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Czechoslovak Mathematical Journal, Vol. 30 (1980), No. 1, 1–6

Persistent URL: `http://dml.cz/dmlcz/101650`

CZECHOSLOVAK MATHEMATICAL JOURNAL
Mathematical Institute of the Czechoslovak Academy of Sciences
V. *30* (*105*), PRAHA 24. 3. 1980, No 1

SOBOLEV MULTIPLIERS

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(Received February 25, 1972, in revised form May 31, 1979)

Let $\mathfrak{S}$ be the space of rapidly decreasing functions on R^n. Write $^\wedge$ and $^\vee$ respectively for the Fourier transform and its inverse on $\mathfrak{S}'$ the space of temperate distributions. For $r \in R$ let $\mathfrak{W}^r$ be the set of all $F \in \mathfrak{S}'$ such that

$$(1) \qquad \|F\|_r = \left[\int_{R^n} (1 + |x|^2)^r \, |F^\vee(x)|^2 \, dx\right]^{1/2} < \infty .$$

Under the norm $\| \ \|_r$ the *Sobolev space* $\mathfrak{W}^r$ is a Hilbert space. If s and r are real numbers, write $\mathfrak{B}(s, r)$ for the Banach space of bounded linear operators from $\mathfrak{W}^s$ into $\mathfrak{W}^r$; write $\mathfrak{M}(s, r)$ for the subspace of $\mathfrak{B}(s, r)$ consisting of those operators which commute with translations. It is the purpose of this paper to characterize the elements of $\mathfrak{M}(s, r)$ in terms of the behavior of their Fourier transforms.

It is clear that $\mathfrak{M}(s, r)$ is a closed subspace of $\mathfrak{B}(s, r)$ in the weak topology. If $r \leqq s$, the translation operators themselves are in $\mathfrak{M}(s, r)$. We shall collect some facts about Sobolev spaces and translations.

$$(2) \qquad \mathfrak{S} \text{ is a dense in } \mathfrak{W}^s .$$

A translation operator on $\mathfrak{W}^s$ is an operator T_x, $x \in R^n$, such that

$$[T_x(V)](f) = V(f_x)$$

for all $V \in \mathfrak{W}^s$ and $f \in \mathfrak{S}$ where f_x is the translate of f by x. Evidently,

$$(3) \qquad \|T_x(V)\|_s = \|V\|_s$$

for all $V \in \mathfrak{W}^s$ and so, if $r \leqq s$,

$$(4) \qquad \|T_x(V)\|_r \leqq \|V\|_s \quad \text{and} \quad T_x \in \mathfrak{B}(s, r) .$$

Let $\langle \, , \rangle_r$ be the bilinear form on $\mathfrak{W}^r \times \mathfrak{W}^{-r}$ defined by

$$(5) \qquad \langle T, V\rangle_r = \int_{R^n} T^\vee(x) \, . \, V^\vee(x) \, dx$$

for all $T \in \mathfrak{W}^r$ and $V \in \mathfrak{W}^{-r}$. The form $\langle \, , \rangle_r$ identifies $\mathfrak{W}^{-r}$ with the dual of W^r (see [2]).

Lemma 1. *Let f be in $\mathfrak{S}$ and define T_f by $T_f(V) = (f^\vee V^\vee)^\wedge$ for all $V \in \mathfrak{W}^s$. Then T_f is in $\mathfrak{B}(s, s)$ and there is a sequence $\{S_n\}$ of linear combinations of translation operators such that*

(i) $\{S_n\} \subset \mathfrak{B}(s, s)$ *for each* n;

(ii) $\sup\limits_{n=1}^{\infty} \|S_n\| \leqq \int_{R^n} |f(x)| \, dx$ ($\| \ \|$ = *norm on* $\mathfrak{B}(s, s)$);

(iii) $\lim\limits_n S_n = T_f$ *in the strong operator topology.*

Proof. For $V \in \mathfrak{W}^s$,

$$\left[\int_{R^n} (1 + |x|^2)^s \, |V^\vee(x)|^2 \, |f^\vee(x)|^2 \, dx\right]^{1/2} \leqq$$

$$\leqq \|V\|_s \, . \sup\{|f^\vee(x)| : x \in R^n\} \leqq \|V\|_s \, . \int_{R^n} |f(x)| \, dx \, .$$

This shows that T_f is in $\mathfrak{B}(s, s)$.

