Q'_4 = -\tfrac{3}{2}P_2^2\,,$$

and the equations (23) assume the form

$$(27)\qquad \begin{aligned} w_1w_1\alpha &= 6\alpha P_1^2\,, & w_1w_1\beta &= 0\,, \\ w_1w_2\alpha &= \tfrac{1}{2}\alpha(2Q - 3P_1P_2)\,, & w_1w_2\beta &= -\beta(2Q + 3P_1P_2)\,, \\ w_2w_1\alpha &= \ \ \alpha(2Q - 3P_1P_2)\,, & w_2w_1\beta &= -\tfrac{1}{2}\beta(2Q + 3P_1P_2)\,, \\ w_2w_2\alpha &= 0\,, & w_2w_2\beta &= \ \ 6\beta P_2^2\,. \end{aligned}$$

Taking into account (25) and (26), we get

$$(28)\qquad w_1w_1w_2\alpha - 2w_1w_2w_1\alpha + w_2w_1w_1\alpha = 0\,,$$
$$w_1w_1w_2\beta - 2w_1w_2w_1\beta + w_2w_1w_1\beta = 0\,,$$
$$-w_1w_2w_2\alpha + 2w_2w_1w_2\alpha - w_2w_2w_1\alpha = 0\,,$$
$$-w_1w_2w_2\beta + 2w_2w_1w_2\beta - w_2w_2w_1\beta = 0\,.$$

The general solution of the system (18) + (28) at $m \in M$ is given by

$$(29)\qquad w_1w_1w_1\alpha = 2\alpha(R_1' + 6P_1^3)\,,$$
$$w_1w_1w_1\beta = -\beta(R_1' - 6P_1^3)\,,$$
$$w_1w_1w_2\alpha = -4\alpha(P_1^2P_2 - P_1Q)\,,$$
$$w_1w_1w_2\beta = 2\beta(P_1^2P_2 + 2P_1Q)\,,$$
$$w_1w_2w_1\alpha = \alpha(2R_2' - 5P_1^2P_2 + 6P_1Q)\,,$$
$$w_1w_2w_1\beta = -\beta(R_2' + \tfrac{1}{2}P_1^2P_2 - 3P_1Q)\,,$$
$$w_1w_2w_2\alpha = \alpha(2R_3' - 3P_1P_2^2 - 2P_2Q)\,,$$
$$w_1w_2w_2\beta = -2\beta(2R_3' + 3P_1P_2^2 + 4P_2Q)\,,$$
$$w_2w_1w_1\alpha = 2\alpha(2R_2' - 3P_1^2P_2 + 4P_1Q)\,,$$
$$w_2w_1w_1\beta = -\beta(2R_2' + 3P_1^2P_2 - 2P_1Q)\,,$$
$$w_2w_1w_2\alpha = \alpha(R_3' - \tfrac{1}{2}P_1P_2^2 - 3P_2Q)\,,$$
$$w_2w_1w_2\beta = -\beta(2R_3' + 5P_1P_2^2 + 6P_2Q)\,,$$
$$w_2w_2w_1\alpha = 2\alpha(P_1P_2^2 - 2P_2Q)\,,$$
$$w_2w_2w_1\beta = -4\beta(P_1P_2^2 + P_2Q)\,,$$
$$w_2w_2w_2\alpha = \alpha(R_4' + 6P_2^3)\,,$$
$$w_2w_2w_2\beta = -2\beta(R_4' - 6P_2^3)\,.$$

Inserting this into (14), we get

$$(30)\qquad [w_1, [w_1, [w_1, w_2]]] = (2R_2' + 3P_1^2P_2 - 2P_1Q + \alpha^2\beta v_1 a)\, w_1 + $$
$$+ (-R_1' + 6P_1^3 + \alpha^3 v_1 b)\, w_2\,,$$
$$[w_2, [w_1, [w_1, w_2]]] = (-2R_3' + 3P_1P_2^2 + 2P_2Q + \alpha\beta^2 v_2 a)\, w_1 +$$
$$+ (-2R_2' - 3P_1^2P_2 + 2P_1Q + \alpha^2\beta v_2 b)\, w_2\,,$$
$$[w_2, [w_2, [w_1, w_2]]] = (-R_4' - 6P_2^3 + \beta^3 v_2 c)\, w_1 +$$
$$+ (2R_3' - 3P_1P_2^2 - 2P_2Q - \alpha\beta^2 v_2 a)\, w_2\,,$$

and, choosing suitable numbers $R_1', \ldots, R_4'$, we obtain the existence of special sections satisfying $(10_{1,2})$ at $m \in M$; let us restrict ourselves to them. Then

(31) $R_1' = 6P_1^3, \quad R_2' = -\frac{3}{2}P_1^2P_2 + P_1Q, \quad R_3' = \frac{3}{2}P_1P_2^2 + P_2Q, \quad R_4' = -6P_2^3,$

and the equations (29) reduce to

(32)
$$\begin{aligned}
w_1w_1w_1\alpha &= 24\alpha P_1^3, & w_1w_1w_1\beta &= 0, \\
w_1w_1w_2\alpha &= -4\alpha P_1(P_1P_2 - Q), & w_1w_1w_2\beta &= 2\beta P_1(P_1P_2 + 2Q), \\
w_1w_2w_1\alpha &= -8\alpha P_1(P_1P_2 - Q), & w_1w_2w_1\beta &= \beta P_1(P_1P_2 + 2Q), \\
w_1w_2w_2\alpha &= 0, & w_1w_2w_2\beta &= -12\beta P_2(P_1P_2 + Q), \\
w_2w_1w_1\alpha &= -12\alpha P_1(P_1P_2 - Q), & w_2w_1w_1\beta &= 0, \\
w_2w_1w_2\alpha &= \alpha P_2(P_1P_2 - 2Q), & w_2w_1w_2\beta &= -8\beta P_2(P_1P_2 + Q), \\
w_2w_2w_1\alpha &= 2\alpha P_2(P_1P_2 - 2Q), & w_2w_2w_1\beta &= -4\beta P_2(P_1P_2 + Q), \\
w_2w_2w_2\alpha &= 0, & w_2w_2w_2\beta &= 24\beta P_2^3.
\end{aligned}$$

From (10_2) and (21), we get

(33)
$$\begin{aligned}
w_1w_1w_1w_2\alpha - 3w_1w_1w_2w_1\alpha + 3w_1w_2w_1w_1\alpha - w_2w_1w_1w_1\alpha &= 0, \\
-2w_2w_1w_2w_1\alpha + w_2w_2w_1w_1\alpha - w_1w_1w_2w_2\alpha + 2w_1w_2w_1w_2\alpha &= 0, \\
w_1w_2w_2w_2\alpha - 3w_2w_1w_2w_2\alpha + 3w_2w_2w_1w_2\alpha - w_2w_2w_2w_1\alpha &= 0, \\
w_1w_1w_1w_2\beta - 3w_1w_1w_2w_1\beta + 3w_1w_2w_1w_1\beta - w_2w_1w_1w_1\beta &= 0, \\
-2w_2w_1w_2w_1\beta + w_2w_2w_1w_1\beta - w_1w_1w_2w_2\beta + 2w_1w_2w_1w_2\beta &= 0, \\
w_1w_2w_2w_2\beta - 3w_2w_1w_2w_2\beta + 3w_2w_2w_1w_2\beta - w_2w_2w_2w_1\beta &= 0.
\end{aligned}$$

