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PRIMITIVE LATTICES

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By a primitive lattice we mean a lattice L such that the class of all lattices that do not contain a sublattice isomorphic to L is a variety. A typical example of a primitive lattice is the pentagon (the lattice pictured in Fig. 18); the corresponding variety is just the variety of modular lattices. In [1] we have found several examples of primitive lattices; the task of the present paper is to find all primitive lattices.

It turns out that the notion of a primitive lattice has several equivalent definitions (see Theorem 11.1). For example, a lattice is primitive iff it is a finite, subdirectly irreducible sublattice of a free lattice. In the study of finite sublattices of free lattices the following three conditions are important:

(I) $a \wedge b = a \wedge c$ implies $a \wedge b = a \wedge (b \vee c)$;

(II) $a \vee b = a \vee c$ implies $a \vee b = a \vee (b \wedge c)$;

(III) $a \wedge b \leqq c \vee d$ implies either $a \leqq c \vee d$ or $b \leqq c \vee d$ or $a \wedge b \leqq c$ or $a \wedge b \leqq d$.

In fact, every sublattice of a free lattice satisfies these three conditions while, on the other hand, it is an open problem if any finite lattice satisfying these three conditions is a sublattice of a free (cf. [2] and [3]). We shall prove (see Theorem 11.1) that the answer to this problem is affirmative in the case of subdirectly irreducible lattices; in more detail, we call a lattice weakly primitive if it is finite, subdirectly irreducible and satisfies (I), (II), (III) and we prove that the notions of primitive lattices and weakly primitive lattices coincide.

The main (and only) results of this paper are Theorems 11.1 and 11.2. Both these theorems give a complete description of the class of primitive lattices. The description found in Theorem 11.1 is simpler, while that in Theorem 11.2 is more transparent. The description formulated in Theorem 11.2 is a list consisting of 33 items, nearly all of them representing infinite families of primitive lattices indexed by finite sequences of numbers from $\{1, 2, 3\}$.

In Section 2 we find an infinite collection of "fundamental" primitive lattices (their typical representants are pictured in Figures 1, 2, ..., 10) and five constructions R, P, P^*, Q, Q^* enabling us to construct a new primitive lattice from a given primitive lattice (and from its given element, satisfying certain conditions). In Sections 3, 4, ..., 10 we are trying to show, conversely, that there are no other primitive lattices (and, moreover, no other weakly primitive lattices) than those that can be obtained by the five constructions from the fundamental primitive lattices and their duals. The results of these sections are summarized in Theorem 11.1.

In several cases we were forced to omit proofs that otherwise would be long but routine.

The acquaintance with [3] would be convenient for the reader.

1. PRELIMINARIES

Let L be a lattice. The least and the greatest elements of L (if they exist) are denoted 0_L and 1_L. If $a, b \in L$ and $a \leqq b$, we put $[a, b]_L = \{x \in L; a \leqq x \leqq b\}$ and $(a, b)_L = \{x \in L; a < x < b\}$. If no misunderstanding threatens, we write only 0, 1, $[a, b]$, (a, b). The dual of L is denoted by L^*. If $a, b \in L$, $a < b$ and there is no c with $a < c < b$, then we write $a \prec b$ and say that b is a cover of a (or that a is covered by b, or that b covers a). Elements a such that $0_L \prec a$ are called atoms of L; atoms of L^* are called coatoms of L. An element $a \in L$ is called $\wedge$-reducible (in L) if $a = b \wedge c$ for some $b, c \in L \setminus \{a\}$; in the opposite case it is called $\wedge$-irreducible. Dually we can define $\vee$-reducible and $\vee$-irreducible elements. Evidently, if L is finite, then an element of L is $\wedge$-reducible iff it has at least two covers and it is $\vee$-reducible iff it covers at least two different elements. If $a, b \in L$ and neither $a \leqq b$ nor $b \leqq a$, we write $a \parallel b$; if either $a \leqq b$ or $b \leqq a$, we write $a \nparallel b$.

Ordered pairs are denoted by $\langle a, b\rangle$. The set $\mathrm{id}_L = \{\langle x, x\rangle; x \in L\}$ is just the identical mapping of L onto itself and, at the same time, the least congruence of L. The set $L \times L = L^2$ is just the greatest congruence of L. If $[a_1, b_1], \ldots, [a_n, b_n]$ are pairwise disjoint intervals of L, then $[a_1, b_1]^2 \cup \ldots \cup [a_n, b_n]^2 \cup \mathrm{id}_L$ is an equivalence on L; the intervals $[a_1, b_1], \ldots, [a_n, b_n]$ are its blocks and all its remaining blocks are one-element. A congruence of L is called non-trivial if it is different from id_L. The intersection of all non-trivial congruences of L is denoted by ω_L. L is called subdirectly irreducible if it contains at least two elements and ω_L is non-trivial.

1.1. Lemma. *Let L be a lattice and $[a, b]$ its interval. Then $[a, b]^2 \cup \mathrm{id}_L$ is a congruence of L iff the following two conditions are satisfied:*

(i) *If $x \in L$ and $x < b$ then $x \nparallel a$.*

(ii) *If $x \in L$ and $a < x$ then $x \nparallel b$.*

Proof. It is evident. ∎

1.2. Lemma. *Let L be a lattice and $[a, b]$, $[c, d]$ its two disjoint intervals such that $a < c$ and $b < d$. Then $[a, b]^2 \cup [c, d]^2 \cup \mathrm{id}_L$ is a congruence of L iff the following four conditions are satisfied:*

(i) *If $x < b$ then $x \not\parallel a$.*

(ii) *If $x > c$ then $x \not\parallel d$.*

(iii) *If $x < d$ and $x \nleqq b$ then $x \not\parallel c$.*

(iv) *If $x > a$ and $x \ngeqq c$ then $x \not\parallel b$.*

Proof. It is evident. ■

1.3. Lemma. *Let L be a sublattice of a lattice K. Let α be a congruence of L such that $\alpha \cup \mathrm{id}_K$ is a congruence of K. Let β be a congruence of L such that $\beta \subseteq \alpha$. Then $\beta \cup \mathrm{id}_K$ is a congruence of K.*

Proof. It is enough to prove that if $\langle a, b\rangle \in \beta$ and $x \in K$ then $\langle a \vee x, b \vee x\rangle \in \in \beta \cup \mathrm{id}_K$. If $a \vee x = b \vee x$, this is (of course) true. Let $a \vee x \neq b \vee x$. Since $\beta \subseteq \alpha$ and $\alpha \cup \mathrm{id}_K$ is a congruence, $\langle a \vee x, b \vee x\rangle \in \alpha \cup \mathrm{id}_K$ and thus $\langle a \vee x, b \vee x\rangle \in \alpha$; hence $a \vee x, b \vee x \in L$. Put $z = (a \vee x) \wedge (b \vee x)$, so that $z \in L$ and consequently $\langle a \vee z, b \vee z\rangle \in \beta$. However, it is easy to see that $a \vee z = a \vee x$ and $b \vee z = b \vee x$. ■

1.4. Lemma. *Let L be a sublattice of a subdirectly irreducible lattice K. Let α be a congruence of L such that $\alpha \cup \mathrm{id}_K$ is a congruence of K. Suppose that whenever $a, b \in L$ and $\langle a, b\rangle \notin \alpha$ then α has a non-trivial intersection with the congruence of L generated by $\langle a, b\rangle$. Then L is subdirectly irreducible.*

Proof. Suppose, on the contrary, that $\beta \cap \gamma = \mathrm{id}_L$ for two non-trivial congruences β, γ of L. By 1.3, $(\alpha \cap \beta) \cup \mathrm{id}_K$ and $(\alpha \cap \gamma) \cup \mathrm{id}_K$ are congruences of K; since their intersection is trivial, one of them, say $(\alpha \cap \beta) \cup \mathrm{id}_K$, is trivial. Hence $\alpha \cap \beta = \mathrm{id}_L$. There exists a pair $\langle a, b\rangle \in \beta$ with $a \neq b$; hence $\langle a, b\rangle \notin \alpha$. Denote by δ the congruence of L generated by $\langle a, b\rangle$, so that $\alpha \cap \delta$ is non-trivial. We get a contradiction, since $\alpha \cap \delta \subseteq \alpha \cap \beta = \mathrm{id}_L$. ■

Consider the following three conditions for a lattice L:

(I) $a \wedge b = a \wedge c$ implies $a \wedge b = a \wedge (b \vee c)$ for all $a, b, c \in L$.

(II) $a \vee b = a \vee c$ implies $a \vee b = a \vee (b \wedge c)$ for all $a, b, c \in L$.

(III) If $a, b, c, d \in L$ and $a \wedge b \lneqq c \vee d$ then either $a \leqq c \vee d$ or $b \leqq c \vee d$ or $a \wedge b \leqq c$ or $a \wedge b \leqq d$.

As is well-known, every sublattice of a free lattice satisfies all these three conditions. On the other hand, it is an open problem if any finite lattice satisfying these three conditions is a sublattice of a free lattice (cf. [2] and [3]).

Evidently, if a lattice satisfies (III), then its every element is either $\wedge$-irreducible or $\vee$-irreducible.

By a fine interval of a finite lattice L we mean an interval $[a, b]$ of L such that $a \prec b$, a is $\wedge$-irreducible and b is $\vee$-irreducible (i.e. such that b is the only cover of a and a is the only element which is covered by b).

1.5. Lemma. *Let $[a, b]$ be a fine interval of a finite lattice L. Then $[a, b]^2 \cup \cup \mathrm{id}_L$ is a congruence of L.*

Proof. It follows from 1.1. ■

1.6. Lemma. *Let L be a finite lattice satisfying (III) and let a, b be its elements such that $a < b$, a is $\wedge$-irreducible and b is $\vee$-irreducible. Then there exists a fine interval of L contained in $[a, b]$.*

Proof. (By induction on Card $[a, b]$.) If Card $[a, b] = 2$ then $[a, b]$ is fine. Let Card $[a, b] > 2$. The element a is covered by exactly one element c; evidently $c < b$. If c is $\vee$-irreducible then $[a, c]$ is fine. If c is $\vee$-reducible then (since L satisfies (III)) c is $\wedge$-irreducible and thus $[c, b]$ contains a fine interval by the induction assumption. ■

1.7. Lemma. *Let L be a finite lattice satisfying (I), (II), (III). Then every element of L has at most four covers. Moreover, if there exists an element of L with four different covers, then L has at least six different fine intervals.*

Proof. The first assertion follows from Corollary 1.3 of [2]. If an element of L has four different covers, then it follows from Theorem 3.3 of [2] that L contains a sublattice isomorphic to a certain 22-element lattice pictured in Fig. 1 of [2]; looking through the picture, we conclude from 1.6 that L has at least six different fine intervals. ■

A lattice is called weakly primitive if it is finite, subdirectly irreducible and satisfies (I), (II), (III).

1.8. Lemma. *A weakly primitive lattice contains at most one fine interval. If $[a, b]$ is a fine interval of a weakly primitive lattice L, then $\omega_L = [a, b]^2 \cup \mathrm{id}_L$.*

Proof. It follows from 1.5. ■

1.9. Lemma. *Let L be a weakly primitive lattice. Then every element of L has at most three covers and every element of L is a cover of at most three different elements.*

Proof. It follows from 1.7, 1.8 and from duality. ■

By a star element of a weakly primitive lattice L we mean an element with exactly three different covers in L. Costar elements are defined dually.

1.10. Lemma. *Let L be a weakly primitive lattice with more than two elements. Then L has at least two atoms and at least two coatoms.*

Proof. Suppose that L has only one coatom a. Then evidently $[a, 1]^2 \cup \mathrm{id}_L$ and $[0, a]^2 \cup \mathrm{id}_L$ are two non-trivial congruences of L with trivial intersection, a contradiction with the subdirect irreducibility of L. Similarly, L can not have only one atom. ■

2. STRONGLY PRIMITIVE LATTICES

Let L be a finite lattice. A finite sequence $x_1, \ldots, x_k$ of elements of L is called admissible (for L) if every element of L is a member of $x_1, \ldots, x_k$ and for every $i \in \{1, \ldots, k\}$ and every non-empty subset U of $\{u \in L;\ u \nleqq x_i\}$ such that $\bigwedge U \leqq x_i$ there exists a non-empty subset V of $\{x_1, \ldots, x_{i-1}\}$ such that $\bigwedge V \leqq x_i$ and such that for every $v \in V$ there exists a $u \in U$ with $u \leqq v$.

A lattice L is called strongly primitive if it is finite, subdirectly irreducible, satisfies (III) and both L and L^* have admissible sequences.

2.1. Lemma. *Let L be a strongly primitive lattice. Then L is a sublattice of a free lattice; consequently, L is weakly primitive.*

Proof. It follows from Theorems 6.4 and 6.3 of [3]. ■

Let us define lattices $A_1, A_2, A_3, A_4, B_n (n \geqq 1), C_n (n \geqq 1), D_n (n \geqq 0), E_n (n \geqq 0), F_n (n \geqq 2), G_n (n \geqq 2)$. They will be defined if we indicate their underlying sets and all the pairs x, y of their elements such that $x \prec y$:

$A_1 = \{0, 1\};\ 0 \prec 1$. (See Fig. 1.)

$A_2 = \{0, 1, \ldots, 9\};\ 0 \prec 2 \prec 3 \prec 4 \prec 1,\ 0 \prec 5 \prec 6 \prec 7 \prec 1,\ 2 \prec 8 \prec 9 \prec 4,$
$5 \prec 8,\ 9 \prec 7$. (See Fig. 2.)

$A_3 = \{0, 1, \ldots, 9\};\ 0 \prec 2 \prec 3 \prec 4 \prec 1,\ 0 \prec 5 \prec 6 \prec 7 \prec 1,\ 2 \prec 8 \prec 9 \prec 4,$
$5 \prec 8,\ 6 \prec 9$. (See Fig. 3.)

Fig. 1: A_1

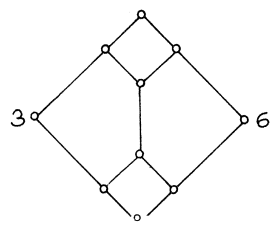

Fig. 2: A_2

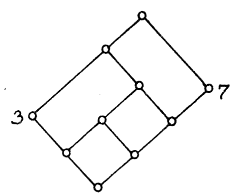

Fig. 3: A_3

$A_4 = \{0, 1, \ldots, 11\}$; $0 \prec 2 \prec 3 \prec 4 \prec 1$, $0 \prec 5 \prec 6 \prec 7 \prec 1$, $2 \prec 9 \prec 11 \prec 4$, $5 \prec 8 \prec 10 \prec 7$, $8 \prec 9$, $10 \prec 11$. (See Fig. 4.)

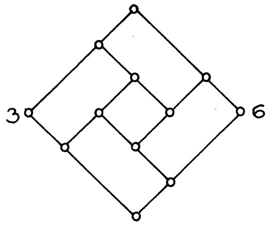

Fig. 4: A_4

$B_n = \{0, 1, \ldots, 11, a_1, \ldots, a_n, b_1, \ldots, b_n, c_1, \ldots, c_n, d_1, \ldots, d_n\}$; $0 \prec 2 \prec 3 \prec 4 \prec 1$, $0 \prec 5 \prec 6 \prec 7 \prec b_1 \prec \ldots \prec b_n \prec 9 \prec 4$, $2 \prec 7$, $6 \prec c_1 \prec a_1 \prec \ldots \prec c_n \prec \prec a_n \prec 8 \prec 9$, $5 \prec 10 \prec d_1 \prec \ldots \prec d_n \prec 11 \prec 1$, $8 \prec 11$, $a_1 \prec b_1, \ldots, a_n \prec \prec b_n$, $c_1 \prec d_1, \ldots, c_n \prec d_n$. (See Fig. 5 for $n = 3$.)