For $x \in R^n$, let $\delta(x)$ be the Dirac measure concentrated at x. Then

$$T_x(V) = \delta(-x) * V$$

for all $V \in \mathfrak{W}^s$. As is well known, we may choose a sequence $\{\mu_\varkappa = \sum_{j=1}^{m(\varkappa)} c_j^{(\varkappa)} \, . \, \delta(x_j^{(\varkappa)})\}_{\varkappa=1}^{\infty}$ such that $\{c_j^\varkappa : \varkappa = 1, \ldots, \infty; j = 1, \ldots, m(\varkappa)\} \subset C$,

(i) $\sum_{j=1}^{m(\varkappa)} |c_j^{(\varkappa)}| \leqq \int_{R^n} |f(x)| \, dx$ for each $\varkappa$;

(ii) $\lim \int_{R^n} h \, d\mu_\varkappa = \int_{R^n} h(x) \, . \, f(x) \, dx$ for all $h \in \mathfrak{S}$.

Let $\{S_\varkappa\}_{\varkappa=1}^{\infty} \subset \mathfrak{B}(s, s)$ be defined by

$$S_\varkappa(V) = \mu_\varkappa * V$$

for all $\varkappa = 1, 2, \ldots$ and $V \in \mathfrak{M}^s$.

By (ii) and (4), we have

$$\|S_\varkappa(V)\|_r = \Big\|\sum_{j=1}^{m(\varkappa)} c_j^{(\varkappa)} \, . \, T_{-x_j}(\varkappa) \, V)\Big\|_r \leqq \|V\|_s \, . \sum_{j=1}^{m(\varkappa)} |c_j| \leqq \|V\|_s \, . \int_{R^n} |f(x)| \, dx$$

for each $V \in \mathfrak{W}^s$ and each $\varkappa = 1, 2, \ldots$. This means

(iii) $\sup_{\varkappa=1}^{\infty} \|S_\varkappa\| \leqq \int_{R^n} |f(x)| \, dx$.

For each h and g in $\mathfrak{S}$, (ii) implies

$$\langle T_f(h), g \rangle_s = \langle f * h, g \rangle_s = \langle f, {}^\sim h * g \rangle_s =$$

$$= \lim_\varkappa \int {}^\sim h * g d\mu_\varkappa = \lim_\varkappa \langle S_\varkappa(h), g \rangle .$$

This, with (iii), implies that $\{S_\varkappa\}_{\varkappa=1}^{\infty}$ converges to T_f in the weak operator topology. It follows then from ([1] VI.1.5) that T_f is the strong operator limit of convex combinations of the $S_\varkappa$. Q.E.D.

Lemma 2. *For each $f \in \mathfrak{S}$ and $r, s \in R$, the operator T_f is in $\mathfrak{M}(s, r)$.*

Proof. We know that $f^\vee$ is in $\mathfrak{S}$ and so

$$M = \sup_{x \in R} |f^\vee(x)| \,.\, (1 + |x|^2)^{(r-s)/2} < \infty .$$

For $V \in \mathfrak{W}^s$, a direct calculation gives

$$\|T_f(V)\|_r \leqq M \,.\, \|V\|_s$$

and so T_f is in $\mathfrak{B}(s, r)$ and

$$\|T_f\| \leqq \sup_{x \in R} |f^\vee(x)| \,.\, (1 + |x|^2)^{(r-s)/2} . \tag{6}$$

For $h \in \mathfrak{S}$ and $x \in R^n$, we have

$$T_f \circ T_x(h) = f * h_x = (f * h)_x = T_x \circ T_f(h)$$

where the subscript x denotes translation by x.

Lemma 3. *For $r, s \in R$, $f \in \mathfrak{S}$, and $T \in \mathfrak{M}(s, r)$,*

$$T_f \circ T = T \circ T_f .$$

Proof. Choose a sequence $\{S_n\}$ for T_f as in Lemma 1. For $V \in \mathfrak{W}^s$, Lemma 1 implies

$$T_f \circ T(V) = \lim_n S_n \circ T(V) = \lim_n T \circ S_n(V) = T(\lim_n S_n(V)) = T \circ T_f(V) .$$

Q.E.D.

We shall now have use for other Sobolev spaces, analogous to the spaces $\mathfrak{W}^s$.

For $s \in R$, let ${}^{\sim}\mathfrak{W}^{s,1}$ be the set of all $F \in \mathfrak{S}'$ such that $F^{\vee}$ is a function and

$$(7) \qquad \|F\|_{s,1} = \int |F^{\vee}(x)| \, (1 + |x|^2)^{s/2} \, dx < \infty$$

and let ${}^{\sim}\mathfrak{W}^{s,\infty}$ be the set of all $F \in \mathfrak{S}'$ such that $F^{\vee}$ is a function and

$$(8) \qquad \|F\|_{s,\infty} = \text{ess. sup} \{|F^{\vee}(x)| \, . \, (1 + |x|^2)^{s/2} : x \in R^n\} < \infty \, .$$