The general solution of the system (19) is

(34)
$$\begin{aligned}
w_1w_1w_1w_1\alpha &= 2\alpha(S_1 + 30P_1^4), \\
w_1w_1w_1w_1\beta &= -\beta(S_1 - 30P_1^4), \\
w_2w_1w_1w_1\alpha &= 2\alpha(S_2 - 15P_1^3P_2 + 18P_1^2Q), \\
w_2w_1w_1w_1\beta &= -\beta(S_2 + 15P_1^3P_2 - 18P_1^2Q), \\
w_1w_2w_1w_1\alpha &= \tfrac{1}{2}\alpha(4S_3 - 45P_1^3P_2 + 54P_1^2Q), \\
w_1w_2w_1w_1\beta &= -\tfrac{1}{4}\beta(4S_3 + 45P_1^3P_2 - 54P_1^2Q), \\
w_2w_2w_1w_1\alpha &= \tfrac{1}{2}\alpha(4S_4 + 15P_1^2P_2^2 - 36P_1P_2Q + 12Q^2), \\
w_2w_2w_1w_1\beta &= -\tfrac{1}{4}\beta(4S_4 - 15P_1^2P_2^2 + 36P_1P_2Q - 12Q^2), \\
w_1w_1w_2w_1\alpha &= \alpha(2S_5 - 15P_1^3P_2 + 18P_1^2Q), \\
w_1w_1w_2w_1\beta &= -\tfrac{1}{2}\beta(2S_5 + 15P_1^3P_2 - 18P_1^2Q),
\end{aligned}$$

$$w_2w_1w_2w_1\alpha = \tfrac{1}{2}\alpha(4S_6 + 15P_1^2P_2^2 - 12P_1P_2Q + 12Q^2),$$
$$w_2w_1w_2w_1\beta = -\tfrac{1}{4}\beta(4S_6 - 15P_1^2P_2^2 + 12P_1P_2Q - 12Q^2),$$
$$w_1w_2w_2w_1\alpha = \tfrac{1}{2}\alpha(4S_7 + 15P_1^2P_2^2 + 12P_1P_2Q + 12Q^2),$$
$$w_1w_2w_2w_1\beta = -\tfrac{1}{4}\beta(4S_7 - 15P_1^2P_2^2 - 12P_1P_2Q - 12Q^2),$$
$$w_2w_2w_2w_1\alpha = \alpha(2S_8 - 15P_1P_2^3 - 18P_2^2Q),$$
$$w_2w_2w_2w_1\beta = -\tfrac{1}{2}\beta(2S_8 + 15P_1P_2^3 + 18P_2^2Q),$$
$$w_1w_1w_1w_2\alpha = \tfrac{1}{2}\alpha(2S_9 - 15P_1^3P_2 + 18P_1^2Q),$$
$$w_1w_1w_1w_2\beta = -\beta(2S_9 + 15P_1^3P_2 - 18P_1^2Q),$$
$$w_2w_1w_1w_2\alpha = \tfrac{1}{4}\alpha(4S_{10} + 15P_1^2P_2^2 - 12P_1P_2Q + 12Q^2),$$
$$w_2w_1w_1w_2\beta = -\tfrac{1}{2}\beta(4S_{10} - 15P_1^2P_2^2 + 12P_1P_2Q - 12Q^2),$$
$$w_1w_2w_1w_2\alpha = \tfrac{1}{4}\alpha(4S_{11} + 15P_1^2P_2^2 + 12P_1P_2Q + 12Q^2),$$
$$w_1w_2w_1w_2\beta = -\tfrac{1}{2}\beta(4S_{11} - 15P_1^2P_2^2 - 12P_1P_2Q - 12Q^2),$$
$$w_2w_2w_1w_2\alpha = \tfrac{1}{2}\alpha(2S_{12} - 15P_1P_2^3 - 18P_2^2Q),$$
$$w_2w_2w_1w_2\beta = -\beta(2S_{12} + 15P_1P_2^3 + 18P_2^2Q),$$
$$w_1w_1w_2w_2\alpha = \tfrac{1}{4}\alpha(4S_{13} + 15P_1^2P_2^2 + 36P_1P_2Q + 12Q^2),$$
$$w_1w_1w_2w_2\beta = -\tfrac{1}{2}\beta(4S_{13} - 15P_1^2P_2^2 - 36P_1P_2Q - 12Q^2),$$
$$w_2w_1w_2w_2\alpha = \tfrac{1}{4}\alpha(4S_{14} - 45P_1P_2^3 - 54P_2^2Q),$$
$$w_2w_1w_2w_2\beta = -\tfrac{1}{2}\beta(4S_{14} + 45P_1P_2^3 + 54P_2^2Q),$$
$$w_1w_2w_2w_2\alpha = \alpha(S_{15} - 15P_1P_2^3 - 18P_2^2Q),$$
$$w_1w_2w_2w_2\beta = -2\beta(S_{15} + 15P_1P_2^3 + 18P_2^2Q),$$
$$w_2w_2w_2w_2\alpha = \alpha(S_{16} + 30P_2^4),$$
$$w_2w_2w_2w_2\beta = -2\beta(S_{16} - 30P_2^4).$$

Inserting this into (33), we get

(35)
$$\begin{aligned}
S_2 &= \tfrac{3}{4}(4S_3 - 4S_5 - 5P_1^3P_2 + 6P_1^2Q),\\
S_4 &= \tfrac{1}{4}(8S_6 + 5P_1^2P_2^2 + 36P_1P_2Q + 4Q^2),\\
S_8 &= \tfrac{3}{2}(5P_1P_2^3 + 6P_2^2Q),\\
S_9 &= -\tfrac{3}{2}(5P_1^3P_2 - 6P_1^2Q),\\
S_{13} &= \tfrac{1}{4}(8S_{11} - 5P_1^2P_2^2 + 36P_1P_2Q - 4Q^2),\\
S_{15} &= \tfrac{3}{4}(4S_{14} - 4S_{12} + 5P_1P_2^3 + 6P_2^2Q).
\end{aligned}$$

Set

$$(36)\qquad T_1 = S_1\,,\quad T_2 = S_5\,,\quad T_3 = S_{11}\,,\quad T_4 = S_3\,,\quad T_5 = S_7\,,\quad T_6 = S_{14}\,,$$
$$T_7 = S_{12}\,,\quad T_8 = S_{10}\,,\quad T_9 = S_6\,,\quad T_{10} = S_{16}\,;$$