$C_n = \{0, 1, \ldots, 9, a_1, \ldots, a_n, b_1, \ldots, b_n, c_1, \ldots, c_n, d_1, \ldots, d_n\}$; $0 \prec 2 \prec 3 \prec 4 \prec 1$, $0 \prec 5 \prec d_1 \prec \ldots \prec d_n$, $2 \prec 6 \prec 7 \prec b_1 \prec c_1 \prec \ldots \prec b_n \prec c_n \prec 9 \prec 1$, $6 \prec \prec a_1 \prec \ldots \prec a_n \prec 8 \prec 4$, $5 \prec 7$, $8 \prec 9$, $a_1 \prec b_1, \ldots, a_n \prec b_n$, $d_1 \prec c_1, \ldots \ldots, d_n \prec c_n$. (See Fig. 6 for $n = 3$.)

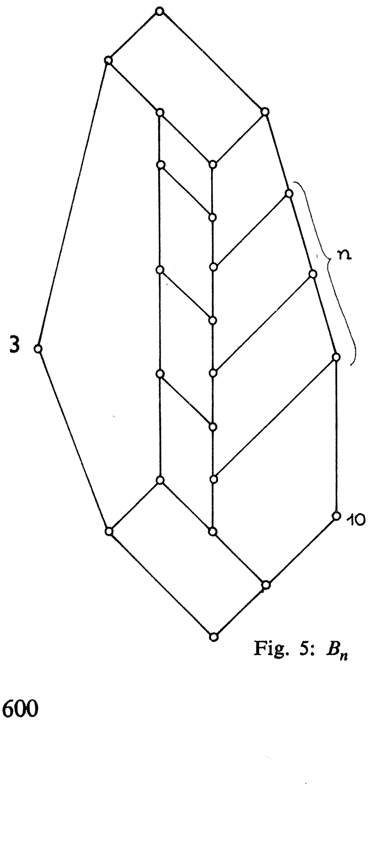

Fig. 5: B_n

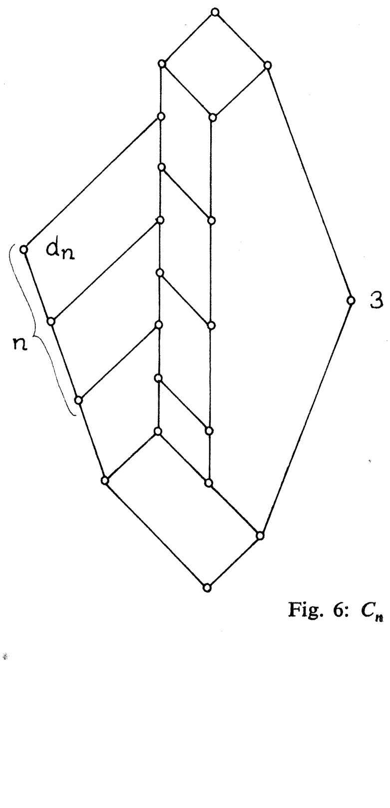

Fig. 6: C_n

$D_n = \{0, 1, \ldots, 8, a_1, \ldots, a_n, b_1, \ldots, b_n\}$; $0 \prec 2 \prec 5 \prec 6 \prec a_1 \prec \ldots \prec a_n \prec 1$, $0 \prec \prec 4 \prec 5$, $0 \prec 3 \prec 7 \prec 1$, $3 \prec 8 \prec 1$, $b_1 \prec a_1, \ldots, b_n \prec a_n$, $2 \prec b_1 \prec b_3 \prec \prec b_5 \prec \ldots \prec 7$, $4 \prec b_2 \prec b_4 \prec b_6 \prec \ldots \prec 8$. (See Fig. 7 for $n = 3$.)

$E_n = \{0, 1, \ldots, 13, a_1, \ldots, a_n, b_1, \ldots, b_n\}$; $0 \prec 2 \prec 5 \prec 6 \prec 7 \prec 9 \prec a_1 \prec \ldots \ldots \prec a_n \prec 1$, $0 \prec 4 \prec 5$, $6 \prec 8 \prec 9$, $2 \prec 10 \prec 7$, $4 \prec 11 \prec 8$, $0 \prec 3 \prec 12 \prec \prec 1$, $3 \prec 13 \prec 1$, $10 \prec b_1 \prec b_3 \prec b_5 \prec \ldots \prec 12$, $11 \prec b_2 \prec b_4 \prec b_6 \prec \ldots \ldots \prec 13$, $b_1 \prec a_1, \ldots, b_n \prec a_n$. (See Fig. 8 for $n = 3$.)

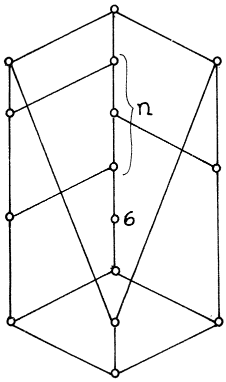

Fig. 7: D_n

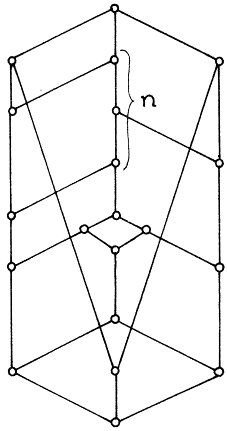

Fig. 8: E_n

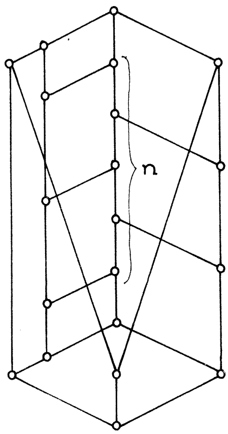

Fig. 9: F_n

$F_n = \{0, 1, \ldots, 9, a_1, \ldots, a_n, b_1, \ldots, b_n\}$; $0 \prec 2 \prec 5 \prec 6 \prec a_1 \prec \ldots \prec a_n \prec 1$, $0 \prec \prec 4 \prec 6$, $0 \prec 3 \prec 7 \prec 8 \prec 1$, $3 \prec 9 \prec 1$, $2 \prec 7$, $5 \prec b_1 \prec b_3 \prec b_5 \prec \ldots \ldots \prec 8$, $4 \prec b_2 \prec b_4 \prec b_6 \prec \ldots \prec 9$, $b_1 \prec a_1, \ldots, b_n \prec a_n$. (See Fig. 9 for $n = 5$.)

$G_n = \{0, 1, \ldots, 7, a_1, \ldots, a_n, b_1, \ldots, b_n\}$; $0 \prec 2 \prec 5 \prec a_1 \prec \ldots \prec a_n \prec 1$, $0 \prec 4 \prec \prec 5$, $0 \prec 3 \prec 6 \prec 1$, $3 \prec 7 \prec 1$, $2 \prec b_1 \prec b_3 \prec b_5 \prec \ldots \prec 6$, $4 \prec b_2 \prec \prec b_4 \prec b_6 \prec \ldots \prec 7$, $b_1 \prec a_1, \ldots, b_n \prec a_n$. (See Fig. 10 for $n = 4$.)

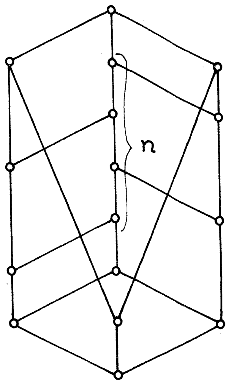

Fig. 10: G_n

2.2. Lemma. *The lattices* A_1, A_2, A_3, A_4, $B_n(n \geqq 1)$, $C_n(n \geqq 1)$, $D_n(n \geqq 0)$, $E_n(n \geqq 0)$, $F_n(n \geqq 2)$, $G_n(n \geqq 2)$ *and their duals are strongly primitive.*

Proof. The verification of all the conditions from the definition of a strongly primitive lattice is a tedious routine work in each of the ten cases. ■

Let L be a finite lattice and $a \in L$. An element $b \in L$ is called the companion of a (in L) if $L = [0_L, a] \cup [b, 1_L]$ is a disjoint union. Evidently, b is uniquely determined by a. We say also that a is the cocompanion of b.

Let L be a finite lattice. An element $a \in L$ is called a perfect element of L if $0_L < < a < 1_L$, a has a companion and the following is satisfied: if $x, y \in L$ and $a \leqq x \vee \vee y < 1_L$ then either $a \leqq x$ or $a \leqq y$. An element $a \in L$ is called a coperfect element of L if it is a perfect element of L^*, i.e. if $0_L < a < 1_L$, a has a cocompanion and whenever $0_L < x \wedge y \leqq a$ for some $x, y \in L$ then either $x \leqq a$ or $y \leqq a$. Evidently, if a is either perfect or coperfect, then a is both $\wedge$-irreducible and $\vee$-irreducible.

For every lattice L we fix four different elements not belonging to L and denote them by o_L, i_L, c_L, d_L.

For every finite lattice L we define a lattice $R(L)$ as follows: $R(L) = L \cup \{o_L, i_L, c_L\}$; L is a sublattice of $R(L)$; o_L is the least and i_L is the greatest element of $R(L)$; the element c_L is incomparable with all elements of L. (See Fig. 11.)

Let L be a finite lattice, $a \in L$ and $0_L < a < 1_L$. Then we define four new lattices $P(L, a)$, $P^*(L, a)$, $Q(L, a)$, $Q^*(L, a)$ as follows:

$P(L, a) = L \cup \{i_L, c_L\}$; L is a sublattice of $P(L, a)$; i_L is the greatest element of $P(L, a)$; $1_L \wedge c_L = a$. (See Fig. 12.)

$P^*(L, a) = L \cup \{o_L, c_L\}$; L is a sublattice of $P^*(L, a)$; o_L is the least element of $P^*(L, a)$; $0_L \vee c_L = a$. (See Fig. 13.)

$Q(L, a) = L \cup \{o_L, i_L, c_L, d_L\}$; L is a sublattice of $Q(L, a)$; o_L is the least and i_L is the greatest element of $Q(L, a)$; $1_L \wedge c_L = a$; $0_L \vee d_L = c_L$. (See Fig. 14.)

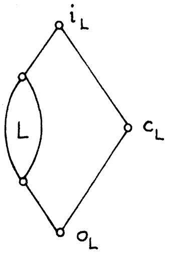

Fig. 11: $R(L)$

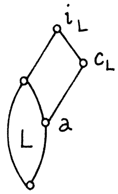

Fig. 12: $P(L, a)$

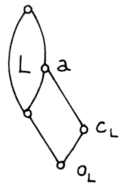

Fig. 13: $P^*(L, a)$

$Q^*(L, a) = L \cup \{o_L, i_L, c_L, d_L\}$; L is a sublattice of $Q^*(L, a)$; o_L is the least and i_L is the greatest element of $Q^*(L, a)$; $0_L \vee c_L = a$; $1_L \wedge d_L = c_L$. (See Fig. 15.)

Evidently, $P^*(L, a)$ is isomorphic to the dual of $P(L^*, a)$, $Q^*(L, a)$ is isomorphic to the dual of $Q(L^*, a)$ and $Q(L, a)$ is isomorphic to $P^*(P(L, a), c_L)$.

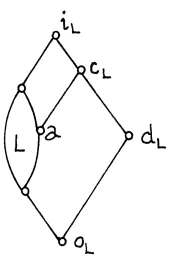

Fig. 14: $Q(L, a)$

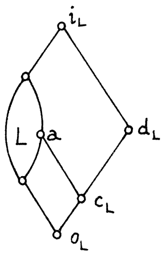

Fig. 15: $Q^*(L, a)$

2.3. Lemma. *Let L be a strongly primitive lattice. Then $R(L)$ is strongly primitive and c_L is both a perfect and coperfect element of $R(L)$.*

Proof. If $x_1, \ldots, x_k$ is an admissible sequence for L then $i_L, 1_L, c_L, x_1, \ldots, x_k, o_L$ is an admissible sequence for $R(L)$. Analogously we can find an admissible sequence for the dual of $R(L)$. We have $\omega_{R(L)} = \omega_L \cup \mathrm{id}_{R(L)}$. The rest is easy. ■

2.4. Lemma. *Let L be a finite lattice satisfying (III), let both L and L^* have admissible sequences and let a be a perfect element of L. Then $P(L, a)$ satisfies (III), both $P(L, a)$ and the dual of $P(L, a)$ have admissible sequences and c_L is both a perfect and coperfect element of $P(L, a)$.*

Proof. Let $c, d, e, f \in P(L, a)$ and $c \wedge d \leqq e \vee f$. If $c, d, e, f \in L$ then either $c \wedge d \leqq e$ or $c \wedge d \leqq f$ or $c \leqq e \vee f$ or $d \leqq e \vee f$, since L satisfies (III). The same conclusion is evident if either one of the elements c, d, e, f equals i_L or one of the elements e, f equals c_L or $c = d$. It remains to consider the case $c = c_L \& d, e, f \in L$ (since the case $d = c_L \& c, e, f \in L$ would be similar). Since $c \wedge d = c_L \wedge d = a \wedge d$ and $a, d, e, f \in L$, we have either $a \wedge d \leqq e$ or $a \wedge d \leqq f$ or $a \leqq e \vee f$ or $d \leqq e \vee f$. To prove that $P(L, a)$ satisfies (III), it is enough to derive a contradiction from $a \leqq e \vee f \& a \wedge d \nleqq e \& a \wedge d \nleqq f \& d \nleqq e \vee f$. Since $d \nleqq e \vee f$, we have $e \vee f \neq 1_L$ and so $a \leqq e \vee f < 1_L$. It follows from the perfectness of a that either $a \leqq e$ or $a \leqq f$, so that either $a \wedge d \leqq e$ or $a \wedge d \leqq f$, a contradiction.

Let $x_1, \ldots, x_k$ be an admissible sequence for L and let us prove that $i_L, c_L, 1_L, a, x_1, \ldots, x_k$ is an admissible sequence for $P(L, a)$. Let $i \in \{1, \ldots, k\}$ and $\emptyset \neq U \subseteq \{x \in P(L, a);\ x \nleqq x_i\}$, $\bigwedge U \leqq x_i$. If $c_L \notin U$ then $\emptyset \neq U \setminus \{i_L\} \subseteq \{x \in L;\ x \nleqq x_i\}$ and $\bigwedge(U \setminus \{i_L\}) \leqq x_i$, so that there exists a non-empty $V \subseteq \{x_1, \ldots, x_{i-1}\}$ such that $\bigwedge V \leqq x_i$ and $(\forall v \in V)(\exists u \in U \setminus \{i_L\})\ u \leqq v$; evidently $V \subseteq \{i_L, c_L, 1_L, a, x_1, \ldots, x_{i-1}\}$ and $(\forall v \in V)(\exists u \in U)\ u \leqq v$. Let $c_L \in U$. If $x_i \geqq a$, we can evidently put $V = \{1_L, c_L\}$. Let $x_i \ngeqq a$. Put $U_0 = (U \setminus \{i_L, c_L\}) \cup \{a\}$. Evidently $\emptyset \neq U_0 \subseteq \{x \in L;\ x \nleqq x_i\}$ and $\bigwedge U_0 \leqq x_i$; hence there exists a non-empty $V_0 \subseteq \{x_1, \ldots, x_{i-1}\}$ with $\bigwedge V_0 \leqq x_i$ and $(\forall v \in V_0)(\exists u \in U_0)\ u \leqq v$. Put $V = (V_0 \setminus [a, 1_L]) \cup \{c_L\}$. Evidently $\emptyset \neq V \subseteq \{i_L, c_L, 1_L, a, x_1, \ldots, x_{i-1}\}$, $\bigwedge V \leqq x_i$ and $(\forall v \in V)(\exists u \in U)\ u \leqq v$.

Denote by b the companion of a.