Then ${}^{\sim}\mathfrak{W}^{s,1}$ and ${}^{\sim}\mathfrak{W}^{s,\infty}$ are Banach spaces under the norms given by (7) and (8) respectively ([2] Theorem 2.2.1). Furthermore, if we define

$$(9) \qquad \langle F, V \rangle_{s,1} = \int_{R^n} F^{\vee}(x) \, . \, V^{\vee}(x) \, dx$$

for all $F \in {}^{\sim}\mathfrak{W}^{s,1}$ and $V \in {}^{\sim}\mathfrak{W}^{-s,\infty}$, then the bilinear form $\langle \, , \, \rangle_{s,1}$ associates ${}^{\sim}\mathfrak{W}^{s,\infty}$ with the Banach space dual of ${}^{\sim}\mathfrak{M}^{s,1}$. We require several more lemmas preparatory to our main theorem.

Lemma 4. *Let $s, r \in R$ and $V \in {}^{\sim}\mathfrak{W}^{s-r,1}$. Then there exist $S \in \mathfrak{W}^s$ and $W \in \mathfrak{W}^{-r}$ such that*

$$S * W = V \quad \text{and} \quad \|S\|_s \, . \, \|W\|_{-r} = \|V\|_{s-r,1} \, .$$

Proof. By hypothesis, the function f defined by

$$f(x) = V^{\vee}(x) \, . \, (1 + |x|^2)^{(s-r)/2}$$

is in $L_1(R^n)$. It follows that there exist functions g and h in $\mathfrak{W}^0 = L_2(R^n)$ for which

$$g \, . \, h = f \, , \quad \|f\|_{0,1} = \|g\|_0 \, . \, \|h\|_0 \, .$$

Choose functions g_0 and h_0 such that

$$g(x) = g_0(x) \, . \, (1 + |x|^2)^{s/2} \, , \quad h(x) = h_0(x) \, . \, (1 + |x|^2)^{-r/2} \, .$$

Let $S = \hat{g}_0$ and $W = \hat{h}_0$. We have

$$(S * W)^{\vee}(x) = g_0(x) \, . \, h_0(x) = f(x) \, . \, (1 + |x|^2)^{(r-s)/2} = V^{\vee}(x) \, ,$$

$$\|S\|_s = \left[\int |g_0(x)|^2 \, (1 + |x|^2)^s \, dx \right]^{1/2} = \|g\|_0 \, .$$

$$\|W\|_r = \|h\|_0 \, , \quad \|V\|_{s-r,1} = \|f\|_{0,1} \, ,$$

and so $\|S\|_s \, . \, \|W\|_r = \|V\|_{s-r,1}$. Q.E.D.

Lemma 5. *Let $r, s \in R$, $M \in \mathfrak{M}(s, r)$, $\{V_m\}_{m=1}^k \subset \mathfrak{W}^s$, and $\{S_m\}_{m=1}^k \subset \mathfrak{W}^{-r}$. Then, if*

$$\sum_{m=1}^{k} V_m * S_m = 0 \, ,$$

we have

$$\sum_{m=1}^{k} \langle M(V_m), S_m\rangle_r = 0\,.$$

Proof. For each positive integer, define $f_j \mid R^n \ni x \to j^{1/2n} \exp(-j\,\pi|x|^2)$. Then $f^\vee{}_j \mid R^n \ni x \to \exp(-\pi/j|x|^2)$ and $\lim_j \sup_{j\to\infty\, x\in R} |(1+|x|^2)^p(1-f^\vee{}_j(x))| = 0$ for all $p > 0$. Thus $\{T_{fj}\}$ converges in norm to the identity in $\mathfrak{M}(s, s)$. In view of (2) there exist sequences $\{h_{m,i}\}_{i=1}^{\infty}$ $(m = 1, 2, \dots, k)$ in $\mathfrak{S}$ which converge in $\mathfrak{W}^s$ respectively to the V_m. For all m, i. and j, we have by Lemma 3,

$$M \circ T_{f_j}(h_{m,i}) = M((f^\vee{}_j h^\vee{}_{m,i})^\wedge) = M((h^\vee{}_{m,i} f^\vee{}_j)^\wedge) = M \circ T_{h_{m,i}}(f_j) = T_{h_{m,i}} \circ M(f_j)\,.$$

Thus

$$\sum_{m=1}^{k} \langle M(V_m), S_m\rangle_r = \lim_j \lim_i \sum_{m=1}^{k} \langle M \circ T_{f_j}(h_{m,i})\; S_m\rangle_r =$$

$$= \lim_j \lim_i \sum_{m=1}^{k} \langle T_{h_{m,i}} \circ M(f_j), S_m\rangle_r =$$

$$= \lim_j \lim_i \sum_{m=1}^{k} \int_{R^n} h^\vee{}_{m,i}(x) \,.\, M(f_j)^\vee(x) \,.\, S^\vee{}_m(x)\, dx =$$

$$= \lim_j \lim_i \int_{R^n} M(f_j)^\vee \sum_{m=1}^{k} h^\vee{}_{m,i}(x) \,.\, S^\vee{}_m(x)\, dx = \lim_j \int_{R^n} M(f_j) \sum_{m=1}^{k} V^\vee{}_m \,.\, S^\vee{}_m(x)\, dx = 0\,.$$

Q.E.D.