then

$$(37)\qquad w_1w_1w_1w_1\alpha = 2\alpha(T_1 + 30P_1^4)\,,$$
$$w_1w_1w_1w_1\beta = -\beta(T_1 - 30P_1^4)\,,$$
$$w_1w_1w_1w_2\alpha = -3\alpha(5P_1^3P_2 - 6P_1^2Q)\,,$$
$$w_1w_1w_1w_2\beta = 0\,,$$
$$w_1w_1w_2w_1\alpha = \alpha(2T_2 - 15P_1^3P_2 + 18P_1^2Q)\,,$$
$$w_1w_1w_2w_1\beta = -\tfrac{1}{2}\beta(2T_2 + 15P_1^3P_2 - 18P_1^2Q)\,,$$
$$w_1w_1w_2w_2\alpha = \tfrac{1}{2}\alpha(4T_3 + 5P_1^2P_2^2 + 36P_1P_2Q + 4Q^2)\,,$$
$$w_1w_1w_2w_2\beta = -2\beta(2T_3 - 15P_1^2P_2^2 - 4Q^2)\,,$$
$$w_1w_2w_1w_1\alpha = \tfrac{1}{2}\alpha(4T_4 - 45P_1^3P_2 + 54P_1^2Q)\,,$$
$$w_1w_2w_1w_1\beta = -\tfrac{1}{4}\beta(4T_4 + 45P_1^3P_2 - 54P_1^2Q)\,,$$
$$w_1w_2w_1w_2\alpha = \tfrac{1}{4}\alpha(4T_3 + 15P_2^1P_2^2 + 12P_1P_2Q + 12Q^2)\,,$$
$$w_1w_2w_1w_2\beta = -\tfrac{1}{2}\beta(4T_3 - 15P_1^2P_2^2 - 12P_1P_2Q - 12Q^2)\,,$$
$$w_1w_2w_2w_1\alpha = \tfrac{1}{2}\alpha(4T_5 + 15P_1^2P_2^2 + 12P_1P_2Q + 12Q^2)\,,$$
$$w_1w_2w_2w_1\beta = -\tfrac{1}{4}\beta(4T_5 - 15P_1^2P_2^2 - 12P_1P_2Q - 12Q^2)\,,$$
$$w_1w_2w_2w_2\alpha = \tfrac{3}{4}\alpha(4T_6 - 4T_7 - 15P_1P_2^3 - 18P_2^2Q)\,,$$
$$w_1w_2w_2w_2\beta = -\tfrac{3}{2}\beta(4T_6 - 4T_7 + 25P_1P_2^3 + 30P_2^2Q)\,,$$
$$w_2w_1w_1w_1\alpha = \tfrac{3}{2}\alpha(4T_4 - 4T_2 - 25P_1^3P_2 + 30P_1^2Q)\,,$$
$$w_2w_1w_1w_1\beta = -\tfrac{3}{4}\beta(4T_4 - 4T_2 + 15P_1^3P_2 - 18P_1^2Q)\,,$$
$$w_2w_1w_1w_2\alpha = \tfrac{1}{4}\alpha(4T_8 + 15P_1^2P_2^2 - 12P_1P_2Q + 12Q^2)\,,$$
$$w_2w_1w_1w_2\beta = -\tfrac{1}{2}\beta(4T_8 - 15P_1^2P_2^2 + 12P_1P_2Q - 12Q^2)\,,$$
$$w_2w_1w_2w_1\alpha = \tfrac{1}{2}\alpha(4T_9 + 15P_1^2P_2^2 - 12P_1P_2Q + 12Q^2)\,,$$
$$w_2w_1w_2w_1\beta = -\tfrac{1}{4}\beta(4T_9 - 15P_1^2P_2^2 + 12P_1P_2Q - 12Q^2)\,,$$
$$w_2w_1w_2w_2\alpha = \tfrac{1}{4}\alpha(4T_6 - 45P_1P_2^3 - 54P_2^2Q)\,,$$
$$w_2w_1w_2w_2\beta = -\tfrac{1}{2}\beta(4T_6 + 45P_1P_2^3 + 54P_2^2Q)\,,$$

$$w_2 w_2 w_1 w_1 \alpha = 2\alpha(2T_9 + 5P_1^2 P_2^2 + 4Q^2)\,,$$
$$w_2 w_2 w_1 w_1 \beta = -\tfrac{1}{2}\beta(4T_9 - 5P_1^2 P_2^2 + 36 P_1 P_2 Q - 4Q^2)\,,$$
$$w_2 w_2 w_1 w_2 \alpha = \tfrac{1}{2}\alpha(2T_7 - 15P_1 P_2^3 - 18 P_2^2 Q)\,,$$
$$w_2 w_2 w_1 w_2 \beta = -\beta(2T_7 + 15 P_1 P_2^3 + 18 P_2^2 Q)\,,$$
$$w_2 w_2 w_2 w_1 \alpha = 0\,,$$
$$w_2 w_2 w_2 w_1 \beta = -3\beta(5P_1 P_2^3 + 6P_2^2 Q)\,,$$
$$w_2 w_2 w_2 w_2 \alpha = \alpha(T_{10} + 30 P_2^4)\,,$$
$$w_2 w_2 w_2 w_2 \beta = -2\beta(T_{10} - 30 P_2^4)\,.$$

Inserting this into (15), we have

(38) $$[w_1, [w_1, [w_1, [w_1, w_2]]]] = \tfrac{1}{2}(12T_2 - 4T_4 + 45P_1^3 P_2 - 54 P_1^2 Q + {} + 2\alpha^3 \beta v_1 v_1 a)\, w_1 + (-T_1 + 30P_1^4 + \alpha^4 v_1 v_1 b)\, w_2\,,$$

$$[w_2, [w_1, [w_1, [w_1, w_2]]]] = \tfrac{1}{4}(-12T_8 + 8T_9 - 5P_1^2 P_2^2 - 36 P_1 P_2 Q + {} + 4Q^2 + 4\alpha^2\beta^2 v_2 v_1 a)\, w_1 + \tfrac{1}{4}(12T_2 - 12T_4 - 45P_1^3 P_2 + 54P_1^2 Q + {} + 4\alpha^3 \beta v_2 v_1 b)\, w_2\,,$$

$$[w_1, [w_2, [w_1, [w_1, w_2]]]] = (-2T_3 + 2T_5 + \alpha^2\beta^2 v_1 v_2 a)\, w_1 + {} + (-2T_2 - 15P_1^3 P_2 + 18 P_1^2 Q - \alpha^3 \beta v_1 v_1 a)\, w_2\,,$$

$$[w_2, [w_2, [w_1, [w_1, w_2]]]] = (-2T_7 + 15P_1 P_2^3 + 18 P_2^2 Q + \alpha\beta^3 v_2 v_2 a)\, w_1 + {} + (2T_8 - 2T_9 - \alpha^2\beta^2 v_2 v_1 a)\, w_2\,,$$

$$[w_1, [w_2, [w_2, [w_1, w_2]]]] = \tfrac{1}{4}(-12T_6 + 12T_7 + 45P_1 P_2^3 + 54 P_2^2 Q + {} + 4\alpha\beta^3 v_1 v_2 c)\, w_1 + \tfrac{1}{4}(8T_3 - 12T_5 - 5P_1^2 P_2^2 - 36 P_1 P_2 Q - 4Q^2 - {} - 4\alpha^2\beta^2 v_1 v_2 a)\, w_2\,,$$

$$[w_2, [w_2, [w_2, [w_1, w_2]]]] = (-T_{10} - 30P_2^4 + \beta^4 v_2 v_2 c)\, w_1 + {} + \tfrac{1}{2}(-4T_6 + 12T_7 - 45P_1 P_2^3 - 54P_2^2 Q - 2\alpha\beta^3 v_2 v_2 a)\, w_2\,.$$

Let us choose

(39) $$T_1 = 30P_1^4 + \alpha^4 v_1 v_1 b\,,$$
$$T_2 = -\tfrac{1}{2}(15P_1^3 P_2 - 18P_1^2 Q + \alpha^3 \beta v_1 v_1 a)\,,$$
$$T_3 = -\tfrac{1}{4}(5P_1^2 P_2^2 + 36 P_1 P_2 Q + 4Q^2 - 2\alpha^2\beta^2 v_1 v_2 a)\,,$$
$$T_4 = -\tfrac{1}{4}(45P_1^3 P_2 - 54 P_1^2 Q + 4\alpha^3 \beta v_1 v_1 a)\,,$$

$$T_5 = -\tfrac{1}{4}(5P_1^2P_2^2 + 36P_1P_2Q + 4Q^2)\,,$$
$$T_6 = \tfrac{1}{4}(45P_1P_2^3 + 54P_2^2Q + 4\alpha\beta^3v_2v_2a)\,,$$
$$T_7 = \tfrac{1}{2}(15P_1P_2^3 + 18P_2^2Q + \alpha\beta^3v_2v_2a)\,,$$
$$T_8 = -\tfrac{1}{4}(5P_1^2P_2^2 + 36P_1P_2Q - 4Q^2)\,,$$
$$T_9 = -\tfrac{1}{4}(5P_1^2P_2^2 + 36P_1P_2Q - 4Q^2 + 2\alpha^2\beta^2v_2v_1a)\,,$$
$$T_{10} = -30P_2^4 + \beta^4v_2v_2c\,.$$

Then

(40) $$[w_1, [w_1, [w_1[w_1, w_2]]]] = 0\,,$$
$$[w_2, [w_2, [w_1, [w_1, w_2]]]] = 0,$$
$$[w_2. [w_1, [w_1, [w_1, w_2]]]] = \tfrac{1}{2}\alpha^3\beta R w_2 = \tfrac{1}{2}\tilde{R} w_2\,,$$
$$[w_1, [w_2, [w_2, [w_1, w_2]]]] = -\tfrac{1}{2}\alpha\beta^3 S w_1 = -\tfrac{1}{2}\tilde{S} w_1\,,$$
$$[w_1, [w_2, [w_1, [w_1, w_2]]]] = 0\,,$$
$$[w_2, [w_2, [w_2, [w_1, w_2]]]] = 0\,.$$

QED.