Let $y_1, \ldots, y_k$ be an admissible sequence for L^* and let us prove that $b, c_L, i_L, y_1, \ldots, y_k$ is an admissible sequence for the dual of $P(L, a)$. Let $i \in \{1, \ldots, k\}$ and $\emptyset \neq U \subseteq \{x \in P(L, a);\ x \ngeqq y_i\}$, $\bigvee U \geqq x_i$. If $c_L \notin U$ then $U \subseteq \{x \in L;\ x \ngeqq y_i\}$, so that there exists a non-empty $V \subseteq \{y_1, \ldots, y_{i-1}\}$ such that $\bigvee V \geqq y_i$ and $(\forall v \in V)(\exists u \in U)\ u \geqq v$; evidently $V \subseteq \{b, c_L, i_L, y_1, \ldots, y_{i-1}\}$. If $c_L \in U$ then evidently $y_i \geqq b$ and it is enough to put $V = \{c_L, b\}$.

The element b is the companion of c_L in $P(L, a)$, 1_L is the cocompanion of c_L in $P(L, a)$ and it is easy to see that c_L is both perfect and coperfect. ∎

2.5. Lemma. *Let L be a strongly primitive lattice and a its perfect element such that a is not a coatom of L. Then $P(L, a)$ is a strongly primitive lattice and c_L is both a perfect and coperfect element of $P(L, a)$.*

Proof. By 2.4 it is enough to show that $P(L, a)$ is subdirectly irreducible. There exists an element $e \in L$ with $a < e < 1_L$. Since L is subdirectly irreducible, there exists a pair $\langle v, w \rangle$ of different elements of L such that $\langle v, w \rangle$ belongs to any non-trivial congruence of L. Let α be a non-trivial congruence of $P(L, a)$. We have $\langle p, q \rangle \in \alpha$ for some $p, q \in P(L, a)$ with $p \neq q$. If $p, q \in L$ then evidently $\langle v, w \rangle \in \alpha$. If $p, q \notin L$ then evidently $\langle a, 1_L \rangle \in \alpha$ and so $\langle v, w \rangle \in \alpha$. If $p \in L$ and $q \notin L$ then evidently $\langle 1_L, i_L \rangle \in \alpha$ and thus $\langle e, 1_L \rangle \in \alpha$, so that $\langle v, w \rangle \in \alpha$ again. We have proved

$\langle v, w\rangle \in \alpha$ for any non-trivial congruence α of $P(L, a)$ and thus $P(L, a)$ is subdirectly irreducible. ∎

2.6. Lemma. *Let L be a strongly primitive lattice and a its coperfect element such that a is not an atom of L. Then $P^*(L, a)$ is a strongly primitive lattice and c_L is both a perfect and coperfect element of $P^*(L, a)$.*

Proof. It is the dual of 2.5. ∎

2.7. Lemma. *Let L be a strongly primitive lattice and a its perfect element such that a is a coatom of L. Then $Q(L, a)$ is strongly primitive and d_L is both a perfect and coperfect element of $Q(L, a)$.*

Proof. By 2.4 and its dual it is enough to show that $Q(L, a)$ is subdirectly irreducible. Since L is subdirectly irreducible, there exists a pair $\langle v, w\rangle$ of different elements of L such that $\langle v, w\rangle$ belongs to any non-trivial congruence of L. Let α be a non-trivial congruence of $Q(L, a)$. We have $\langle p, q\rangle \in \alpha$ for some $p, q \in Q(L, a)$ with $p \neq q$. If $p, q \in L$ then evidently $\langle v, w\rangle \in \alpha$. Since $\{o_L, 0_L, a, c_L, d_L\}$ is a sublattice isomorphic to the pentagon, $p, q \in \{o_L, 0_L, a, c_L, d_L\}$ implies $\langle 0_L, a\rangle \in \alpha$ and so $\langle v, w\rangle \in \alpha$. If $p = i_L$ and $q = c_L$ then $\langle a, 1_L\rangle \in \alpha$ and so $\langle v, w\rangle \in \alpha$. If $p \in L$ and $q = c_L$ then $\langle p \vee 1_L, c_L \vee 1_L\rangle \in \alpha$, i.e. $\langle 1_L, i_L\rangle \in \alpha$, so that $\langle a, c_L\rangle \in \alpha$ and consequently $\langle v, w\rangle \in \alpha$. If $p \in L$ and $q = d_L$, then $\langle v, w\rangle \in \alpha$ similarly. ∎

2.8. Lemma. *Let L be a strongly primitive lattice and a its coperfect element such that a is an atom of L. Then $Q^*(L, a)$ is strongly primitive and d_L is both a perfect and coperfect element of $Q^*(L, a)$.*

Proof. It is the dual of 2.7. ∎

3. WEAKLY PRIMITIVE LATTICES: NO STAR OR COSTAR ELEMENTS AND ONE OF THE COATOMS $\vee$-IRREDUCIBLE

In this section let L be a weakly primitive lattice such that L has neither star nor costar elements, L has exactly two coatoms u, r and r is $\vee$-irreducible.

Put $a = u \wedge r$, $U = \{x \in L;\ x \leqq u,\ x \nleqq a\}$ (so that e.g. $u \in U$) and denote by t the meet of all elements of U; put $b = t \vee a$.

3.1. Lemma. *t is the companion of r in L; we have $a \prec r$, $L = [0, u] \cup \{r, 1\}$ and $a \prec b < u$.*

Proof. If $x, y \in U$ then $x \nleqq r$ and $y \nleqq r$, so that $x \vee r = y \vee r = 1$; by (II) we get $1 = (x \wedge y) \vee r$, so that $x \wedge y \nleqq a$ and thus $x \wedge y \in U$. This implies that $t \in U$ and t is the companion of a in $[0, u]$; evidently, t is the companion of r in L.

Suppose that a is not covered by r. There exists a unique element s with $s \prec r$ and we have $a < s$. Evidently s is $\wedge$-irreducible and so $[s, r]$ is a finite interval of L. Put $U_1 = \{x \in L;\ x \leqq s,\ x \nleqq a\}$. We have $s \in U_1$. Denote by t_1 the meet of all elements of U_1. If $x, y \in U_1$ then $x, y \nleqq u$, so that $x \vee u = y \vee u = 1$; by (II) we get $1 = (x \wedge y) \vee u$, so that $x \wedge y \nleqq a$ and thus $x \wedge y \in U_1$. This implies that $t_1 \in U_1$ and t_1 is the companion of a in $[0, s]$. Evidently $t \vee t_1 = 1$. Now it is easy to see that $[0, a]^2 \cup [t, u]^2 \cup [t_1, s]^2 \cup \mathrm{id}_L$ is a congruence of L. However, $\omega_L = [s, r]^2 \cup \mathrm{id}_L$ by 1.8 and we get a contradiction with the subdirect irreducibility of L.

Since $a \prec r$ and r is $\vee$-irreducible, we have $L = [0, u] \cup \{r, 1\}$.

We have evidently $a \prec b \leqq u$. Suppose $b = u$. Then it is easy to see that $[u, 1]^2 \cup [a, r]^2 \cup \mathrm{id}_L$ and $[0, u]^2 \cup [r, 1]^2 \cup \mathrm{id}_L$ are congruences of L, a contradiction with the subdirect irreducibility of L. ■

3.2. Lemma. *If $a = 0$ then $[t, u]$ is weakly primitive and $L \simeq R([t, u])$.*

Proof. Evidently $L \simeq R([t, u])$. Since L satisfies (I), (II), (III) and $[t, u]$ is a sublattice of L, $[t, u]$ satisfies (I), (II), (III) as well. Hence, to prove that $[t, u]$ is weakly primitive, it is sufficient to show that it is subdirectly irreducible. However, this follows from the fact that if α is a congruence of $[t, u]$ then evidently $\alpha \cup \mathrm{id}_L$ is a congruence of L. ■

3.3. Lemma. *If $a \neq 0$ and if $[0, u]$ is subdirectly irreducible, then $[0, u]$ is a weakly primitive lattice, a is a perfect element of $[0, u]$, a is not a coatom of $[0, u]$ and $L \simeq P([0, u], a)$.*

Proof. Only the perfectness of a needs an explanation. Evidently $0 < a \prec b < u$ and t is the companion of a in $[0, u]$. Let $x, y \in [t, u]$ and $a \leqq x \vee y < u$. We have $u \wedge r = a \leqq x \vee y$; since L satisfies (III), either $u \leqq x \vee y$ or $r \leqq x \vee y$ or $a \leqq x$ or $a \leqq y$. The first two cases are impossible and thus either $a \leqq x$ or $a \leqq y$. ■

3.4. Lemma. *Let α be a congruence of $[0, u]$. The following assertions are equivalent:*

(i) *$\alpha \cup \mathrm{id}_L$ is a congruence of L.*

(ii) *$\langle a, b\rangle \notin \alpha$.*

(iii) *If $\langle x, y\rangle \in \alpha$ then either $x, y \geqq t$ or $x, y \leqq a$.*

Proof. It is easy to prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). ■

3.5. Lemma. *Let $a \neq 0$ and suppose that $[0, u]$ is not subdirectly irreducible. Denote by γ the congruence of $[0, u]$ generated by $\langle a, b\rangle$. Let $\langle p, q\rangle \in \gamma$ and $p \neq q$. Then either $p \geqq t \,\&\, q \leqq a$ or $p \leqq a \,\&\, q \geqq t$.*

Proof. Suppose, on the contrary, that either $p, q \geqq t$ or $p, q \leqq a$. Since $[0, u]$ is not subdirectly irreducible, there exist two non-trivial congruences α, β of $[0, u]$ with trivial intersection. We have either $\langle a, b \rangle \notin \alpha$ or $\langle a, b \rangle \notin \beta$; it is enough to consider the case $\langle a, b \rangle \notin \alpha$. It follows from 3.4 and from the subdirect irreducibility of L that $\langle a, b \rangle \in \beta$. Denote by δ the congruence of $[0, u]$ generated by $\langle p, q \rangle$. Since $[0, a]^2 \cup [t, u]^2$ is evidently a congruence of $[0, u]$ containing δ, we get $\langle a, b \rangle \notin \delta$. By 3.4, $\delta \cup \mathrm{id}_L$ and $\alpha \cup \mathrm{id}_L$ are non-trivial congruences of L and so $\omega_L \subseteq (\delta \cup \mathrm{id}_L) \cap \cap (\alpha \cup \mathrm{id}_L)$; consequently, $\delta \cap \alpha$ is non-trivial. However, $\delta \cap \alpha \subseteq \beta \cap \alpha = \mathrm{id}_{[0,u]}$ (since $\delta \subseteq \gamma \subseteq \beta$), a contradiction. ■

3.6. Lemma. *If $0 \prec a$ and if $[0, u]$ is not subdirectly irreducible, then $[t, u]$ is a weakly primitive lattice, b is a coperfect element of $[t, u]$, b is an atom of $[t, u]$ and $L \simeq Q^*([t, u], b)$.*

Proof. If it were $b = t$ then $[0, a]^2 \cup \mathrm{id}_L$ and $[a, 1]^2 \cup \mathrm{id}_L$ would be two non-trivial congruences of L with trivial intersection, a contradiction. Hence $t < b$. If $x \in L$ and $t \leqq x < b$, then evidently $\langle t, x \rangle$ belongs to the congruence of $[0, u]$ generated by $\langle a, b \rangle$, so that $t = x$ by 3.5. We have proved that b is an atom of $[t, u]$. Put $U_0 = \{x \in [t, u]; x \ngeqq b\}$ and denote by p the meet of all element of U_0. If $x, y \in U_0$ then $x \wedge a = y \wedge a = 0$, so that $0 = (x \vee y) \wedge a$ by (I) and thus $x \vee y \in \in U_0$. This implies that $p \in U_0$ and p is the cocompanion of b in $[t, u]$. Since b is $\vee$-reducible in L, b is $\wedge$-irreducible and b is a coperfect element of $[t, u]$. Evidently $L \simeq Q^*([t, u], b)$ and it remains to show that $[t, u]$ is subdirectly irreducible. Suppose, on the contrary, that there are two non-trivial congruences α_1, α_2 of $[t, u]$ with trivial intersection. For every $i \in \{1, 2\}$ define an equivalence β_i on L as follows: $\langle x, y \rangle \in \beta_i$ iff either $\langle x, y \rangle \in \alpha_i$ or $x = y$ or $\{x, y\} = \{0, a\}$ and $\langle t, b \rangle \in \alpha_i$. It is easy to see that β_1, β_2 are congruences of L. Since L is subdirectly irreducible, there exist two elements $x, y \in L$ such that $x \neq y$ and $\langle x, y \rangle \in \beta_1 \cap \beta_2$. If it were $x, y \in [t, u]$, then we should have $\langle x, y \rangle \in \alpha_1 \cap \alpha_2$, a contradiction. Hence, by the definition of β_i, there is no other possibility than $\{x, y\} = \{0, a\}$. But then $\langle t, b \rangle \in \alpha_1 \cap \alpha_2$, a contradiction. ■

3.7. Lemma. *Suppose that $[0, u]$ is not subdirectly irreducible, $a \neq 0$ and a is not an atom of L. Then $L \simeq A_3$.*

Proof. There exists exactly one element a_1 with $a_1 \prec a$ and we have $0 < a_1$. Put $b_1 = t \vee a_1$. If it were $b_1 = b$, then we should have $u \wedge r = a < b = b_1 = = t \vee a_1$, so that either $u \leqq b$ or $r \leqq b$ or $a \leqq a_1$ or $a \leqq t$; since the first three cases are evidently impossible, we get $a \leqq t$, but then $[0, a]^2 \cup \mathrm{id}_L$ and $[a, 1]^2 \cup \mathrm{id}_L$ would be congruences of L, a contradiction. Hence $b_1 < b$. If $x \in L$ and $b_1 \leqq x < b$, then evidently $\langle b_1, x \rangle$ belongs to the congruence of $[0, u]$ generated by $\langle a, b \rangle$, so that $b_1 = x$ by 3.5. We have proved $b_1 \prec b$. If it were $t = b_1$ then evidently $[0, a_1]^2 \cup \mathrm{id}_L$ and $[a_1, 1]^2 \cup \mathrm{id}_L$ would be non-trivial congruences of L with trivial intersection, a contradiction. Hence $t < b_1$. Especially, b_1 is $\vee$-reducible and b is the only cover of b_1. If $x \in L$ and $a_1 < x \leqq b_1$ then $x \geqq t$ and evidently $\langle x, b_1 \rangle$ belongs to the congruence of $[0, u]$ generated by $\langle a, b \rangle$, so that $x = b_1$ by 3.5. Hence $a_1 \prec b_1$.

It follows from 1.2 that $[a_1, a]^2 \cup [b_1, b]^2 \cup \mathrm{id}_L$ is a congruence of L, so that $\omega_L = [a_1, a]^2 \cup [b_1, b]^2 \cup \mathrm{id}_L$.

Denote by a_2 the only element with $a_2 \prec a_1$ and put $b_2 = t \vee a_2$. If it were $b_2 = b_1$, then evidently a_2 would be $\wedge$-irreducible and $[a_2, a_1]$ would be a fine interval, a contradiction with $\omega_L = [a_1, a]^2 \cup [b_1, b]^2 \cup \mathrm{id}_L$. Hence $b_2 < b_1$. We can prove $a_2 \prec b_2 \prec b_1$ similarly as we have proved $a_1 \prec b_1 \prec b$. If it were $a_2 \neq 0$, then we should have $t \prec b_2$ (otherwise $[0, a_2]^2 \cup \mathrm{id}_L$ and $[a_2, 1]^2 \cup \mathrm{id}_L$ would be congruences), so that b_2 would be $\vee$-reducible and by 1.2 we could show that $[a_2, a_1]^2 \cup [b_2, b_1]^2 \cup \mathrm{id}_L$ is a congruence, a contradiction with $\omega_L = [a_1, a]^2 \cup \cup [b_1, b]^2 \cup \mathrm{id}_L$. Hence $a_2 = 0$, i.e. $0 \prec a_1$; evidently $0 \prec t \prec b_1$.