Theorem 1. *Let r and s be real numbers. Then $\mathfrak{M}(s, r)$ is linearly isometric with $^\sim\mathfrak{W}^{r-s,\infty}$. More precisely, let $\Psi \mid {}^\sim\mathfrak{W}^{r-s,\infty} \to \mathfrak{M}(s, r)$ be defined by, for each $F \in {}^\sim\mathfrak{M}^{r-s,\infty}$,*

$$\Psi_F(V) = F * V$$

for all $V \in \mathfrak{W}^s$. Then Ψ is well-defined and a linear isometry of $^\sim\mathfrak{W}^{r-s,\infty}$ onto $\mathfrak{M}(s, r)$.

Proof. Let F be in $^\sim\mathfrak{W}^{r-s,\infty}$. For $V \in \mathfrak{W}^s$, Hölder's Inequality yields

$$\|F * V\|_r = \left[\int_{R^n} |F^\vee(x)|^2 \,.\, |V^\vee(x)|^2 \,.\, (1 + |x|^2)^r\, dx\right]^{1/2} \leqq \tag{10}$$

$$\leqq \left[\int_{R^n} |V^\vee(x)|^2 \,.\, (1 + |x|^2)^s\, dx\right]^{1/2} .\ \text{ess. sup}\,\{|F^\vee(x)| \,.\, (1 + |x|^2)^{(r-s)/2}\} =$$

$$= \|V\|_s \,.\, \|F\|_{r-s,\infty}\,.$$

This shows that Ψ_F is in $\mathfrak{B}(s, r)$. For each $x \in R$, $T^\vee{}_x$ is in $L_\infty(R^n)$ and so $F^\vee$ commutes with $T^\vee{}_x$. Hence, convolution by F defines an element Ψ_F of $\mathfrak{M}(s, r)$.

Now let M be an arbitrary element of $\mathfrak{M}(s, r)$. In view of Lemmas 4 and 5, we may define a linear functional η on $\mathfrak{W}^{s-r,1}$ by letting

$$\eta(S * W) = \langle M(S), W\rangle_r$$

for all $S \in \mathfrak{W}^s$ and $W \in \mathfrak{W}^{-r}$. Let V be in $\mathfrak{W}^{s-r,1}$ and choose $S \in \mathfrak{W}^s$ and $W \in \mathfrak{W}^{-r}$ such that $\|V\|^{s-r,1} = \|S\|^s \, . \, \|W\|^{-r}$. We have, by (5),

$$\begin{aligned} (11) \qquad & |\eta(V)| = |\langle M(S), W\rangle_r| \leqq \\ & \leqq \|M(S)\|_r \, \|W\|_{-r} \leqq \|S\|_s \, \|M\| \, \|W\|_{-r} = \|M\| \, \|V\|^{s-r,1} \, . \end{aligned}$$

This means that η is in the conjugate space of ${}^{\sim}\mathfrak{W}^{s-r,1}$ and so, by the remark following (9), there is some $F \in {}^{\sim}\mathfrak{W}^{r-s,\infty}$ such that

$$(12) \qquad \eta(J) = \langle J, F\rangle_{s-r,1} \, , \quad \|\eta\| = \|F\|^{r-s,\infty}$$

for all $J \in {}^{\sim}\mathfrak{W}^{s-r,1}$. It follows from (11) that

$$(13) \qquad \|F\|^{r-s,\infty} \leqq \|M\| \, .$$

For $V \in \mathfrak{W}^s$ and $W \in \mathfrak{W}^{-r}$, we have

$$\begin{aligned} & M(V), W\rangle_r = \eta(V * W) = \langle V * W, F\rangle_{s-r,1} = \\ & = \int_{R^n} V^\vee(x) \, . \, W^\vee(x) \, . \, F^\vee(x) \, \mathrm{d}\lambda = \langle F * V, W\rangle_r = \langle \Psi_F(V), W\rangle_r \, . \end{aligned}$$

so that $M = \Psi_F$. By (10), we have

$$\|M\| \leqq \|F\|^{r-s,\infty}$$

which, with (13), implies that Ψ is an isometry. Q.E.D.

Bibliography

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