2. Let B_G be a G-structure on M of the type considered. A tangent vector field v on M is called an *infinitesimal motion* of B_G is the vector fields $\mathscr{L}_v v_1$, v_1 and $\mathscr{L}_v v_2$ v_2, are dependent for each section $\{v_1, v_2, v_3\}$ of B_G; here, $\mathscr{L}_v u = [v, u]$ is the Lie derivative of u. We are going to investigate the infinitesimal motions of B_G.

Let $\{v_1, v_2, v_3\}$ be a section of B_G satisfying (3), and let

(41) $$v = Av_1 + Bv_2 + Cv_3$$

be an arbitrary tangent vector field on M. Then

(42) $$[v_1, v] = (v_1A + aC)\,v_1 + (v_1B + bC)\,v_2 + (v_1C + B)\,v_3\,,$$
$$[v_2, v] = (v_2A + cC)\,v_1 + (v_2B - aC)\,v_2 + (v_2C - A)\,v_3\,,$$
$$[v_3, v] = (v_3A - aA - cB)\,v_1 + (v_3B - bA + aB)\,v_2 + v_3Cv_3\,.$$

Let $T(M)$ be the tangent vector bundle of M, $J^k\,T(M)$ its k-th prolongation. For a given section $\{v_1, v_2, v_3\}$ of B_G, the 1-jet $j_m^1(v)$ of the vector field v at $m \in M$ is given by the vectors $v(m)$, $[v_1, v]\,(m)$, $[v_2, v]\,(m)$, $[v_3, v]\,(m) \in T_m(M)$.

The vector field v (41) is an infinitesimal motion of B_G if and only if

(43) $$v_2A = -cC\,, \quad v_1B = -bC\,, \quad v_1C = -B\,, \quad v_2C = A\,;$$

obviously, (43) is a linear first order differential equation $\mathscr{R} \subset J^1\,T(M)$ on $T(M) \to M$. Denote by $\mathscr{R}^{(k)} \subset J^{k+1}\,T(M)$ the k-th prolongation of $\mathscr{R}$; for $k \geqq l$, let $\pi_l^k : J^k\,T(M) \to J^k\,T(M)$ be the natural projection. Now, our problem may be presented as follows: Let $m \in M$ be a fixed point, k natural; how "big" is the projection $\pi_1^{k+1}\mathscr{R}_m^{(k)} \subset J^1 T(M)_m$? The answer will be given at the end of this section.

First of all, let us carry out some auxiliary calculations. Consider the sedond order differential equation for a, b, c given by

(44)
$$v_1 c - v_2 a = 0\,, \quad v_2 b + v_1 a = 0\,,$$

(45)
$$v_1 v_1 a - 2v_1 v_2 b + 2v_2 v_1 b - 3ab = R\,,$$
$$v_2 v_2 a - 2v_1 v_2 c + 2v_2 v_1 c + 3ac = S\,.$$

Set

(46)
$$v_1 a = p_1\,, \quad v_2 a = p_2\,, \quad v_1 b = p_3\,, \quad v_2 c = p_4\,, \quad v_1 p_1 = q_1\,, \quad v_2 p_2 = q_2\,;$$

the system (44) + (45) is then equivalent to (46) and

(47)
$$v_2 b = -p\,, \quad v_1 c = p_2\,, \quad v_2 p_3 = \tfrac{1}{2}(R + 3ab - 3q_1)\,,$$
$$v_1 p_4 = -\tfrac{1}{2}(S - 3ac - 3q_2)\,.$$

The integrability conditions of the couples $(46_1) + (46_2)$, $(46_3) + (47_1)$, $(46_4) + (47_2)$ are

(48)
$$v_3 a = v_1 p_2 - v_2 p_1\,,$$

(49)
$$v_3 b = -\tfrac{1}{2}(R + 3ab - q_1)\,, \quad v_3 c = -\tfrac{1}{2}(S - 3ac - q_2)\,.$$

Set

(50)
$$v_1 p_2 = q_3\,, \quad v_2 p_1 = q_4\,.$$

The equation (48) assumes the form

(51)
$$v_3 a = q_3 - q_4\,.$$

The integrability conditions of the couples $(46_1) + (51)$, $(46_2) + (51)$, $(46_3) + (49_1)$, $(47_1) + (49_1)$, $(47_2) + (49_2)$, $(46_4) + (49_2)$, $(46_5) + (50_2)$ and $(50_1) + (46_6)$ are

(52)
$$\begin{aligned}
v_3 p_1 - v_1 q_3 + v_1 q_4 &= -ap_1 - bp_2\,,\\
v_3 p_2 - v_2 q_3 + v_2 q_4 &= -cp_1 + ap_2\,,\\
2v_3 p_3 - v_1 q_1 &= -v_1 R - bp_1 - 5ap_3\,,\\
2v_3 p_1 + v_2 q_1 &= v_2 R - ap_1 + 3bp_2 + 2cp_3\,,\\
2v_3 p_2 - v_1 q_2 &= -v_1 S + 3cp_1 + cp_2 - 2bp_4\,,\\
2v_3 p_4 - v_2 q_2 &= -v_2 S + cp_2 + 5ap_4\,,\\
v_3 p_1 + v_2 q_1 - v_1 q_4 &= 0\,,\\
v_3 p_2 - v_1 q_2 + v_2 q_3 &= 0\,.
\end{aligned}$$

Set

(53) $$v_1q_1 = r_1\,, \quad v_2q_2 = r_2\,, \quad v_2q_3 = r_3\,, \quad v_1q_4 = r_4\,;$$

we then have

(54) $$\begin{aligned} v_3p_1 &= -r_4 + v_2R - ap_1 + 3bp_2 + 2cp_3\,, & v_3p_2 &= r_3 - v_1S + 3cp_1 + ap_2 - 2bp_4\,,\\ v_2q_1 &= 2r_4 - v_2R + ap_1 - 3bp_2 - 2cp_3\,, & v_1q_2 &= 2r_3 - v_1S + 3cp_1 + ap_2 - 2bp_4\,,\\ v_1q_3 &= v_2R + 4bp_2 + 2cp_3\,, & v_2q_4 &= v_1S - 4cp_1 + 2bp_4\,,\\ v_3p_3 &= \tfrac{1}{2}(r_1 - v_1R - bp_1 - 5ap_3)\,, & v_3p_4 &= \tfrac{1}{2}(r_2 - v_2S + cp_2 + 5ap_4)\,. \end{aligned}$$

The integrability conditions of the couples $(46_5) + (54_1)$, $(50_2) + (54_1)$, $(50_1) + (54_2)$, $(46_6) + (54_2)$, $(53_1) + (54_3)$, $(54_4) + (53_2)$, $(54_5) + (53_3)$ and $(53_4) + (54_6)$ are