If t were $\wedge$-irreducible, then evidently $[0, a_1]^2 \cup [t, b_1]^2 \cup \mathrm{id}_L$ would be a congruence, a contradiction with $\omega_L = [a_1, a]^2 \cup [b_1, b]^2 \cup \mathrm{id}_L$. Hence there exists a unique p with $t \prec p$ and $p \neq b_1$. Evidently $p < u$. We have $p \vee a_1 \geqq b > u \wedge r$; it follows from (III) that $p \vee a_1 = u$. There exists an element q with $p \leqq q \prec u$ and an element c with $b \leqq c \prec u$. Since $p \vee a_1 = u$, it is easy to see (using 1.1) that $[p, q]^2 \cup \mathrm{id}_L$ and $[b, c]^2 \cup \mathrm{id}_L$ are congruences of L; $\omega_L = [a_1, a]^2 \cup [b_1, b]^2 \cup \cup \mathrm{id}_L$ thus implies $p = q$ and $b = c$, so that $p \prec u$ and $b \prec u$.

Now it is clear that $L = \{1, r, 0, a_1, a, t, b_1, b, p, u\}$ and $L \simeq A_3$. ∎

4. WEAKLY PRIMITIVE LATTICES: NO STAR OR COSTAR ELEMENTS, BOTH ATOMS $\wedge$-REDUCIBLE AND BOTH COATOMS $\wedge$-REDUCIBLE. PART I.

In this section let L be a weakly primitive lattice such that L has neither star nor costar elements, L has exactly two coatoms r, s and exactly two atoms u, v, the elements r, s are $\vee$-reducible and the elements u, v are $\wedge$-reducible.

Put $a = r \wedge s$ and $b = u \vee v$.

4.1. Lemma. *The elements r and s have companions and the elements u and v have cocompanions in L.*

Proof. Put $U = \{x \in L;\ x \nleqq r\}$. If $x, y \in U$ then $x \vee r = y \vee r = 1$, so that $(x \wedge y) \vee r = 1$ by (II) and so $x \wedge y \in U$. This implies that the meet of all elements of U is the companion of r. Analogously we can construct the companion of s and the cocompanions of u and v. ∎

Denote by t the companion of r, by t_0 the companion of s, by q the cocompanion of v and by q_0 the cocompanion of u.

4.2. Lemma. *We do not have* $a \leqq b$.

Proof. Suppose $a \leqq b$, i.e. $r \wedge s \leqq u \vee v$. By (III) we get either $r \leqq b$ or $s \leqq b$ of $a \leqq u$ or $a \leqq v$. It is enough to derive a contradiction from $r \leqq b$.

Suppose $r < b$, so that $b = 1$. We have either $u \leqq r$ or $u \leqq s$; for the reasons of symmetry only the case $u \leqq r$ will be considered. Then $u \nleqq s$, $v \nleqq r$, $v \leqq s$. It is easy to see that $[u, r]^2 \cup \mathrm{id}_L$ and $[v, s]^2 \cup \mathrm{id}_L$ are congruences of L with trivial intersection and so either $u = r$ or $v = s$. But then either u or v is $\wedge$-irreducible, a contradiction.

Suppose $r = b$. We have either $u \leqq s$ or $v \leqq s$; only the case $v \leqq s$ will be considered. We have $u \nleqq s$ and evidently $u \leqq q \prec r$; since u is $\wedge$-reducible, it is $u < < q \prec r$. Now it is clear that the equivalence $\alpha = [u, q]^2 \cup \mathrm{id}_L$ is a non-trivial congruence of L. Since $u \nleqq s$, evidently $q_0 = s$. This implies by 1.2 that the equivalence $\beta = [v, s]^2 \cup [r, 1]^2 \cup \mathrm{id}_L$ is a non-trivial congruence of L. However, $\alpha \cap \beta$ is trivial, a contradiction. ∎

4.3. Lemma. *If* $b < a$ *then* $L \simeq A_2$.

Proof. Since b is $\vee$-reducible, a is $\wedge$-reducible and $b < a$, there exists a fine interval $[f, g]$ of L and $b \leqq f \prec g \leqq a$. By 1.8 there is no fine interval other than $[f, g]$ and so $[b, a]$ is a chain, every element of $[b, f]$ is $\vee$-reducible (so that it has a unique cover) and every element of $[g, a]$ is $\wedge$-reducible. Since $a \wedge q \ngeqq b$ and $a \wedge q \leqq a$, it is easy to see that $a \wedge q < f$. Analogously $a \wedge q_0 < f$, $b \vee t > g$ and $b \vee t_0 > g$. Evidently $t \nleqq r$ and $t_0 \nleqq s$, so that $t \vee t_0 \nleqq r$ and $t \vee t_0 \nleqq s$; this proves $t \vee t_0 = 1$. Analogously $q \wedge q_0 = 0$. Using these relations, it is a routine and tedious work to verify that the equivalence $\alpha = [b, f]^2 \cup [g, a]^2 \cup \cup [u, a \wedge q]^2 \cup [v, a \wedge q_0]^2 \cup [b \vee t, s]^2 \cup [b \vee t_0, r]^2 \cup [u \vee t, q \wedge s]^2 \cup \cup [u \vee t_0, q \wedge r]^2 \cup [v \vee t, q_0 \wedge s]^2 \cup [v \vee t_0, q_0 \wedge r]^2 \cup \{0\}^2 \cup \{1\}^2$ is a congruence of L. However, $\omega_L = [f, g]^2 \cup \mathrm{id}_L$ and so α is trivial. Consequently, L has at most twelve elements. It is a routine work to find all (up to isomorphism) lattices with at most twelve elements and then to decide which of them satisfy the conditions imposed on L; A_2 is the only lattice with these properties. ∎

5. WEAKLY PRIMITIVE LATTICES: NO STAR OR COSTAR ELEMENTS, BOTH ATOMS $\wedge$-REDUCIBLE AND BOTH COATOMS $\vee$-REDUCIBLE. PART II.

In this section let L be a weakly primitive lattice such that L has neither star nor costar elements, L has exactly two coatoms r, s and exactly two atoms u, v, the elements r, s are $\vee$-reducible, the elements u, v are $\wedge$-reducible, the elements $a = r \wedge s$ and $b = u \vee v$ are not comparable, $a \vee b \leqq r$ and $a \wedge b \geqq v$.

Similarly as in Section 4, denote by t the companion of r and by q the cocompanion of v. Moreover, put

$$d = b \vee q\,, \quad k = a \wedge t\,, \quad h = b \vee k\,, \quad l = a \wedge d\,.$$

5.1. Lemma. *$v < t < s$, $u < q < r$, $u \nleqq t$, $q \nleqq s$, $t \parallel a$, $q \parallel b$ and u is the companion of s.*

Proof. Since $u \nleqq s$, u is the companion of s by 4.1. Since s is $\vee$-reducible, $t < s$. Evidently $t \ngeqq u$ and so $t > v$. Now it remains to prove $t \parallel a$, since duality implies the remaining assertions. Suppose, on the contrary, that $a < t < s$. It is easy to verify that $[t, s]^2 \cup \mathrm{id}_L$ and $[v, a]^2 \cup [b, r]^2 \cup [u, q]^2 \cup \mathrm{id}_L$ are two non-trivial congruences of L with trivial intersection, a contradiction with the subdirect irreducibility of L. ■

5.2. Lemma. *Let $x, y \in L$ and suppose that $a \wedge b \leqq x \leqq a$, $b \leqq y \leqq a \vee b$, $y = b \vee x$ and $x = a \wedge y$. Then $x \prec y$. Especially, we have $a \prec a \vee b$ and $a \wedge b \prec b$.*

Proof. Suppose, on the contrary, that $x < z < y$ for some z. If it were $z \leqq s$ then $z \leqq a$, so that $z \leqq y \wedge a = x$; hence $z \nleqq s$. Dually, we can prove $z \ngeqq u$. We get a contradiction, since u is the companion of s. ■

5.3. Lemma. *If $a \vee b = r$ and $a \wedge b = v$ then $b < d < r$, $v < k < a$ and $d \parallel k$.*

Proof. $b < d < r$ and $v < k < a$ are evident. Suppose $d \nparallel k$. Then evidently $k < d$, so that $a \wedge t < b \vee q$, a contradiction by (III). ■

5.4. Lemma. *Let $a \vee b = r$ and $a \wedge b = v$. Then it is not possible that $d < h$ and simultaneously $l < k$.*

Proof. Suppose that $d < h$ and $l < k$.

Suppose that $[u, d]^2 \cup [0, l]^2 \cup \mathrm{id}_L$ is not a congruence of L. By 1.2 either there exists an $x \geqq u$ with $x \parallel d$ or there exists a y with $y \ngeqq u$ and $y \parallel l$. In the first case evidently $[b, x \wedge d]$ must contain a fine subinterval. Consider the second case and take a minimal y with the required property. Using $l < k$, it is easy to see that $v < y < a$; using the minimality of y we see that $y \wedge l \prec y$. If it were $y \wedge k \neq y$ then $y \wedge k = y \wedge l < l < d = b \vee q$ and we should get a contradiction by (III). Hence $y \leqq k$ and $[y \vee l, k]$ contains a fine subinterval.

We have proved: either $[u, d]^2 \cup [0, l]^2 \cup \mathrm{id}_L$ is a congruence or $[b, d]$ or $[v, k]$ contains a fine subinterval. Dually: either $[k, s]^2 \cup [h, 1]^2 \cup \mathrm{id}_L$ is a congruence or $[k, a]$ or $[d, r]$ contains a fine subinterval. However, this is a contradiction, since L is subdirectly irreducible. ■

5.5. Lemma. *Let $a \vee b = r$ and $a \wedge b = v$. Then $d \nparallel h$.*

Proof. Suppose $d \parallel h$. Since there is a fine interval contained in $[b, d \wedge h]$, the interval $[0, a]$ is a chain. Denote by x the only element covering h. We have $a \wedge x > a \wedge h$, since otherwise $a \wedge x = a \wedge h < h = b \vee k$, a contradiction by (III). We have $a \wedge x \prec x$, since if there existed z with $a \wedge x < z < x$, we should have $z \nleqq a$, $z \nleqq s$, $z \geqq u$, $z \geqq b$, $z \geqq h$, $z \geqq x$, a contradiction. Since $a \wedge x \neq h$, the elements $a \wedge x$ and h are the only two elements which are covered by x. Since $d \wedge x < x$, we have either $d \wedge x \leqq h$ or $d \wedge x \leqq a \wedge x$. In the first case $d \wedge x \leqq b \vee k$, a contradiction with (III); in the second case $b \leqq d \wedge x \leqq a$, a contradiction again. ■

5.6. Lemma. *It is not possible that $a \vee b = r$ and simultaneously $a \wedge b = v$.*

Proof. It follows from 5.3, 5.4, 5.5 and the dual of 5.5. ■

6. WEAKLY PRIMITIVE LATTICES: NO STAR OR COSTAR ELEMENTS, BOTH ATOMS $\wedge$-REDUCIBLE AND BOTH COATOMS $\vee$-REDUCIBLE. PART III.

In this section let L be a weakly primitive lattice satisfying the conditions of Section 5 and, moreover, $a \vee b = r$ and $a \wedge b > v$.

Define elements $t, q, d = b \vee q$ in the same way as in Section 5. Evidently, we have $d < r$ and $d \parallel a$. There exists exactly one element b_1 with $b \prec b_1$. Put $a_1 = a \wedge b_1$.

6.1. Lemma. $a \wedge b < a_1 \prec b_1 < d$.

Proof. If it were $a_1 = a \wedge b$ then we should have $a \wedge b_1 < u \vee v$, a contradiction by (III). Hence $a \wedge b < a_1$. Since $b \prec b_1$, we have $b \vee a_1 = b_1$ and so $a_1 \prec b_1$ by 5.2. Evidently $b_1 \leqq d$; if it were $b_1 = d$, then d would cover three different elements. ■

6.2. Lemma. *Either $[b, b_1]^2 \cup [a \wedge b, a_1]^2 \cup \mathrm{id}_L$ is a congruence of L or $[a \wedge b, a]$ contains a fine subinterval of L.*

Proof. Suppose that neither the first nor the second case occurs. Using 1.2, it is easy to see that there exists an $x \in L$ with $a \wedge b \leqq x \leqq s$ and $x \parallel a_1$. Take a minimal x with these properties. Since $[a \wedge b, a]$ does not contain a fine subinterval, $x \nleqq a$ and so $x \wedge a < x$; by the minimality of x we get $x \wedge a < a_1$. If it were $x \wedge a = a \wedge b$, we should have $x \wedge a < u \vee v$, a contradiction with (III). Hence $a \wedge b < x \wedge a < a_1$. We have $b \vee q = d > b_1 > a_1 > a \wedge x$, a contradiction by (III). ■

6.3. Lemma. $[b, 1]$ *is a chain.*

Proof. It follows from 6.2 that $[b, 1]$ does not contain a fine subinterval. Since b is $\vee$-reducible, $[b, 1]$ must be a chain. ■

Evidently there exists exactly one finite sequence $e_1, d_1, e_2, d_2, \ldots, e_n, d_n, e_{n+1}, d_{n+1}$ of elements of $[b, d]$ such that $n \geqq 1$, $e_1 = d$, $d_{n+1} = b$, $d_i = (e_i \wedge a) \vee b$ for all $i = 1, \ldots, n + 1$ and $e_{i+1} \prec d_i$ for all $i = 1, \ldots, n$. We have $d = e_1 \geqq d_1 \succ e_2 \geqq \geqq d_2 \succ \ldots \succ e_n \geqq d_n = b_1 \succ e_{n+1} = d_{n+1} = b$.

6.4. Lemma. *If* $1 \leqq i \leqq n$ *and if* $d_i < e_i$ *then* $a \wedge d_{i+1} \prec a \wedge d_i$.

Proof. Suppose $a \wedge d_{i+1} < x < a \wedge d_i$ for some x. If it were $x \leqq e_{i+1}$ then $x \leqq e_{i+1} \wedge a$ and so $d_{i+1} \geqq x \vee b \geqq x$, $x \leqq a \wedge d_{i+1}$, a contradiction. Hence $b \vee x = d_i > e_i \wedge a$, a contradiction by (III). ■

6.5. Lemma. *If* $1 \leqq i \leqq n - 1$ *and if* $a \wedge d_{i+1} \prec a \wedge d_i$ *then* $d_{i+1} < e_{i+1}$.

Proof. Suppose, on the contrary, that $d_{i+1} = e_{i+1}$, so that $d_{i+1} \prec d_i$. Then it follows from 1.2 that $[d_{i+1}, d_i]^2 \cup [a \wedge d_{i+1}, a \wedge d_i]^2 \cup \mathrm{id}_L$ is a congruence; by 6.2, $[a \wedge d_{i+1}, a \wedge d_i]$ is a fine interval, a contradiction. ■

6.6. Lemma. $d = e_1 > d_1 \succ e_2 > d_2 \succ \ldots \succ e_n > d_n = b_1 \succ e_{n+1} = d_{n+1} = b$ *and* $a \wedge d_1 \succ a \wedge d_2 \succ \ldots \succ a \wedge d_n = a_1 \succ a \wedge d_{n+1} = a \wedge b$.

Proof. We have $d_1 < d$; in fact, otherwise d would cover three different elements. Now we can use 6.4 and 6.5 step-by-step. ■

6.7. Lemma. $\omega_L = [b, b_1]^2 \cup [a \wedge b, a_1]^2 \cup \mathrm{id}_L$ *and* $[0, a]$ *is a chain.*

Proof. By 6.6, $a \wedge b \prec a_1$. Now it follows from 1.2 that $[b, b_1]^2 \cup [a \wedge b, a_1]^2$ is a congruence. Hence L contains no fine interval; since a is $\wedge$-reducible, $[0, a]$ is a chain. ■

6.8. Lemma. $L = [u, d] \cup \{1, r, s, a, t, 0, v, d_1 \wedge a, \ldots, d_{n+1} \wedge a\}$.

Proof. Since v is $\wedge$-reducible, there exists a w with $v \prec w$ and $w \nleqq a$. If it were $w \geqq u$ then $w \geqq b$ and so w could not cover v; hence $w \leqq s$. Since $w \nleqq a$, we get $w \nleqq r$ and so $w \geqq t$. Consequently, $w = t$.

We have proved $v \prec t$. This implies that $a \wedge t = v$. There exists exactly one $\bar{r}$ with $d \prec \bar{r} \leqq r$ and exactly one $\bar{a}$ with $a \wedge d \prec \bar{a} \leqq a$. If it were $\bar{a} \nleqq \bar{r}$ then $\bar{a} \wedge \bar{r} = a \wedge d < d = b \vee q$, a contradiction by (III). Hence $\bar{a} < \bar{r}$. This yields evidently $b \vee \bar{a} = \bar{r}$.

If x is an arbitrary element of $[t, s] \setminus [t \vee \bar{a}, s]$ then $a \wedge x \leqq a \wedge d < d = q \vee v$, so that $a \wedge x = v$ by (III); hence by (I) the subset $[t, s] \setminus [t \vee \bar{a}, s]$ contains a largest element $\bar{t}$, $[t, s] = [t, \bar{t}] \cup [t \vee \bar{a}, s]$ is a disjoint union and $a \wedge \bar{t} = v$. Since every element from $[v, a \wedge b]$ must be $\wedge$-reducible, we conclude $v \prec a \wedge b$.

Now it is not difficult to verify that the equivalence $\alpha = [\bar{r}, r]^2 \cup [\bar{a}, a]^2 \cup \cup [\bar{a} \vee t, s]^2 \cup [t, \bar{t}]^2 \cup \mathrm{id}_L$ is a congruence of L. By 6.7 and the subdirect irreducibility of L, α is trivial. Hence $\bar{r} = r$, $\bar{a} = a$, $\bar{a} \vee t = s$, $\bar{t} = t$. This, together with $v \prec a \wedge b$, 6.6 and 6.7 implies the assertion. ∎

6.9. Lemma. $L \simeq C_n$.

Proof. Since $q \wedge b_1 = q \wedge b < b = u \vee v$, it follows from (III) that $q \wedge b = u$. This implies that $u \prec b$. For every $i = 1, \ldots, n$, denote by f_i the only element with $d_i \prec f_i \leqq e_i$ and, if x is an arbitrary element of $[f_i, e_i]$, denote by $\bar{x}$ the only element with $\bar{x} \prec x$ and $\bar{x} \ngeqq b$. Evidently, there is no other possibility than $\bar{x} \leqq q$, so that $\bar{x} = q \wedge x$ and $b \vee \bar{x} = x$.

Define an equivalence α on L as follows: $\langle x, y\rangle \in \alpha$ iff either $x = y$ or $x, y \in [f_i, e_i]$ for some $i = 1, \ldots, n$ or $x, y \in [u, q]$ and $x \vee b$, $y \vee b \in [f_i, e_i]$ for some $i = 1, \ldots, n$.

Let us prove that if $\langle x, y\rangle \in \alpha$, $x \neq y$ and $z \in L$ then $\langle z \wedge x, z \wedge y\rangle \in \alpha$. First suppose $x, y \in [u, q]$, so that $b \vee x$, $b \vee y \in [f_i, e_i]$ for some $i = 1, \ldots, n$. If $q \wedge z \in \in \{0, u\}$ then $z \wedge x = z \wedge y$. Otherwise $(q \wedge z) \vee b \in [f_j, e_j]$ for some $j = 1, \ldots, n$. Put $m = \mathrm{Max}(i, j)$. If it were $(z \wedge x) \vee b < f_m$ then $z \wedge x \leqq d_m = b \vee (a \wedge d_m)$, so that $z \wedge x \leqq b$ by (III); however, if w denotes the only element with $u \prec w \leqq q$ then $w \leqq x$ and $w \leqq q \wedge z$, so that $w \leqq z \wedge x \leqq b$, a contradiction. Hence $(z \wedge x) \vee b \geqq f_m$; evidently $(z \wedge x) \vee b \leqq e_m$. Quite similarly, $(z \wedge y) \vee b$ belongs to $[f_m, e_m]$, too, and so $\langle z \wedge x, z \wedge y\rangle \in \alpha$.

Now suppose $x, y \in [f_i, e_i]$ for some $i = 1, \ldots, n$. Evidently $q \wedge x \prec x$ and so $(q \wedge x) \vee b = x$. Similarly $(q \wedge y) \vee b = y$, so that $\langle q \wedge x, q \wedge y\rangle \in \alpha$. If $z \leqq q$ then $\langle z \wedge x, z \wedge y\rangle = \langle z \wedge (q \wedge x), z \wedge (q \wedge y)\rangle \in \alpha$ by the preceding case. If $z \nleqq q$ then $\langle z \wedge x, z \wedge y\rangle \in \alpha$ is easy.

It is easy to prove that $\langle x, y\rangle \in \alpha$ implies $\langle z \vee x, z \vee y\rangle \in \alpha$ for all $z \in L$. Now, α is a congruence; by 6.7 it is trivial and so $[u, d] = \{u, \bar{e}_1, \ldots, \bar{e}_n, b, e_1, d_1, \ldots, e_n, d_n\}$. Now $L \simeq C_n$ is immediate. ∎

7. WEAKLY PRIMITIVE LATTICES: NO STAR OR COSTAR ELEMENTS, BOTH ATOMS $\wedge$-REDUCIBLE AND BOTH COATOMS $\vee$-REDUCIBLE. PART IV.

In this section let L be a weakly primitive lattice satisfying the conditions of Section 5 and, moreover, $a \vee b < r$ and $a \wedge b > v$.

Define elements t, q in the same way as in Section 5. Denote by b_1 the only element with $b \prec b_1$, by $\bar{a}$ the only element with $\bar{a} \prec a$ and put $a_1 = b_1 \wedge a$ and $\bar{b} = b \vee \bar{a}$. We have $b_1 \leqq a \vee b$ and $a \wedge b \leqq \bar{a}$.

7.1. Lemma. *$a \wedge b < a_1 \prec b_1$ and $\bar{a} \prec \bar{b} < a \vee b$.*

Proof. If it were $a \wedge b = a_1$, then $b_1 \wedge a < b = u \vee v$, a contradiction by (III). Hence $a \wedge b < a_1$. Consequently $b \vee a_1 = b_1$; we get $a_1 \prec b_1$ by 5.2 .The rest is dual. ■

7.2. Lemma. *If $b \prec a \vee b$ then $a \wedge b \prec a$.*

Proof. If there existed an x with $a \wedge b < x < a$ then $b \vee x = a \vee b > r \wedge s$, a contradiction by (III). ■

7.3. Lemma. *Suppose that $[a \wedge b, a]$ does not contain a fine subinterval. Then $[a \wedge b, a]$ is a chain, every element of $[a \wedge b, a]$ is $\wedge$-reducible and we have either $a \wedge b \prec a_1$ or $a \wedge b \prec c_1 \prec a_1$ for exactly one c_1. Moreover, if $a \wedge b \prec a_1$, then $\omega_L = [b, b_1]^2 \cup [a \wedge b, a_1]^2 \cup \mathrm{id}_L$.*

Proof. The first two assertions are easy. Suppose that there are elements x, y with $a \wedge b < x \prec y \prec a_1$. Since y is $\wedge$-reducible, $y = a_1 \wedge z$ for some $z \parallel a$. We have $b \vee x = b_1 > a_1 > y = a \wedge z$ and so, by (III), either $b_1 \geqq a$ or $b_1 \geqq z$. If $b_1 \geqq a$ then $b \prec a \vee b$, so that $a \wedge b \prec a$ by 7.2, a contradiction. If $b_1 \geqq z$ then, since b_1 may cover only b and a_1, we get a contradiction with $y = a \wedge z$.

The rest follows from 1.2. ■

7.4. Lemma. *Suppose that $[a \wedge b, a]$ does not contain a fine subinterval and $a \wedge b \prec c_1 \prec a_1$. Then there exists an $n \geqq 1$ and elements $b \prec b_1 \prec b_2 \prec \ldots \ldots \prec b_n \prec a \vee b, a \wedge b \prec c_1 \prec a_1 \prec c_2 \prec a_2 \ldots \prec c_n \prec a_n \prec a$ with $a_i = b_i \wedge a$ and $b_i = a_i \vee b$ for all $i = 1, \ldots, n$. Moreover, $\omega_L = [a_n, a]^2 \cup [b_n, b]^2 \cup \mathrm{id}_L$.*

Proof. Suppose that $k \geqq 1$ and that we have constructed elements $b = b_0 \prec \prec b_1 \prec b_2 \prec \ldots \prec b_k < a \vee b$, $a \wedge b \prec c_1 \prec a_1 \prec \ldots \prec c_k \prec a_k < a$ such that $a_i = b_i \wedge a$ and $b_i = a_i \vee b$ for all $i = 1, \ldots, k$. Suppose further that b_k is not covered by $a \vee b$. Since b_k is $\vee$-reducible, there exists exactly one element covering b_k; denote it by b_{k+1} and put $a_{k+1} = b_{k+1} \wedge a$. If it were $a_{k+1} = a_k$ then we should have $b_{k-1} \vee c_k = b_k > a_k = b_{k+1} \wedge a$, a contradiction by (III). Hence $a_k < < a_{k+1} < a$. Now evidently $b \vee a_{k+1} = b_{k+1}$ and so $a_{k+1} \prec b_{k+1}$.

Put $\alpha = [a_{k+1}, a]^2 \cup [b_{k+1}, a \vee b]^2 \cup \mathrm{id}_L$ and let us prove $\omega_L \subseteq \alpha$. If $[b, a \vee b]$ contains a fine subinterval $[f, g]$, then evidently $[f, g] \subseteq [b_{k+1}, a \vee b]$ and so $\omega_L = [f, g]^2 \cup \mathrm{id}_L \subseteq \alpha$. Now let $[b, a \vee b]$ contain no fine subinterval, so that $[b, a \vee b]$ is a chain. It is enough to prove that $[\bar{a}, a]^2 \cup [\bar{b}, a \vee b]^2 \cup \mathrm{id}_L$ is a congruence. By 1.2 it is enough to prove that there does not exist an x with $x < < a \vee b$, $x \nleqq a$ and $x \parallel \bar{b}$. Suppose that there is such an x. Evidently $x \nleqq s$, so that $x \geqq u$. Since $x \ngeqq b$, we get $x \ngeqq v$ and so $x \leqq q$. Evidently $\bar{b} < x \vee b < a \vee b$. Hence $x \vee b > \bar{b} > \bar{a} > c_1 = a \wedge z$ for some $z \parallel a$ and so $x \vee b \geqq z$ by (III);

consequently $z < a \vee b$. However, evidently $z \not\geqq b$, so that $z \leqq s$ and we get $z \leqq a$, a contradiction.

Thus we have proved $\omega_L \subseteq \alpha$. Suppose $a_k \prec a_{k+1}$. Then it follows from 1.2 that $[a_k, a_{k+1}]^2 \cup [b_k, b_{k+1}]^2 \cup \mathrm{id}_L$ is a congruence, a contradiction. Next suppose that there exist elements x, y with $a_k < x \prec y \prec a_{k+1}$. We have $y = a \wedge z$ for some $z \parallel a$; since $b_k \vee x = b_{k+1} > y = a \wedge z$, we get $b_{k+1} \geqq z$ by (III). However, $z \neq b_{k+1}$ and thus either $z \leqq a_{k+1}$ or $z \leqq b_k$, so that either $z \leqq a$ or $y = a \wedge z \leqq \leqq a_k$, a contradiction.

We have proved that there exists exactly one element c_{k+1} with $a_k \prec c_{k+1} \prec a_{k+1}$. Now we can construct elements $b \prec b_1 \prec b_2 \prec \ldots \prec b_n \prec a \vee b$ and $a \wedge b \prec \prec c_1 \prec a_1 \prec \ldots \prec c_n \prec a_n < a$ with $a_i = b_i \wedge a$ and $b_i = a_i \vee b$ by induction. Suppose $a_n < x < a$ for some x. Then $b_n \vee x = a \vee b > r \wedge s$, a contradiction by (III). Hence $a_n \prec a$. The rest is easy. ∎

7.5. Lemma. *L does not contain a fine interval.*

Proof. If $[a \wedge b, a]$ does not contain a fine subinterval then it follows from 7.3 and 7.4 that L does not contain a fine interval. Now, if $[a \wedge b, a]$ contained a fine subinterval, then the subdirect irreducibility of L would imply that $[b, a \vee b]$ does not contain a fine subinterval and by the duals of 7.3 and 7.4 L would contain no fine interval again. ∎

7.6. Lemma. *Let $n \geqq 0$ and $b \prec b_1 \prec \ldots \prec b_n \prec a \vee b$, $a \wedge b \prec c_1 \prec a_1 \prec \ldots \ldots \prec c_n \prec a_n \prec a$ where $a_i = b_i \wedge a$ and $b_i = a_i \vee b$ for all $i = 1, \ldots, n$. If $n = 0$ then $L \simeq A_4$; if $n \geqq 1$ then $L \simeq B_n$.*

Proof. Since v is $\wedge$-reducible, there are exactly two elements v_1, v_2 covering v. Evidently $v_1 \not\geqq u$ and so $v_1 \leqq s$. Similarly $v_2 \leqq s$. If it were $v_1, v_2 \leqq r$ then $v_1, v_2 \leqq a$, a contradiction with 7.5. Hence either $v_1 \geqq t$ or $v_2 \geqq t$. We have proved $v \prec t$. Quite similarly $q \prec r$.

We have $t \not\leqq \bar{a}$, so that $t \vee \bar{a} \geqq a = r \wedge s$; by (III), $t \vee \bar{a} = s$. Hence $a \prec s$. We have $q \not\geqq b_1$, so that $q \wedge b_1 \leqq b = u \vee v$; by (III), $q \wedge b_1 = u$. Hence $u \prec b$.

There exists exactly one $q' \succ u$ with $q' \leqq q$ and exactly one $r' \succ a \vee b$. Since $[a \vee b, r']$ is not a fine interval, there exists an x with $x \prec r'$ and $x \neq a \vee b$. It is easy to see that $x \leqq q$, so that $x \geqq q'$. Hence $q' \leqq r'$. Now we can verify by 1.2 that $[q', q]^2 \cup [r', r]^2 \cup \mathrm{id}_L$ is a congruence, so that it must be trivial and we get $u \prec q$ and $a \vee b \prec r$.

For every $i = 1, \ldots, n$ denote by d_n the element covering c_n and different from a_n. It is easy to see that $t < d_n < s$. Suppose that we have proved $s \succ d_n \succ d_{n-1} \succ \ldots \ldots \succ d_k$ for some $k \in \{2, \ldots, n+1\}$. There exists an x with $t \leqq x \prec d_k$. Since $d_{k-1} \notin \{s, d_n, d_{n-1}, \ldots, d_k\}$ and every element of $[t, s] \setminus \{s, d_n, d_{n-1}, \ldots, d_k\}$ is $\leqq x$, we get $d_{k-1} \leqq x$. We have $a_{k-2} \vee t \geqq c_{k-1} = a \wedge x$ (where $a_0 = a \wedge b$), so that $a_{k-2} \vee t \geqq x$ by (III); but $a_{k-2} \vee t \leqq d_{k-1}$, so that $x = d_{k-1}$ and $d_{k-1} \prec d_k$.