(55) $$\begin{aligned} 2cv_1p_3 - v_3q_1 - v_1r_4 &= -v_1v_2R + p_1^2 - 5p_2p_3 + 2aq_1 - 3bq_3 + bq_4\,,\\ v_3q_4 + v_2r_4 &= v_2v_2R - 4p_1p_2 + 2p_3p_4 - 4cq_1 + 3bq_2 + cR + 3abc\,,\\ v_3q_3 - v_1r_3 &= -v_1v_1S + 4p_1p_2 - 2p_3p_4 + 3cq_1 - 4bq_2 + bS - 3abc\,,\\ 2bv_2p_4 + v_3q_2 - v_2r_3 &= -v_2v_1S + 5p_1p_4 + p_2^2 + 2aq_2 - cq_3 + 3cq_4\,,\\ 2cv_1p_3 + v_2r_1 - 2v_1r_4 + v_3q_1 &= -v_1v_2R + p_1^2 - 5p_2p_3 + aq_1 - 3bq_3\,,\\ 2bv_2p_4 - v_3q_2 + v_1r_2 - 2v_2r_3 &= -v_2v_1S + 5p_1p_4 + p_2^2 + aq_2 + 3cq_4\,,\\ v_3q_3 - v_1r_3 &= -v_2v_2R + 4p_1p_2 - 2p_3p_4 + 3cq_1 - 4bq_2 - cR - 3abc\,,\\ v_3q_4 + v_2r_4 &= v_1v_1S - 4p_1p_2 + 2p_3p_4 - 4cq_1 + 3bq_2 - bS + 3abc\,, \end{aligned}$$

From $(55_{2,8})$ or $(55_{3,7})$ we get an important relation

(56) $$v_2v_2R + cR = v_1v_1S - bS\,.$$

Set

(57) $$v_1p_3 = s_1\,, \quad v_2p_4 = s_2\,, \quad v_3q_1 = s_3\,, \quad v_3q_2 = s_4\,, \quad v_3q_3 = s_5\,, \quad v_3q_4 = s_6\,.$$

Then

$$(58)\quad \begin{aligned} v_2 r_1 &= 2cs_1 - 3s_3 + v_1 v_2 R - p_1^2 + 5p_2 p_3 - 3aq_1 + 3bq_3 - 2bq_4\,,\\ v_1 r_2 &= 2bs_2 + 3s_4 + v_2 v_1 S - 5p_1 p_4 - p_2^2 - 3aq_2 + 2cq_3 - 3cq_4\,,\\ v_1 r_3 &= s_5 + v_1 v_1 S - 4p_1 p_2 + 2p_3 p_4 - 3cq_1 + 4bq_2 - bS + 3abc\,,\\ v_2 r_3 &= 2bs_2 + s_4 + v_2 v_1 S - 5p_1 p_4 - p_2^2 - 2aq_2 + cq_3 - 3cq_4\,,\\ v_1 r_4 &= 2cs_1 - s_3 + v_1 v_2 R - p_1^2 + 5p_2 p_3 - 2aq_1 + 3bq_3 - bq_4\,,\\ v_2 r_4 &= -s_6 + v_2 v_2 R - 4p_1 p_2 + 2p_3 p_4 - 4cq_1 + 3bq_2 + cR + 3abc\,. \end{aligned}$$

Now, let us study the differential equation $\mathscr{R}$ (43). The integrability condition of the couple $(43_3) + (43_4)$ is

$$(59)\qquad v_1 A + v_2 B - v_3 C = 0\,.$$

Setting

$$v_1 A = D\,,\quad v_2 B = E\,,$$

we have

$$(61)\qquad v_3 C = D + E\,.$$

Thus $\pi_1^2 \mathscr{R}_m^{(1)}$ consists, in the way explained above, of the quadruples of tangent vectors at m of the form

$$(62)\quad \begin{aligned} &v = Av_1 + Bv_2 + Cv_3\,,\quad [v_1, v] = (D + aC)\, v_1\,,\quad [v_2, v] = (E - aC)\, v_2\,,\\ &[v_3, v] = (v_3 A - aA - cB)\, v_1 + (v_3 B - bA + aB)\, v_2 + (D + E)\, v_3\,, \end{aligned}$$

where $A, B, C, D, E, v_3 A, v_3 B$ are arbitrary numbers.

The integrability conditions of the couples $(43_1) + (60_1)$, $(60_2) + (43_3)$ and $(61) + = (43_4)$ are

$$(63)\quad \begin{aligned} &v_3 A + v_2 D = cB - v_2 a\,.\,C\,, &&v_3 B - v_1 E = bA - v_1 aC\,,\\ &v_3 B + v_1 D + v_1 E = bA - aB\,, &&v_3 A - v_2 D - v_2 E = aA + cB\,. \end{aligned}$$

Set

$$(64)\qquad v_1 D = F\,,\quad v_2 E = G\,;$$

then

$$(65)\quad \begin{aligned} &v_3 A = \tfrac{1}{2}(aA + 2cB - v_2 aC + G)\,, &&v_3 B = \tfrac{1}{2}(2bA - aB - v_1 aC - F)\,,\\ &v_2 D = -\tfrac{1}{2}(aA + v_2 aC + G)\,, &&v_1 E = -\tfrac{1}{2}(aB - v_1 aC + F)\,. \end{aligned}$$

Thus $\pi_1^3 \mathscr{R}_m^{(2)}$ consists of the quadruples of the tangent vectors

$$(66)\quad \begin{aligned} &v = Av_1 + Bv_2 + Cv_3\,,\quad [v_1, v] = (D + aC)\, v_1\,,\quad [v_2, v] = (E - aC)\, v_2\,,\\ &[v_3, v] = -\tfrac{1}{2}(G - aA - v_2 a\,.\,C)\, v_1 - \tfrac{1}{2}(F - aB + v_1 a\,.\,C)\, v_2 + (D + E)\, v_3\,, \end{aligned}$$

where A, B, C, D, E, F, G are arbitrary numbers. Thus $\pi_1^3 \mathscr{R}_m^{(2)} = \pi_1^2 \mathscr{R}_m^{(1)}$.

The integrability conditions of the couples $(65_1) + (60_1)$, $(65_1) + (43_1)$, $(65_2) + (43_2)$, $(65_2) + (60_2)$, $(65_3) + (64_1)$ and $(64_2) + (65_4)$ are

$$(67) \qquad \begin{aligned} 2v_3 D - v_1 G &= v_1 a \,.\, A + 3v_2 a \,.\, B - v_1 v_2 a \,.\, C - aD \,, \\ v_2 G &= -2v_2 c \,.\, B + SC - 4cE \,, \\ v_1 F &= 2v_1 b \,.\, A - RC + 4bD \,, \\ 2v_3 E + v_2 F &= -3v_1 a \,.\, A - v_2 a \,.\, B - v_2 v_1 a \,.\, C + aE \,, \\ 2v_3 D + 2v_2 F + v_1 G &= -v_1 a \,.\, A + v_2 a \,.\, B - v_1 v_2 a \,.\, C - aD \,, \\ 2v_3 E - v_2 F - 2v_1 G &= -v_1 a \,.\, A + v_2 a \,.\, B - v_2 v_1 a \,.\, C + aE \,. \end{aligned}$$

Set

$$(68) \qquad H = v_2 F + v_1 a \,.\, A = -v_1 G - v_2 a \,.\, B \,;$$

then

$$(69) \qquad \begin{aligned} v_3 D &= \tfrac{1}{2}(v_1 a \,.\, A + 2v_2 a \,.\, B - v_1 v_2 a \,.\, C - aD - H) \,, \\ v_3 E &= -\tfrac{1}{2}(2v_1 a \,.\, A + v_2 a \,.\, B + v_2 v_1 a \,.\, C - aE + H) \,, \\ v_1 F &= 2v_1 b \,.\, A - RC + 4bD \,, \quad v_2 F = -v_1 a \,.\, A + H \,, \\ v_1 G &= -v_2 a \,.\, B - H \,, \quad v_2 G = -2v_2 c \,.\, B + SC - 4cE \,, \end{aligned}$$

and we see that $\pi_1^4 \mathscr{R}_m^{(3)} = \pi_1^2 \mathscr{R}_m^{(1)}$.