It follows by induction that $s \succ d_n \succ d_{n-1} \succ \ldots \succ d_1$. There exists an x with $t \leqq x \prec d_1$ and a $y \prec a \wedge b$. By 1.2 it is easy to see that $[v, y]^2 \cup [t, x]^2 \cup \mathrm{id}_L$ is a congruence, so that it is trivial, so that $t \prec d_1$ and $v \prec a \wedge b$. ∎

7.7. Lemma. *L is isomorphic either to A_4 or to B_n for some $n \geqq 1$ or to the dual of B_n for some $n \geqq 1$.*

Proof. It follows from the preceding lemmas and their duals. ∎

8. WEAKLY PRIMITIVE LATTICES: UNICITY OF STAR ELEMENTS

In this section let L be a weakly primitive lattice and z and $\bar{z}$ its two star elements; our aim is to prove $z = \bar{z}$. Denote by p_1, p_2, p_3 the three elements covering z and put

$$b_1 = p_2 \vee p_3\,, \quad b_2 = p_1 \vee p_3\,, \quad b_3 = p_1 \vee p_2\,, \quad a_1 = b_2 \wedge b_3\,,$$
$$a_2 = b_1 \wedge b_3\,, \quad a_3 = b_1 \wedge b_2\,,$$
$$u = p_1 \vee p_2 \vee p_3 = p_1 \vee b_1 = p_2 \vee b_2 = p_3 \vee b_3 = b_1 \vee b_2 =$$
$$= b_1 \vee b_3 = b_2 \vee b_3\,.$$

By (I), $p_1 \wedge b_1 = p_2 \wedge b_2 = p_3 \wedge b_3 = z$. This implies that $b_1 \parallel b_2$, $b_1 \parallel b_3$, $b_2 \parallel b_3$. There exists exactly one triple q_1, q_2, q_3 such that $b_1 \leqq q_1 \prec u$, $b_2 \leqq q_2 \prec u$, $b_3 \leqq q_3 \prec u$. Evidently $p_1 \parallel q_1$, $p_2 \parallel q_2$, $p_3 \parallel q_3$ and consequently $q_1 \wedge q_2 \wedge q_3 = z$. Denote by $\bar{p}_1, \bar{p}_2, \bar{p}_3$ the three elements covering $\bar{z}$ and define $\bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{u}, \bar{q}_1, \bar{q}_2, \bar{q}_3$ analogously.

8.1. Lemma. *Let $i \in \{1, 2, 3\}$. Then p_i is the companion of q_i in $[z, u]$.*

Proof. Let $x \in [z, u]$ and $x \ngeqq p_i$. Then $p_i \wedge x = z$. Since, moreover, $p_i \wedge q_i = z$, we get $p_i \wedge (x \vee q_i) = z$ by (I), so that $x \vee q_i \neq u$ and thus $x \leqq q_i$. ∎

8.2. Lemma. *If there exist elements $a, b \in L \smallsetminus (z, u)$ with $a \wedge b \in (z, u)$ then there exists an element $c \in L \smallsetminus (z, u)$ with $u \wedge c \in (z, u)$.*

Proof. We have $u \wedge b \in [z, u]$ and $u \wedge b \neq z$. If $u \wedge b \in (z, u)$ then we can put $c = b$. If $u \wedge b = u$ then $u \wedge a = (u \wedge b) \wedge a = u \wedge (a \wedge b) = a \wedge b \in (z, u)$ and we can put $c = a$. ∎

8.3. Lemma. *If there exists an element $c \in L \smallsetminus (z, u)$ with $u \wedge c \in (z, u)$ then there exists an element $d \in L \smallsetminus (z, u)$ with $u \wedge d \in \{q_1, q_2, q_3\}$.*

Proof. Suppose $c \vee q_1 \geqq u$, $c \vee q_2 \geqq u$, $c \vee q_3 \geqq u$. Then $c \vee q_1 = c \vee q_2 = c \vee q_3 = c \vee u$ and so $c \vee u = c \vee (q_1 \wedge q_2 \wedge q_3) = c \vee z = c$ by (II), a contradiction. Hence $c \vee q_i \ngeq u$ for some $i \in \{1, 2, 3\}$. Put $d = c \vee q_i$. Evidently $d \notin (z, u)$ and $u \wedge d = q_i$. ■

8.4. Lemma. *Let there exist an element $d \in L \setminus (z, u)$ with $u \wedge d = q_i$ for some $i \in \{1, 2, 3\}$. Then L contains a fine interval $[f, g]$ and $b_i \leqq f \prec g \leqq q_i$.*

Proof. It follows from 1.6. ■

8.5. Lemma. *Let there exist elements $a, b \in L \setminus (z, u)$ with $a \wedge b \in (z, u)$. Then there do not exist elements $m, n \in L \setminus (z, u)$ with $m \vee n \in (z, u)$.*

Proof. Suppose that there exist elements $a, b, m, n \in L \setminus (z, u)$ with $a \wedge b \in (z, u)$ and $m \vee n \in (z, u)$. By 8.2, 8.3, 8.4 and their duals L contains a fine interval $[f, g]$ and $b_i \leqq f \prec g \leqq q_i$ and $p_j \leqq f \prec g \leqq a_j$ for some $i, j \in \{1, 2, 3\}$. Since $p_i \parallel q_i$, we have $i \neq j$. Denote by k the only element of $\{1, 2, 3\} \setminus \{i, j\}$. We have $p_k \leqq b_i \leqq \leqq f < g < q_k$, a contradiction. ■

8.6. Lemma. *Suppose that there are elements $a, b \in L \setminus (z, u)$ with $a \wedge b \in (z, u)$ and elements $m, n \in L \setminus (\bar{z}, \bar{u})$ with $m \wedge n \in (\bar{z}, \bar{u})$. Then $z = \bar{z}$.*

Proof. By 8.2, 8.3 and 8.4 there exists a fine interval $[f, g]$ of L and we can suppose $b_1 \leqq f \prec g \leqq q_1$ and $\bar{b}_1 \leqq f \prec g \leqq \bar{q}_1$ (the remaining eight possibilities would be quite similar). Since $z \vee \bar{z} \leqq f < u$, it follows from 8.5 that $z, \bar{z}$ are comparable.

Suppose $\bar{z} < z$. Since $z \vee \bar{p}_2 \leqq f < u$ and $z \vee \bar{p}_3 \leqq f < u$, it follows from 8.5 that both $\bar{p}_2$ and $\bar{p}_3$ are comparable with z; since $\bar{z} < z$ and $\bar{z} \prec \bar{p}_2$, $\bar{z} \prec \bar{p}_3$, we get $\bar{p}_2, \bar{p}_3 \leqq z$. Since $\bar{p}_2 \parallel \bar{p}_3$ and z is $\wedge$-reducible, L must a contain a fine interval contained in $[\bar{p}_2 \vee \bar{p}_3, z]$, so that $[f, g] \subseteq [\bar{p}_2 \vee \bar{p}_3, z]$, a contradiction.

Hence $\bar{z} < z$ is impossible. Quite similarly, $z < \bar{z}$ is impossible. Since $z, \bar{z}$ are comparable, we get $z = \bar{z}$. ■

8.7. Lemma. *The following two conditions can not take place simultaneously:*

(i) *There exist elements $a, b \in L \setminus (z, u)$ with $a \wedge b \in (z, u)$.*

(ii) *There exist elements $m, n \in L \setminus (\bar{z}, \bar{u})$ with $m \vee n \in (\bar{z}, \bar{u})$.*

Proof. Suppose that (i), (ii) are satisfied. By 8.2, 8.3 and their duals there exist elements $d \in L \setminus (z, u)$ and $\bar{d} \in L \setminus (\bar{z}, \bar{u})$ with $u \wedge d \in \{q_1, q_2, q_3\}$ and $\bar{z} \vee \bar{d} \in \in \{\bar{p}_1, \bar{p}_2, \bar{p}_3\}$. We shall suppose $u \wedge d = q_1$ and $\bar{z} \vee \bar{d} = \bar{p}_1$; the remaining eight cases are analogous. By 1.6 L contains a fine interval $[f, g]$ and $b_1 \leqq f \prec g \leqq q_1$ and $\bar{p}_1 \leqq f \prec g \leqq \bar{a}_1$.

Since $z \vee \bar{z} \leqq f < u$, it follows from 8.5 that either $z < \bar{z}$ or $\bar{z} \leqq z$. Suppose $\bar{z} \leqq z$. It follows from 8.5 that $z \neq \bar{z}$. Hence $\bar{z} < z$. Since $z \vee \bar{p}_1 \leqq f < u$, it follows from 8.5 that $z, \bar{p}_1$ are comparable; since $\bar{z} < z$ and $\bar{z} \prec \bar{p}_1$, we get $\bar{p}_1 \leqq z$. Since $\bar{p}_1$ is $\vee$-reducible and z is $\wedge$-reducible, L contains a fine interval contained in $[\bar{p}_1, z]$; since L contains only one fine interval, $[f, g] \subseteq [\bar{p}_1, z]$, a contradiction.

We have proved $z < \bar{z}$. Hence either $p_1 \leqq \bar{z}$ or $p_2 \leqq \bar{z}$ or $p_3 \leqq \bar{z}$. We can not have $p_1 \leqq \bar{z}$, since $\bar{z} < q_1$. We shall assume $p_3 \leqq \bar{z}$, since the case $p_2 \leqq \bar{z}$ can be considered analogously.

Quite similarly, either $u \leqq \bar{q}_3$ or $u \leqq \bar{q}_2$. We shall assume $u \leqq \bar{q}_3$, since the other case can be considered analogously.

Let us prove $p_1 \parallel \bar{q}_2$. If it were $\bar{q}_2 \leqq p_1$, we should have $\bar{q}_2 \leqq p_1 < u \leqq \bar{q}_3$, a contradiction. If it were $p_1 \leqq \bar{q}_2$, then, since $b_1 \leqq f < \bar{q}_2$, we should have $u = p_1 \vee b_1 \leqq \bar{q}_2$, so that the interval $[u, \bar{q}_2 \wedge \bar{q}_3]$ would contain a fine interval of L and thus the interval $[f, g]$, a contradiction.

Let us prove $p_1 < \bar{p}_2$. Suppose that this is not true. If it were $\bar{p}_2 \leqq p_1$, then $p_3 \leqq \bar{z} < \bar{p}_2 \leqq p_1$, a contradiction. Thus $p_1 \parallel \bar{p}_2$. Since $p_1 \parallel \bar{q}_2$, too, as we have proved above, $p_1 \wedge \bar{p}_2 = p_1 \wedge \bar{q}_2 = z$. By (I) we get $p_1 \wedge (\bar{p}_2 \vee \bar{q}_2) = z$, i.e. $p_1 \wedge \bar{u} = z$, i.e. $p_1 = z$, a contradiction.

Since $p_1 < \bar{p}_2$ and $p_3 \leqq \bar{z} < \bar{p}_2$, we get $b_2 = p_1 \vee p_3 \leqq \bar{p}_2 \leqq \bar{q}_1 \wedge \bar{q}_3$. Hence $[b_2, \bar{q}_1 \wedge \bar{q}_3]$ must contain a fine interval of L and thus the interval $[f, g]$, so that $b_2 \leqq f$, a contradiction. ∎

8.8. Lemma. *Suppose that $L \setminus (z, u)$ is a sublattice. Then $[z, u]^2 \cup \mathrm{id}_L$ is a congruence of L.*

Proof. It is easy. ∎

8.9. Lemma. *Let $z \leqq \bar{z} < \bar{u} \leqq u$. Then $z = \bar{z}$.*

Proof. Suppose $z < \bar{z}$, so that $\bar{u} < u$. We have $\bar{z} \geqq p_i$ for some $i \in \{1, 2, 3\}$; we can suppose $\bar{z} \geqq p_1$. We have $\bar{u} \leqq q_j$ for some j and $j \neq 1$; we can suppose $\bar{u} \leqq q_3$.

Suppose $\bar{q}_1 \nleqq q_2$, $\bar{q}_2 \nleqq q_2$, $\bar{q}_3 \nleqq q_2$. Then, by 8.1, $\bar{q}_1 \geqq p_2$, $\bar{q}_2 \geqq p_2$, $\bar{q}_3 \geqq p_2$, so that $\bar{z} = \bar{q}_1 \wedge \bar{q}_2 \wedge \bar{q}_3 \geqq p_2$. Hence $\bar{z} \geqq p_1 \vee p_2$ and L contains a fine interval contained in $[p_1 \vee p_2, \bar{z}]$, a contradiction, since by 8.8, 8.2, 8.3, 8.4 and the duals of 8.2, 8.3, 8.4 we have $\omega_L \subseteq [\bar{z}, \bar{u}]^2 \cup \mathrm{id}_L$.

Hence there exists a $k \in \{1, 2, 3\}$ with $\bar{q}_k \leqq q_2$. If it were $\bar{u} \leqq q_2$ then we should have $\bar{u} \leqq q_2 \wedge q_3$, so that there would exist a fine interval of L contained in $[\bar{u}, q_2 \wedge q_3]$, again a contradiction with 8.8, 8.2, 8.3, 8.4 and the duals of 8.2, 8.3, 8.4. Hence $\bar{u} \nleqq q_2$ and thus evidently $\bar{u} \parallel q_2$. Since $\bar{q}_k \prec \bar{u}$ and $\bar{q}_k \leqq q_2$, we get $\bar{q}_k = \bar{u} \wedge q_2$. Quite similarly $\bar{p}_l = \bar{z} \vee p_2$ for some l. We get a contradiction with 8.5. ∎

8.10. Lemma. *Suppose that $L \setminus (z, u)$ is a sublattice of L. Then $L \setminus (\bar{z}, \bar{u})$ is a sublattice of L.*

Proof. Suppose that $L \setminus (\bar{z}, \bar{u})$ is not a sublattice. By 8.2 and 8.3 we can suppose $\bar{q}_1 = \bar{u} \wedge d$ for some $d \notin (\bar{z}, \bar{u})$. There exists a fine interval $[f, g]$ of L contained in $[\bar{b}_1, \bar{q}_1]$. By 8.8, $f, g \in [z, u]$. Since $z \vee \bar{z} \leqq f < u$, the elements $z, \bar{z}$ are comparable.

Of course $z \neq \bar{z}$. Suppose $\bar{z} < z$. Since $\bar{p}_2 \vee z \leqq f < u$ and $\bar{p}_3 \vee z \leqq f < u$, the elements $\bar{p}_2, \bar{p}_3$ are both comparable with z; since $\bar{z} < z$, $\bar{z} \prec \bar{p}_2$ and $\bar{z} \prec \bar{p}_3$, we get $\bar{p}_2, \bar{p}_3 \leqq z$ and thus L contains a fine interval contained in $[\bar{p}_2 \vee \bar{p}_3, z]$, a contradiction. Hence $z < \bar{z}$.

Since $u \wedge \bar{u} \geqq g > z$, we have either $u < \bar{u}$ or $\bar{u} < u$. Suppose $u < \bar{u}$. Then $u \leqq \bar{q}_i$ for some $i \in \{1, 2, 3\}$. We have $i \neq 1$, since if $i = 1$, the interval $[u, \bar{q}_1]$ would contain a fine interval, a contradiction. We have $i \neq 2$ since if $i = 2$ then $\bar{p}_2 < \bar{b}_1 \leqq f < u \leqq \bar{q}_2$, a contradicition. Hence $u \leqq \bar{q}_3$. But then $\bar{p}_3 < \bar{b}_1 \leqq \leqq f < u \leqq \bar{q}_3$, a contradiction.

We have proved $z < \bar{z} < \bar{u} < u$. This is a contradiction with 8.9. ∎

8.11. Lemma. *Suppose that $L \setminus (z, u)$ and $L \setminus (\bar{z}, \bar{u})$ are sublattices of L. Then $z = \bar{z}$.*

Proof. By 8.8 there exist two elements $f, g \in [z, u] \cap [\bar{z}, \bar{u}]$ with $f < g$. Since $z \vee \bar{z} \leqq f < u$, the elements $z, \bar{z}$ are comparable. Suppose $z < \bar{z}$ and $u < \bar{u}$. If $i \in \{1, 2, 3\}$ then $p_i \vee \bar{z} \leqq u < \bar{u}$, so that p_i is comparable with $\bar{z}$; since $z < \bar{z}$ and $z \prec p_i$, we get $p_i \leqq \bar{z}$. Hence $u = p_1 \vee p_2 \vee p_3 \leqq \bar{z}$, a contradiction. If $\bar{z} < z$ and $\bar{u} < u$, a contradiction is obtained similarly. Now we get $z = \bar{z}$ by 8.9. ∎

8.12. Lemma. *L has at most one star and at most one costar element.*

Proof. It follows from 8.6, 8.7, 8.10, 8.11 and duality. ∎

9. WEAKLY PRIMITIVE LATTICES: 0 STAR AND 1 COSTAR

In this section let L be a weakly primitive lattice such that 0 is a star element of L and 1 is a costar element of L. Denote by p_1, p_2, p_3 the there atoms of L, put

$$b_1 = p_2 \vee p_3\,, \quad b_2 = p_1 \vee p_3\,, \quad b_3 = p_1 \vee p_2\,,$$
$$a_1 = b_2 \wedge b_3\,, \quad a_2 = b_1 \wedge b_3\,, \quad a_3 = b_1 \wedge b_2$$

and denote by q_1, q_2, q_3 the three coatoms of L (ordered so that $q_1 \geqq b_1$, $q_2 \geqq b_2$, $q_3 \geqq b_3$).