The integrability conditions of the couples $(69_1) + (64_1)$, $(69_1) + (65_3)$, $(69_2) + (65_4)$ and $(69_2) + (64_2)$ are

$$(70) \qquad \begin{aligned} 2v_3 F + v_1 H &= (v_1 v_1 a + ab) A + \\ &\quad + 3v_1 v_2 a \,.\, B - (v_1 v_1 v_2 a + bv_2 a) C - 3aF + bG \,, \\ v_3 G - v_2 H &= -(2v_2 v_2 a + ac) B + \\ &\quad + (2v_2 v_1 v_2 a - v_1 v_2 v_2 a + cv_1 a + av_2 a) C - 3v_2 a \,.\, E + 2cF \,, \\ v_3 F - v_1 H &= (2v_1 v_2 a - ab) A + \\ &\quad + (2v_1 v_2 v_1 a - v_2 v_1 v_1 a + av_1 a - bv_2 a) C + 3v_1 a \,.\, D + 2bG \,, \\ 2v_3 G + v_2 H &= -3v_2 v_1 a \,.\, A - (v_2 v_2 v_2 a - ac) B - \\ &\quad - (v_2 v_2 v_1 a - cv_1 a) C + cF + 3aG \,; \end{aligned}$$

the integrability conditions of the couples $(69_4) + (69_3)$ and $(69_6) + (69_5)$ are (70_3) and (70_2), respectively. From (70), we obtain

$$(71) \qquad \begin{aligned} v_3 F &= v_1 v_1 a \,.\, A + v_1 v_2 a \,.\, B - bv_2 a \,.\, C + v_1 a \,.\, D - aF + bG \,, \\ v_3 G &= -v_2 v_1 a \,.\, A - v_2 v_2 a \,.\, B + cv_1 a \,.\, C - v_2 a \,.\, E + cF + aG \,, \end{aligned}$$

$$v_1H = \ (-v_1v_1a + ab)\,A + v_1v_2a\,.\,B + \tfrac{1}{3}(-v_1v_1v_2a - 4v_1v_2v_1a + $$
$$+ 2v_2v_1v_1a - 2av_1a + bv_2a)\,C - 2v_1a\,.\,D - aF - bG\,,$$
$$v_2H = -v_2v_1a\,.\,A + (v_2v_2a + ac)\,B + \tfrac{1}{3}(-4v_2v_1v_2a + 2v_1v_2v_2a -$$
$$- v_2v_2v_1a - cv_1a - 2av_2a)\,C + 2v_2a\,.\,E - cF + aG\,,$$

thus $\pi_1^5\mathscr{R}_m^{(4)} = \pi_1^2\mathscr{R}_m^{(1)}$.

The integrability conditions of the couples $(71_1) + (69_3)$, $(71_2) + (69_6)$ are

$$(72)\qquad v_1R\,.\,A + v_2R\,.\,B + (v_3R + 2aR)\,C + 3R\,.\,D + RE = 0\,,$$
$$v_1S\,.\,A + v_2S\,.\,B + (v_3S - 2aS)\,C + SD + 3SE = 0\,,$$

the integrability conditions of the couples $(71_1) + (69_4)$, $(71_2) + (69_5)$ and $(71_4) + (71_3)$ are given by the equation

$$(73)\qquad v_3H = (r_4 - bp_2 - 2cp_3)\,A + (r_3 - 2bp_4 + cp_1)\,B +$$
$$+ (cR + bS - cq_1 - bq_2)\,C + (q_4 - 4bc)\,D + (q_3 - 4bc)\,E - p_2F - p_1G\,.$$

In the case $RS \neq 0$ at m, $\pi_1^6\mathscr{R}_m^{(5)}$ is given by the vectors

$$(74)\qquad v = Av_1 + Bv_2 + Cv_3\,,\quad [v_1, v] = \tfrac{1}{8}(S^{-1}vS - 3R^{-1}vR)\,v_1\,,$$
$$[v_2, v] = \ \tfrac{1}{8}(R^{-1}vR - 3S^{-1}vS)\,v_2\,,$$
$$[v_3, v] = -\tfrac{1}{2}(G - aA - v_2aC)\,v_1 - \tfrac{1}{2}(F - aB + v_1a\,.\,C)\,v_2 -$$
$$- \tfrac{1}{4}(R^{-1}vR + S^{-1}vS)\,v_3\,,$$

where A, B, C, F, G are arbitrary. For $R = 0$, $S \neq 0$ at m, $\pi_1^6\mathscr{R}_m^{(5)}$ is given by

$$(75)\qquad v = Av_1 + Bv_2 + Cv_3 \quad \text{satisfying} \quad vR = 0\,,$$
$$[v_1, v] = \ 3(X + aC)\,v_1\,,\quad [v_2, v] = -(X + \tfrac{1}{3}S^{-1}vS + aC)\,v_2\,,$$
$$[v_3, v] = -\tfrac{1}{2}(G - aA - v_2a\,.\,C)\,v_1 - \tfrac{1}{2}(F - aB + v_1a\,.\,C)\,v_2 +$$
$$+ (2X + 2aC - \tfrac{1}{3}S^{-1}vS)\,,$$

where X, F, G are arbitrary and A, B, C are restricted by the condition $vR = 0$. The case $R \neq 0$, $S = 0$ is symmetric. For $R = S = 0$, $\pi_1^6\mathscr{R}_m^{(5)}$ is given by

$$(76)\qquad v = Av_1 + Bv_2 + Cv_3 \quad \text{satisfying} \quad vR = vS = 0\,,$$
$$[v_1, v] = \ (D + aC)\,v_1\,,\quad [v_2, v] = (E - aC)\,v_2\,,$$
$$[v_3, v] = -\tfrac{1}{2}(G - aA - v_2a\,.\,C)\,v_1 - \tfrac{1}{2}(F - aB + v_1a\,.\,C)\,v_2 +$$
$$+ (D + E)\,v_3\,.$$

Thus we have proved the following

Theorem 2. *Let B_G be a G-structure on M of the type considered. Let $\mathscr{R} \subset J^1 T(M)$ be the differential equation of the infinitesimal motions of B_G. Then $\pi_1^5 \mathscr{R}_m^{(4)} = \pi_1^2 \mathscr{R}_m^{(1)}$ for each point $m \in M$ and* $\dim \pi_1^2 \mathscr{R}_m^{(1)} = 7$. (i) *If $RS \neq 0$ on M, we have* $\dim \pi_1^6 \mathscr{R}_m^{(5)} = 5$. (ii) *If $R = 0 \neq S$ or $R \neq 0 = S$ on M, then* $\dim \pi_1^6 \mathscr{R}_m^{(5)} = 6$. (iii) *If $R = S = 0$ on M, then* $\dim \pi_1^6 \mathscr{R}_m^{(5)} = 7$.

Let us recall the following fact, see [1]: If $R = S = 0$ on M, the system (43) + (60) + (61) + (64) + (65) + (69) + (71) + (73) is completely integrable and there are sections of B_G such that

$$(77) \qquad [v_1, v_2] = v_3 \,, \quad [v_1, v_3] = [v_2, v_3] = 0 \,.$$

3. Let B_G be our G-structure on M; let us suppose

$$(78) \qquad RS \neq 0 \quad \text{on} \quad M \,.$$

Because of (8), we are in the position to choose the special frames in such a way that

$$(79) \qquad R = 1 \,, \quad S = \varepsilon = \operatorname{sgn} S \,;$$

the identity (56) implies

$$(80) \qquad c = -\varepsilon b \,.$$

Thus there is a section $\{v_1, v_2, v_3\}$ of B_G satisfying

$$(81) \qquad [v_1, v_2] = v_3 \,, \quad [v_1, v_3] = av_1 + bv_2 \,, \quad [v_2, v_3] = -\varepsilon bv_1 - av_2 \,.$$

Other sections satisfying the equations of this form and (79) are given by

$$(82) \qquad \{-v_1, -v_2, v_3\} \,, \quad \{v_2, v_1, -v_3\} \,, \quad \{-v_2, -v_1, -v_3\} \,;$$

obviously

$$(83) \qquad \begin{aligned} [-v_1, -v_2] &= v_3 \,, \qquad [-v_1, v_3] = a\,.(-v_1) + b\,.(-v_2) \,, \\ [-v_2, v_3] &= -\varepsilon b(-v_1) - a(-v_2) \,, \\ [v_2, v_1] &= -v_3 \,, \qquad [v_2, -v_3] = av_2 + \varepsilon b\,.\,v_1 \,, \\ [v_1, -v_3] &= -b\,.\,v_2 - av_1 \,, \\ [-v_2, -v_1] &= -v_3 \,, \quad [-v_2, -v_3] = a\,.(-v_2) + \varepsilon b(-v_1) \,, \\ [-v_1, -v_3] &= -b(-v_2) - a(-v_1) \,. \end{aligned}$$

Thus a is an invariant. The quantity b may be replaced by εb, i.e., $b + \varepsilon b$ and b^2 are other invariants of our structure.