9.1. Lemma. (i) *$L = \{0\} \cup \{1\} \cup [p_1, q_2 \wedge q_3] \cup [p_2, q_1 \wedge q_3] \cup [p_3, q_1 \wedge q_2] \cup \cup [b_1, q_1] \cup [b_2, q_2] \cup [b_3, q_3]$ is a union of eight pairwise disjoint sets and the corresponding equivalence is a congruence of L.*

(ii) *If $q_1 \wedge q_2 < b_1$ and $q_1 \wedge q_3 < b_1$ then $[b_1, q_1]^2 \cup \mathrm{id}_L$ is a congruence of L.*

(iii) *If $q_2 \wedge q_3 = a_1$ then $[p_1, a_1]^2 \cup \mathrm{id}_L$ is a congruence of L.*

(iv) *If $q_2 \wedge q_1 < b_2$ and $q_2 \wedge q_3 < b_3$ then $[p_1, q_2 \wedge q_3]^2 \cup [b_2, q_2]^2 \cup \mathrm{id}_L$ is a congruence of L.*

Proof. It is easy. ∎

9.2. Lemma. *Suppose that* $q_1 \wedge q_2 = a_3$, $q_1 \wedge q_3 = a_2$ *and* $q_2 \wedge q_3 = a_1$. *Then* L *is isomorphic either to* D_0 *or to the dual of* D_0.

Proof. By 9.1 (ii), (iii) for every $i \in \{1, 2, 3\}$ both $[p_i, a_i]^2 \cup \mathrm{id}_L$ and $[b_i, q_i]^2 \cup \cup \mathrm{id}_L$ are congruences. Since L is subdirectly irreducible, at most one of the intervals $[p_1, a_1]$, $[p_2, a_2]$, $[p_3, a_3]$, $[b_1, q_1]$, $[b_2, q_2]$, $[b_3, q_3]$ contains more than one element. All of these intervals can not be one-element, since then L would be the 8-element Boolean algebra and thus not a weakly primitive lattice. Let e.g. $p_1 < a_1$. There exists exactly one element x with $p_1 \leqq x \prec a_1$. Evidently $[p_1, x]^2 \cup \mathrm{id}_L$ and $[x, a_1]^2 \cup \mathrm{id}_L$ are congruences of L, so that $p_1 = x$ and thus $p_1 \prec a_1$; thus $L \simeq D_0^*$. Analogously, if either $p_2 < a_2$ or $p_3 < a_3$ then $L \simeq D_0^*$ and if either $b_1 < q_1$ or $b_2 < q_2$ or $b_3 < q_3$ then $L \simeq D_0$. ∎

Consider the following condition:

(C) For every pair i, j of different numbers from $\{1, 2, 3\}$ we have either $q_i \wedge q_j < < b_i$ or $q_i \wedge q_j < b_j$.

9.3. Lemma. *Let* (C) *be satisfied. Then* $b_i < q_i$ *for at most one* $i \in \{1, 2, 3\}$.

Proof. If $q_1 \wedge q_2 = a_3$, $q_1 \wedge q_3 = a_2$ and $q_2 \wedge q_3 = a_1$, this assertion follows from 9.2. In the opposite case we can suppose $a_1 < q_2 \wedge q_3 < b_3$, since all the remaining possibilities are similar. Evidently $b_2 < q_2$. From $q_2 \wedge q_3 < b_3 = = p_1 \vee p_2$ we get $q_3 = b_3$ by (III). It remains to prove $q_1 = b_1$. Suppose, on the contrary, that $b_1 < q_1$. We have either $q_1 \wedge q_2 < b_1 = p_2 \vee p_3$ or $q_1 \wedge q_2 < b_2 = = p_1 \vee p_3$; by (III) in any one of these cases $q_1 \wedge q_2 = p_3$ and so $q_1 \wedge q_2 = a_3 = = p_3$. Since $q_3 = b_3$, we have $q_1 \wedge q_3 < b_3$. Now applying 9.1 (iv) twice we get that $[p_1, q_2 \wedge q_3]^2 \cup [b_2, q_2]^2 \cup \mathrm{id}_L$ and $[p_2, q_1 \wedge q_3]^2 \cup [b_1, q_1]^2 \cup \mathrm{id}_L$ are congruences of L, a contradiction with the subdirect irreducibility of L. ∎

9.4. Lemma. *Let* K *be a weakly primitive lattice without star and costar elements. Let* K *have two atoms* a, b; *suppose that* b *is* $\wedge$-*irreducible and that* $K \setminus [b, 1]$ *is a chain. Then* K *is isomorphic to the pentagon.*

Proof. (By induction on Card (K).) It is evident that K can not be isomorphic to any one of the lattices A_1, A_2, A_3, A_4, B_n $(n \geqq 1)$, C_n $(n \geqq 1)$ and their duals. By the results of Sections 3, 4, 5, 6, 7, K results from a weakly primitive lattice M by one of the five constructions R, P, P^*, Q, Q^*. If it results from M by the construction R, then evidently M has only two elements and K is isomorphic to the pentagon. Evidently, K can result from M by neither Q nor Q^*. Suppose that K results by P^* from M, so that $M = K \setminus \{0, b\}$ is a weakly primitive lattice. Then we do not have $a \prec a \vee b$; M must have two atoms, both of them belonging to $K \setminus [b, 1]$, a contradiction. Finally, suppose that K results from M by P, $K = M \cup \{1, r\}$. Then evidently M satisfies the conditions imposed on K, so that by the induction assumption M is isomorphic to the pentagon. But then evidently K can not be weakly primitive, a contradiction. ■

9.5. Lemma. *Suppose that* (C) *is satisfied,* $a_1 < q_2 \wedge q_3 < b_3$ *and* $q_1 \wedge q_2 = a_3$. *Then* L *is isomorphic either to* D_1 *or to* D_1^*.

Proof. Evidently $b_2 < q_2$. By 9.3, $q_1 = b_1$ and $q_3 = b_3$. By 9.1 (iv), $[p_1, q_2 \wedge \wedge q_3]^2 \cup [b_2, q_2]^2 \cup \mathrm{id}_L$ is a congruence of L. By 9.1 (iii), $p_2 = a_2$ and $p_3 = a_3$. Define elements a and b by $a_1 \prec a \leqq q_2 \wedge q_3$ and $b_2 \prec b \leqq q_2$.

First suppose that $b_3 \wedge b = a$. If there existed an element z with $p_1 < z \parallel a_1$, then evidently $z \leqq q_2 \wedge q_3$ and $z \vee a_1 \geqq a = b_3 \wedge b$, a contradiction with (III). Hence there is no such z and thus evidently $[p_1, a_1]^2 \cup \mathrm{id}_L$ and $[a_1, q_2 \wedge q_3]^2 \cup \cup [b_2, q_2]^2 \cup \mathrm{id}_L$ are congruences of L, so that $p_1 = a_1$. Now it is easy to see that $[a_1, a]^2 \cup [b_2, b]^2 \cup \mathrm{id}_L$ and $[a, q_2 \wedge q_3]^2 \cup [b, q_2]^2 \cup \mathrm{id}_L$ are congruences of L, so that $b = q_2$, $a = q_2 \wedge q_3$ and L is the ten-element lattice which is evidently not subdirectly irreducible, a contradiction.

Next suppose $b_3 \wedge b = a_1$. Then evidently $[b_2, b]$ is a fine interval, so that $[0, q_2 \wedge q_3]$ is a chain; evidently $p_1 = a_1$. Put $\alpha = [a_1, q_2 \wedge q_3]^2 \cup [b_2, q_2]^2$, so that α is a congruence of $[a_1, q_2]$ and $\alpha \cup \mathrm{id}_L$ is a congruence of L. It is easy to see that if $x \in [a_1, q_2 \wedge q_3]$ and $y \in [b_2, q_2]$ then $\langle b_2, b\rangle$ belongs to the congruence of $[a_1, q_2]$ generated by $\langle x, y\rangle$; it follows from 1.4 that $[a_1, q_2]$ is subdirectly irreducible. Hence $[a_1, q_2]$ is a weakly primitive lattice without star and costar elements; b_2 is its atom, b_2 is $\wedge$-irreducible and $[a_1, q_2] \setminus [b_2, q_2]$ is a chain; it follows from 9.4 that $[a_1, q_2]$ is isomorphic to the pentagon. Hence $L \simeq D_1$.

Finally suppose that $b_3 \wedge b > a$. It is easy to see that if $x \in [p_1, q_2 \wedge q_3]$ and $y \in [b_2, q_2]$ then the congruence of $[p_1, q_2]$ generated by $\langle x, y\rangle$ contains $\langle a, b_3 \wedge b\rangle$; now it follows from 1.4 that $[p_1, q_2]$ is subdirectly irreducible. Hence $[p_1, q_2]$ is a weakly primitive lattice without star and costar elements.

Suppose that $[0, q_2 \wedge q_3]$ is not a chain. Then there is a fine interval contained in $[0, q_2 \wedge q_3]$, so that there is no fine interval contained in $[b_2, 1]$ and $[b_2, 1]$ is a chain; thus evidently $q_2 = (q_2 \wedge q_3) \vee b_2$ and $q_2 \wedge q_3 \prec q_2$. Hence $q_2 \wedge q_3$ is a coatom of $[p_1, q_2]$ and $q_2 \wedge q_3$ is $\vee$-irreducible; moreover, $[p_1, q_2] \setminus [p_1, q_2 \wedge \wedge q_3]$ is a chain. It follows from the dual of 9.4 that $[p_1, q_2]$ is isomorphic to the pentagon, a contradiction with the assumption that $[0, q_2 \wedge q_3]$ is not a chain.

Hence $[0, q_2 \wedge q_3]$ is a chain. Evidently $[p_1, a_1]^2 \cup \mathrm{id}_L$ and $[a_1, q_2 \wedge q_3]^2 \cup \cup [b_2, q_2]^2 \cup \mathrm{id}_L$ are congruences of L and thus $p_1 = a_1$. Now $[a_1, q_2]$ is a weakly primitive lattice without star and costar elements, b_2 is its atom, b_2 is $\wedge$-irreducible and $[a_1, q_2] \setminus [b_2, q_2]$ is a chain; it follows from 9.4 that $[a_1, q_2]$ is isomorphic to the pentagon. Hence $L \simeq D_1^*$. ■

9.6. Lemma. *Denote by b the only element with $b_2 \prec b$. Suppose that* (C) *is satisfied, $a_1 < q_2 \wedge q_3 < b_3$, $a_3 < q_2 \wedge q_1 < b_1$ and b is $\vee$-reducible. Then $L \simeq G_n$ for some $n \geqq 2$.*

Proof. Evidently $b \leqq q_2$. By 9.3, $q_1 = b_1$ and $q_3 = b_3$. Denote by $\bar{b}$ the only element with $\bar{b} \prec b$ and $\bar{b} \neq b_2$. We have either $a_1 < \bar{b} \leqq q_2 \wedge q_3$ or $a_3 < \bar{b} \leqq \leqq q_2 \wedge q_1$; without loss of generality we shall consider only the case $a_1 < \bar{b} \leqq \leqq q_2 \wedge q_3$. Evidently $\bar{b} = q_3 \wedge b$. Denote by e the only element with $e \prec \bar{b}$, so that $a_1 \leqq e$. We have either $a_1 < e$ or $a_1 = e$; in the first case evidently there is a fine interval contained in $[e, q_2 \wedge q_3]$ and in the second case $\omega_L = [a_1, \bar{b}]^2 \cup \cup [b_2, b]^2 \cup \mathrm{id}_L$; hence in any case there is no fine interval contained in $[b_2, q_2]$ and thus $[b_2, q_2]$ is a chain and every element of $[b_2, q_2]$ is $\vee$-reducible. Denote by $c_1 \prec \ldots \prec c_n$ the elements of $[b, q_2]$ (so that $n \geqq 1$, $c_1 = b$ and $c_n = q_2$) and, for any $i \in \{1, \ldots, n\}$, by $\bar{c}_i$ the only element with $\bar{c}_i \prec c_i$ and $\bar{c}_i \ngeqq b_2$ (hence either $\bar{c}_i = q_3 \wedge c_i$ or $\bar{c}_i = q_1 \wedge c_i$). Since $q_1 \wedge q_2 > a_3$, we have $n \geqq 2$ and $\bar{c}_k < q_1$ for some $k \in \{2, \ldots, n\}$; let k be the smallest number with $\bar{c}_k < q_1$.

Suppose $a_1 < e$. If it were $k = 2$, then $e \vee b_2 = b > \bar{b} = q_3 \wedge c_2$, a contradiction with (III). Hence $k \geqq 3$. We have $\bar{c}_{k-2} < \bar{c}_{k-1} < q_3$. If it were $\bar{c}_{k-2} < x < \bar{c}_{k-1}$ for some x, then $x \vee c_{k-2} = c_{k-1} > \bar{c}_{k-1} = q_3 \wedge c_k$, a contradiction by (III). Hence $\bar{c}_{k-2} \prec \bar{c}_{k-1}$ and it is evident that $[\bar{c}_{k-2}, \bar{c}_{k-1}]^2 \cup [c_{k-2}, c_{k-1}]^2 \cup \mathrm{id}_L$ is a congruence of L. But then L has no fine interval; this evidently leads to a contradiction with $a_1 < e$.

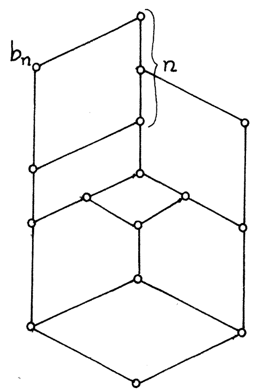

Fig. 16: H_n

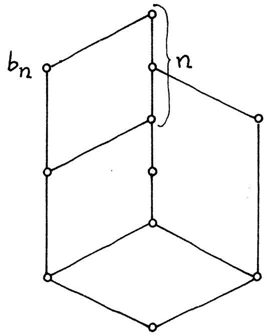

Fig. 17: I_n

We have proved $a_1 = e$, so that $a_1 \prec \bar{b}$ and $[a_1, \bar{b}]^2 \cup [b_2, b]^2 \cup \mathrm{id}_L$ is a congruence. Consequently L has no fine interval. Evidently $p_2 = a_2$, $[0, q_2 \wedge q_3]$ is a chain, every element of $[0, q_2 \wedge q_3]$ is $\wedge$-reducible, $[0, q_2 \wedge q_1]$ is a chain, every element of $[0, q_2 \wedge q_1]$ is $\wedge$-reducible, $p_1 = a_1$, $p_3 = a_3$. If it were $\bar{c}_i < \bar{c}_{i+1}$ for some $i \in \{1, \ldots, n-1\}$, then evidently $\bar{c}_i \prec \bar{c}_{i+1}$, so that $[\bar{c}_i, \bar{c}_{i+1}]^2 \cup [c_i, c_{i+1}]^2 \cup \mathrm{id}_L$

would be a congruence of L, a contradiction. Hence $\bar{c}_2 < q_1$, $\bar{c}_3 < q_3$, $\bar{c}_4 < q_1$, $\bar{c}_5 < q_3, \ldots$; evidently $L \simeq G_n$. ∎

Let us define lattices H_n ($n \geqq 0$) and I_n ($n \geqq 2$) so that we indicate their underlying sets and all the pairs x, y with $x \prec y$:

$H_n = \{0, 2, 3, \ldots, a_1, \ldots, a_n, b_1, \ldots, b_n\}$; $0 \prec 2 \prec 5 \prec 6 \prec 10 \prec a_1 \prec \ldots \prec a_n$, $0 \prec 3 \prec 7 \prec 8 \prec 10, 2 \prec 4 \prec 9 \prec 6, 3 \prec 4, 9 \prec 8, 5 \prec b_1 \prec b_3 \prec b_5 \prec \ldots$, $7 \prec b_2 \prec b_4 \prec b_6 \prec \ldots, b_1 \prec a_1, \ldots, b_n \prec a_n$. (See Fig. 16 for $n = 3$.)