Thus, let us consider the system (4), (79) and (80), i.e.,

$$(84) \qquad \begin{gathered} v_1 a + v_2 b = 0, \quad v_2 a + \varepsilon v_1 b = 0 \,, \\ v_1 v_1 a - 2v_3 b - 3ab = 1 \,, \quad v_2 v_2 a + 2\varepsilon v_3 b - 3\varepsilon ab = \varepsilon \,; \quad \varepsilon = \pm 1 \,. \end{gathered}$$

Write

$$(85) \qquad v_1 a = P_1 , \quad v_2 a = P_2 , \quad v_3 b = P_3 .$$

Then

$$(86) \qquad v_1 b = -\varepsilon P_2 , \quad v_2 b = -P_1 ,$$
$$v_1 P_1 = 2P_3 + 3ab + 1 , \quad v_2 P_2 = \varepsilon(-2P_3 + 3ab + 1) .$$

The integrability conditions of $(85_1) + (85_2)$, $(86_1) + (86_2)$, $(86_1) + (85_3)$ and $(86_2) + (85_3)$ are

$$(87) \qquad v_3 a = v_1 P_2 - v_2 P_1, \quad P_3 = 0 ,$$
$$v_3 P_2 = -(\varepsilon v_1 P_3 + \varepsilon b P_1 + a P_2) , \quad v_3 P_1 = -v_2 P_3 + a P_1 + b P_2 .$$

Using these relations, we may write

$$(88) \qquad \begin{array}{llllll} v_1 a = & P_1 , & v_2 a = & P_2 , & v_3 a = & Q_1 - Q_2 , \\ v_1 b = & -\varepsilon P_2 , & v_2 b = & -P_1 , & v_3 b = & 0 , \\ v_1 P_1 = & 3ab + 1 , & v_2 P_1 = & Q_2 , & v_3 P_1 = & a P_1 + b P_2 , \\ v_1 P_2 = & Q_1 , & v_2 P_2 = & \varepsilon(3ab + 1) , & v_3 P_2 = & -(\varepsilon b P_1 + a P_2) . \end{array}$$

The integrability conditions of $(88_9) + (88_7)$ and $(88_{12}) + (88_{11})$ reduce to

$$(89) \qquad 2b(Q_1 - Q_2) = P_1^2 - \varepsilon P_2^2 .$$

The integrability conditions of $(88_3) + (88_1)$, $(88_3) + (88_2)$, $(88_8) + (88_7)$, $(88_9) + (88_8)$, $(88_{11}) + (88_{10})$ and $(88_{12}) + (88_{10})$ are

$$(90) \qquad v_1 Q_1 - v_1 Q_2 = 2(a P_1 + b P_2) , \quad v_2 Q_1 - v_2 Q_2 = -2(\varepsilon b P_1 + a P_2) ,$$
$$v_1 Q_2 = -2(a P_1 - 2b P_2) , \quad v_3 Q_2 = 2a Q_2 + 2\varepsilon b(3ab + 1) ,$$
$$v_2 Q_1 = -2(a P_2 - 2\varepsilon b P_1) , \quad v_3 Q_1 = -2a Q_1 - 2\varepsilon b(3ab + 1) .$$

Hence

$$(91) \qquad v_1 Q_1 = 6b P_2 , \quad v_2 Q_1 = 2(2\varepsilon b P_1 - a P_2) ,$$
$$v_3 Q_1 = -2\{a Q_1 + \varepsilon b(3ab + 1)\} ,$$
$$v_1 Q_2 = -2(a P_1 - 2b P_2) , \quad v_2 Q_2 = 6\varepsilon b P_1 ,$$
$$v_3 Q_2 = 2\{a Q_2 + \varepsilon b(3ab + 1)\} .$$

Suppose $b = 0$ on M. Then, see $(86_{1,2})$, $P_1 = P_2 = 0$ on M and we get $0 = 1$ from (88_7), a contradiction. Thus $b = 0$ cannot be satisfied on any open subset of M. Similarly for a. For this reason, let us suppose $ab \neq 0$ on M.

Suppose

$$(92) \qquad v_3 a = 0 \quad \text{on} \quad M .$$

Then, see (88_3) and (89),

$$(93) \qquad Q_1 - Q_2 = 0\,, \quad P_1^2 - \varepsilon P_2^2 = 0\,.$$

Applying v_i to these equations, we get

$$(94) \quad aP_1 + bP_2 = 0\,, \quad \varepsilon bP_1 + aP_2 = 0\,, \quad a(Q_1 + Q_2) + 2\varepsilon b(3ab + 1) = 0\,,$$
$$P_2Q_1 = \varepsilon(3ab + 1)\,P_1\,, \quad P_1Q_2 = (3ab + 1)\,P_2\,, \quad aP_1^2 + 2bP_1P_2 + \varepsilon aP_2^2 = 0\,.$$

Let $a^2 - \varepsilon b^2 \neq 0$ on M. From $(94_{1,2})$, we obtain $P_1 = P_2 = 0$, and the other equations (94) imply $Q_1 = Q_2 = 0$ and

$$(95) \qquad 3ab + 1 = 0\,.$$

From (88_{1-3}), we conclude $v_1a = v_2a = v_3a = 0$, thus a is a constant.

Now, let us have (92) and $a^2 - \varepsilon b^2 = 0$ on M. Then

$$(96) \qquad \varepsilon = 1\,; \quad a = \varepsilon_0 b\,, \quad \varepsilon_0 = \pm 1\,.$$

The system (84) reduces to

$$(97) \qquad \varepsilon_0 v_1 b + v_2 b = 0\,,$$
$$\varepsilon_0 v_1 v_1 b - 2v_3 b - 3\varepsilon_0 b^2 = 1\,, \quad \varepsilon_0 v_2 v_2 b + 2v_3 b - 3\varepsilon_0 b^2 = 1\,,$$

which may be rewritten as

$$(98) \qquad v_1 b = P\,, \quad v_2 b = -\varepsilon_0 P\,, \quad v_3 b = 0\,,$$
$$v_1 P = 3b^2 + \varepsilon_0\,, \quad v_2 P = -\varepsilon_0(3b^2 + \varepsilon_0)\,;$$

here (98_1) is the definition of P and (98_3) is the integrability condition of $(98_1) + (98_2)$. The integrability condition of $(98_4) + (98_5)$ is

$$(99) \qquad v_3 P = 0\,;$$

the system (98) + (99) is completely integrable.

Finally, suppose $abv_3a \neq 0$ on M, which implies $(Q_1 - Q_2)(P_1^2 - \varepsilon P_2^2) \neq 0$ on M. Our starting point are the equations (88), (90) and (89). Applying v_i to (89), we get

$$(100) \quad P_1Q_1 = -2\varepsilon b^2 P_1 + (ab + 1)\,P_2\,, \quad P_2Q_2 = \varepsilon(ab + 1)\,P_1 - 2\varepsilon b^2 P_2\,,$$
$$-2ab(Q_1 + Q_2) = aP_1^2 + 2bP_1P_2 + \varepsilon aP_2^2 + 4\varepsilon b^2(3ab + 1)\,;$$

repeating this procedure, we obtain

$$(101) \qquad 2abQ_1 = -bP_1P_2 - \varepsilon aP_2^2 - 2\varepsilon b^2(3ab + 1)\,,$$
$$2abQ_2 = -bP_1P_2 - aP_1^2 - 2\varepsilon b^2(3ab + 1)\,,$$

$$
\begin{aligned}
Q_1 Q_2 + 2\varepsilon b^2 Q_2 &= \quad aP_1P_2 + bP_2^2 + \varepsilon(ab+1)(3ab+1)\,,\\
Q_1 Q_2 + 2\varepsilon b^2 Q_1 &= \quad aP_1P_2 + \varepsilon bP_1^2 + \varepsilon(ab+1)(3ab+1)\,,\\
2bP_2Q_1 &= \quad 2\varepsilon b(1+ab)\,P_1 - 4\varepsilon b^3 P_2 + P_2(P_1^2 - \varepsilon P_2^2)\,,\\
2bP_1Q_2 &= -4\varepsilon b^3 P_1 + 2b(1+ab)\,P_2 - P_1(P_1^2 - \varepsilon P_2^2)\,.
\end{aligned}
$$

The elimination of Q_1 from (100_1) and (101_5) implies

(102) $$P_1P_2 = -2\varepsilon b(ab+1)\,;$$

we get the same result eliminating Q_2 from (100_2) and (101_6). The substitution of (102) into $(101_{1,2})$ yileds

(103) $$2bQ_1 = -\varepsilon P_2^2 - 4\varepsilon b^3\,, \quad 2bQ_2 = -P_1^2 - 4\varepsilon b^3\,.$$

Thus the system (88) reduces to

(104) $$
\begin{gathered}
v_1 a = P_1\,, \quad v_2 a = P_2\,, \quad v_3 a = \tfrac{1}{2}b^{-1}(P_1^2 - \varepsilon P_2^2)\,,\\
v_1 b = -\varepsilon P_2\,, \quad v_2 b = -P_1\,, \quad v_3 b = 0\,,\\
v_1 P_1 = 3ab + 1\,, \quad v_2 P_1 = -\tfrac{1}{2}(b^{-1}P_1^2 + 4\varepsilon b^2)\,, \quad v_3 P_1 = aP_1 + bP_2\,,\\
v_1 P_2 = -\tfrac{1}{2}\varepsilon(b^{-1}P_2^2 + 4b^2)\,, \quad v_2 P_2 = \varepsilon(3ab+1)\,, \quad v_3 P_2 = -(\varepsilon bP_1 + aP_2)\,.
\end{gathered}
$$

The system (104) as well as the equation (102) are completely integrable. From (102), we conclude

(105) $$a = -\tfrac{1}{2}b^{-2}(\varepsilon P_1 P_2 + 2b)\,.$$

Next, let us consider the case

(106) $$R = 1\,, \quad S = 0\,,$$

the case $R = 0$, $S = 1$ being symmetric. The supposition together with (56) implies

(107) $$c = 0\,,$$

i.e., the system (4) + (106) reduces to

(108) $$v_2 a = 0\,, \quad v_2 b = -v_1 a\,, \quad v_3 b + \tfrac{3}{2}ab = \tfrac{1}{2}v_1 v_1 a - \tfrac{1}{2}\,.$$

It is easy to see that the integrability condition of $(108_2) + (108_3)$ is

(109) $$av_1 a + 2v_1 v_2 v_1 a - v_2 v_1 v_1 a = 0\,.$$

But, quite generally, $2v_1v_2v_1 - v_2v_1v_1 = v_1v_1v_2 - av_1 - bv_2$, and (109) is satisfied because of (105).

Our results are collected in

Theorem 3. *Let B_G be a G-structure on M. Let $\sigma = \{v_1, v_2, v_3\}$ be its special section satisfying*

$$(110) \qquad [v_1, v_2] = v_3 , \quad [v_1, v_3] = av_1 + bv_2 , \quad [v_2, v_3] = cv_1 - av_2 ;$$

consider the associated functions

$$(111) \qquad R = v_1v_1a - 2v_3b - 3ab , \; S = v_2v_2a - 2v_3c + 3ac .$$

I. *If $R = S = 0$, σ may be chosen in such a way that $a = b = c = 0$.*

II. *If* $\operatorname{sgn} RS = \varepsilon = \pm 1$, *$\sigma$ may be chosen in such a way that $R = 1$, $S = \varepsilon$. This being the case, we have:*

1° *If $ab(a^2 - \varepsilon b^2) \neq 0$ and $v_3a = 0$ on M, then*

$$(112) \qquad a = -\tfrac{1}{3}b^{-1} , \quad c = -\varepsilon b , \quad b = \text{const.}$$

2° *If $ab \neq 0$, $a^2 - \varepsilon b^2 = v_3a = 0$ on M, then*

$$(113) \qquad \varepsilon = 1 , \quad a = \varepsilon_0 b , \quad c = -b , \quad \varepsilon_0 = \pm 1$$

and b is a solution of the completely integrable system

$$(114) \; v_1b + \varepsilon_0 v_2 b = 0 , \quad v_3b = 0 , \quad v_1v_1b = v_2v_2b = 3b^2 + \varepsilon_0 , \quad v_3v_1b = 0 .$$

3° *If $abv_3a \neq 0$ on M, then*

$$(115) \qquad a = -\tfrac{1}{2}b^{-2}(v_1bv_2b + 2b) , \quad c = -\varepsilon b$$

and b is a solution of the completely integrable system

$$(116) \qquad \begin{gathered} v_1v_1b = \tfrac{1}{2}b^{-1}(v_1b)^2 + 2b^2 , \quad v_2v_2b = \tfrac{1}{2}b^{-1}(v_2b)^2 + 2\varepsilon b^2 , \\ v_1v_2b = v_2v_1b = -(3ab + 1) , \\ v_3v_1b = -(av_1b + bv_2b) , \quad v_3v_2b = \varepsilon bv_1b + av_2b . \end{gathered}$$

III. *If $R \neq 0$, $S = 0$, σ may be chosen in such a way that $R = 1$. This being the case, we have $c = 0$, a is a solution of $v_2a = 0$ and b a solution of the completely integrable system* (106).

As an application, let us present just one global result.

Theorem 4. *Let B_G be our G-structure on a compact manifold, let B_G be of the type* II,3° *of the preceding theorem. Then $\varepsilon = 1$ and $b < 0$ on M.*

Proof. From (104), we get easily

$$(110) \qquad (v_1v_1 + v_2v_2 + v_3v_3)\, b^2 = 3(P_1^2 + P_2^2) + 4(b + \varepsilon b)\, b^2 ;$$

as mentioned above, $b + \varepsilon b$ and b^2 are invariants. Choosing arbitrarily local coordinates on M, it is easy to see that $v_1 v_1 + v_2 v_2 + v_3 v_3$ is an elliptic operator. Suppose $(b + \varepsilon b)\, b^2 \geqq 0$ on M. Applying Hopf's lemma, we get $b = \text{const.}$ on M. But this means, see (104), $P_1 = P_2 = 0$ and $3ab + 1 = 0$ on M, i.e., a is constant on M and $v_3 a = 0$, a contradiction. Thus there is a point $m \in M$ such that $(1 + \varepsilon)\,.\,.\, b(m) < 0$. This implies $\varepsilon = 1$ and $b(m) < 0$. From $b \neq 0$ on M we get $b < 0$ on all of M. QED.

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