$I_n = \{0, 2, 3, 4, 5, a_1, \ldots, a_n, b_1, \ldots, b_n\}$; $0 \prec 2 \prec 4 \prec 5 \prec a_1 \prec \ldots \prec a_n$, $0 \prec 3 \prec 4$, $2 \prec b_1 \prec b_3 \prec b_5 \prec \ldots$, $3 \prec b_2 \prec b_4 \prec b_6 \prec \ldots$, $b_1 \prec a_1, \ldots$, $b_n \prec a_n$. (See Fig. 17 for $n = 3$.)

9.7. Lemma. *Let K be a weakly primitive lattice without star and costar elements. Suppose that K has two atoms a, b, that a, b are $\wedge$-reducible, that $K \setminus [a, 1]$ and $K \setminus [b, 1]$ are chains and that $[a \vee b, 1]$ contains a fine interval of K. Then either $K \simeq H_n$ for some $n \geqq 0$ or $K \simeq I_n$ for some $n \geqq 2$.*

Proof. (By induction on Card (K).) Evidently, if K is isomorphic to one of the lattices $A_1, A_2, A_3, A_4, B_n, C_n$ and their duals, then K is isomorphic to A_2, i.e. to H_0. By the results of Sections 3, 4, 5, 6, 7 it now remains to consider the case of K resulting from a weakly primitive lattice M by one of the five constructions R, P, P^*, Q, Q^*. Evidently, K can result by neither R nor P^* nor Q. Suppose that K results from M by Q^*, $K = M \cup \{0, 1, b, z\}$. Evidently M has two atoms and one of them, namely $a \vee b$, is $\wedge$-irreducible; $M \setminus [a \vee b, 1]$ is a chain. By 9.4, M is isomorphic to the pentagon, so that $K \simeq I_2$. Finally, suppose that K results from M by P, $K = M \cup \cup \{1, r\}$. If the meet of the two coatoms of K were an atom, then M would have an atom c such that c is $\wedge$-irreducible in M and $M \setminus [c, 1]$ is a chain; by 9.4, M would be isomorphic to the pentagon, a contradiction, since the construction P applied in any way to the pentagon does not give a weakly primitive lattice. Hence the two atoms of K are $\wedge$-reducible in M and M satisfies the conditions imposed on K; by the induction assumption either $M \simeq H_n$ for some $n \geqq 0$ or $M \simeq I_n$ for some $n \geqq 2$. But then it is easy to see that $K \simeq H_{n+1}$ in the first case and $K \simeq I_{n+1}$ in the second case. ∎

9.8. Lemma. *Denote by b the only element with $b_2 \prec b$. Suppose that (C) is satisfied, $a_1 < q_2 \wedge q_3 < b_3$, $a_3 < q_2 \wedge q_1 < b_1$ and b is $\vee$-irreducible. Then either $L \simeq D_n$ for some $n \geqq 2$ or $L \simeq E_n$ for some $n \geqq 0$.*

Proof. Evidently $b_2 < q_2$. By 9.3, $q_1 = b_1$ and $q_3 = b_3$. Put $\alpha = [p_1, q_2 \wedge q_3]^2 \cup \cup [p_3, q_1 \wedge q_2]^2 \cup [b_2, q_2]^2$. It is easy to see that $\alpha \cup \mathrm{id}_{[0,q_2]}$ is a congruence of $[0, q_2]$ and $\alpha \cup \mathrm{id}_L$ is a congruence of L. It is easy to see that if $x, y \in [0, q_2]$ and $\langle x, y\rangle \notin \alpha \cup \mathrm{id}_{[0,q_2]}$, then $\langle b_2, b\rangle$ belongs to the congruence of $[0, q_2]$ generated by $\langle x, y\rangle$. By 1.4, $[0, q_2]$ is subdirectly irreducible. Hence $[0, q_2]$ is a weakly primitive lattice without star and costar elements. Since $[b_2, b]$ is a fine interval of L, $[0, q_2 \wedge q_3]$ is a chain, every element of $[0, q_2 \wedge q_3]$ is $\wedge$-reducible and thus evidently $p_1 = a_1$; analogously, $[0, q_1 \wedge q_2]$ is a chain and $p_3 = a_3$. By 9.1 (iii), $p_2 = a_2$. Now $[0, q_2]$ is a weakly primitive lattice without star and costar elements, a_1 and a_3 are its atoms, a_1 and a_3 are $\wedge$-reducible in $[0, q_2]$, $[0, q_2] \setminus [a_1, q_2]$ and $[0, q_2] \setminus [a_3, q_2]$ are chains and $[a_1 \vee a_3, q_2]$ contains a fine interval of $[0, q_2]$. By 9.7, either $L \simeq D_n$ for some $n \geqq 2$ or $L \simeq E_n$ for some $n \geqq 0$, since either $[0, q_2] \simeq I_n$ for some $n \geqq 2$ or $[0, q_2] \simeq H_n$ for some $n \geqq 0$. ∎

9.9. Lemma. *Suppose that L satisfies either the condition (C) or the condition dual to (C). Then L is isomorphic to some of the lattices D_n $(n \geqq 0)$, D_n^* $(n \geqq 0)$, E_n $(n \geqq 0)$, E_n^* $(n \geqq 0)$, G_n $(n \geqq 2)$, G_n^* $(n \geqq 2)$.*

Proof. It follows from 9.2, 9.5, 9.6, 9.8. ∎

9.10. Lemma. *Suppose that neither (C) nor the dual of (C) is satisfied.* Then L is isomorphic either to F_n or to F_n^* for some $n \geqq 2$.

Proof. Since the dual of (C) is not satisfied, it is enough to consider the case $q_2 \wedge q_3 \nleqq b_2$ and $q_2 \wedge q_1 \nleqq b_2$ (the remaining two cases are quite similar). Denote by b the only element with $b_2 \prec b$.

First assume that b is $\vee$-reducible. Denote by $\bar{b}$ the only element with $\bar{b} \prec b$ and $\bar{b} \neq b_2$. We have either $\bar{b} \leqq q_3$ or $\bar{b} \leqq q_1$. It is enough to consider the case $\bar{b} \leqq q_3$. Denote by k the largest non-negative integer such that there exist elements $c_0, \ldots, c_k$, $d_0, \ldots, d_k$, $e_1, \ldots, e_k$ with

$$b_2 = c_0 \prec c_1 \prec \ldots \prec c_k \leqq q_2\,,$$
$$d_i = q_3 \wedge c_i\,, \quad d_i \prec c_i \quad (i = 0, \ldots, k)\,,$$
$$d_{i-1} < e_i \prec d_i \quad (i = 1, \ldots, k)\,.$$

Evidently, the elements $c_0, \ldots, c_k$, $d_0, \ldots, d_k$, $e_1, \ldots, e_k$ are uniquely determined and so we shall keep this notation. It follows from $q_2 \wedge q_1 \nleqq b_2$ that $c_k \neq q_2$. Denote by c_{k+1} the only element with $c_k \prec c_{k+1}$, so that $c_{k+1} \leqq q_2$. Put $d_{k+1} = q_3 \wedge c_{k+1}$. Let us prove $d_k \prec d_{k+1} \prec c_{k+1}$. If $k = 0$, this is evident. Let $k \geqq 1$. Of course, we have $d_k \leqq d_{k+1}$. If it were $d_k = d_{k+1}$, then $q_3 \wedge c_{k+1} = d_k < c_k = e_k \vee c_{k-1}$, a contradiction by (III). Hence $d_k < d_{k+1}$. Now evidently $d_{k+1} \prec c_{k+1}$. By the maximality of k we have $d_k \prec d_{k+1}$. Thus we have proved $d_k \prec d_{k+1} \prec c_{k+1}$. Now evidently $[d_k, d_{k+1}]^2 \cup [c_k, c_{k+1}]^2 \cup \mathrm{id}_L$ is a congruence and $\omega_L = [d_k, d_{k+1}]^2 \cup \cup [c_k, c_{k+1}]^2 \cup \mathrm{id}_L$. Especially, L has no fine interval.

Suppose that b is $\vee$-irreducible. Then $[b_2, b]$ is a fine interval. Since (C) is not satisfied, there exist numbers $i, j \in \{1, 2, 3\}$ such that $i \neq j$ and neither $q_i \wedge q_j \leqq \leqq b_i$ nor $q_i \wedge q_j \leqq b_j$. Denote by g the only element with $g \prec q_i \wedge q_j$. Since L can not have two fine intervals, g is $\wedge$-reducible. But then we can dualize the above considerations that were made under the assumption of $\vee$-reducibility of b and prove that L has no fine interval. However, this is a contradiction.

Thus we have proved that b is $\vee$-reducible and consequently all that was proved under this assumption is valid absolutely.

Since L has no fine interval, it is easy to see that $p_1 = a_1$, $p_2 = a_2$, $p_3 = a_3$.

Dualizing the above considerations we see: if $b_1 \nleqq q_1 \wedge q_3$ and $b_3 \nleqq q_1 \wedge q_3$ then $\omega_L \subseteq [p_2, q_1 \wedge q_2]^2 \cup [b_1, q_1]^2 \cup [b_3, q_3]^2 \cup \mathrm{id}_L$; if $b_1 \nleqq q_2 \wedge q_1$ and $b_2 \nleqq q_2 \wedge q_1$ then $\omega_L \subseteq [p_3, q_1 \wedge q_2]^2 \cup [b_1, q_1]^2 \cup [b_2, q_2]^2 \cup \mathrm{id}_L$. However, these cases are not possible, since $\omega_L = [d_k, d_{k+1}]^2 \cup [c_k, c_{k+1}]^2 \cup \mathrm{id}_L$. We get $q_2 \wedge q_1 < b_1$ and either $q_1 \wedge q_3 < b_1$ or $q_1 \wedge q_3 < b_3$.

Since (C) is not satisfied, we get: $q_3 \wedge q_2 \nleqq b_3$ and $q_3 \wedge q_2 \nleqq b_2$.

Let us prove $q_1 = b_1$. Suppose, on the contrary, that $b_1 < q_1$. Denote by z the only element with $b_1 \prec z$, so that $z \leqq q_1$. Since L has no fine interval, there exists an element y with $y \prec z$ and $y \neq b_1$. Since $q_2 \wedge q_1 < b_1$, we have $y \nleqq q_2$ and so evidently $a_2 < y \leqq q_1 \wedge q_3$. We have either $q_1 \wedge q_3 < b_1$ or $q_1 \wedge q_3 < b_3$; however, evidently only $q_1 \wedge q_3 < b_3$ is possible. Evidently $a_2 \prec b_1$. Since $[a_2, y]^2 \cup [b_1, z]^2 \cup \mathrm{id}_L$ can not be a congruence, there exists an x with $a_2 < x \prec y$; we have $x \prec \bar{x}$ for some $\bar{x} \neq y$. Evidently $\bar{x} \leqq q_1 \wedge q_3$, so that there is a fine interval contained in $[\bar{x} \vee y, q_1 \wedge q_3]$, a contradiction.

Denote by g the only element with $g \prec q_3 \wedge q_2$ and by $\bar{g}$ the only element with $g \prec \bar{g}$ and $\bar{g} \neq q_3 \wedge q_2$. We have either $\bar{g} \geqq p_2$ or $\bar{g} \geqq p_3$. If it were $\bar{g} \geqq p_2$, then the appropriate dualization of the above proof of $(q_2 \wedge q_3 \nleqq b_2 \,\&\, q_2 \wedge q_1 \nleqq b_2 \,\&\, \bar{b} \leqq q_3) \Rightarrow q_1 = b_1$ would give $p_3 = q_1 \wedge q_2$, a contradiction with $q_1 \wedge q_2 \nleqq b_2$. Hence $\bar{g} \geqq p_3$. The appropriate dualization of the above proof of $(q_2 \wedge q_3 \nleqq b_2 \,\&\, q_2 \wedge q_1 \nleqq b_2 \,\&\, \bar{b} \leqq q_3) \Rightarrow q_1 = b_1$ gives $p_2 = q_1 \wedge q_3$.

Denote by l the largest non-negative integer such that there exist elements $r_0, \ldots, r_l$, $s_0, \ldots, s_l$, $t_1, \ldots, t_l$ with

$$q_2 \wedge q_3 = r_0 \succ r_1 \succ \ldots \succ r_l \geqq p_1 ,$$
$$s_i = p_3 \vee r_i , \quad s_i \succ r_i \quad (i = 0, \ldots, l) ,$$
$$s_{i-1} > t_i \succ s_i \quad (i = 1, \ldots, l) .$$

Denote by r_{l+1} the only element with $r_{l+1} \prec r_l$ and put $s_{l+1} = p_3 \vee r_{l+1}$. Quite similarly as we have proved $\omega_L = [d_k, d_{k+1}]^2 \cup [c_k, c_{k+1}]^2 \cup \mathrm{id}_L$ we can prove $\omega_L = [s_{l+1}, s_l]^2 \cup [r_{l+1}, r_l]^2 \cup \mathrm{id}_L$. Hence $s_{l+1} = c_k$, $s_l = c_{k+1}$, $r_{l+1} = d_k$, $r_l = d_{k+1}$.

Let us prove that if $k = 0$ then $p_1 < r_{l+1}$. Suppose, on the contrary, that $p_1 = r_{l+1}$. If $i \in \{0, \ldots, l\}$ and if $r_{i+1} < b_3$, then r_{i+1} must have a cover $x \leqq b_3$; since r_i and s_{i+1} are the only covers of r_{i+1}, we get $r_i \leqq b_3$ and so $r_i < b_3$, since r_i is $\wedge$-reducible and b_3 is $\vee$-reducible. We get $r_{l+1} < b_3$, $r_l < b_3$, $r_{l-1} < b_3, \ldots$ $\ldots, r_1 < b_3$, $r_0 < b_3$. However, $r_0 = q_3 \wedge q_2$ and we get a contradiction with $q_3 \wedge q_2 \nleqq b_3$.

Let $k = 0$. Evidently $p_1 \prec b_3$. It is easy to see that if $x \in (b_3, q_3)$ then the only element y with $y \prec x$ and $y \nleq b_3$ belongs to (p_1, r_{l+1}) and if $u \in (p_1, r_{l+1})$ then the only element v with $u \prec v$ and $v \nleq r_{l+1}$ belongs to (b_3, q_3). Now, since ω_L is not contained in $[p_1, r_{l+1}]^2 \cup [b_3, q_3]^2 \cup \mathrm{id}_L$, we get $p_1 \prec r_{l+1}$ and $b_3 \prec q_3$. For every $i \in \{1, \ldots, l\}$ denote by u_i the only element with $u_i \prec t_i$ and $u_i \neq s_i$. Evidently $p_3 < u_i < b_1$, $u_i = t_i \wedge b_1$. We have $t_i \prec s_{i-1}$ since if there existed an x with $t_i < x < s_{i-1}$ then we would have $b_2 \vee u_i = t_i > r_i = r_{i-1} \wedge x$, a contradiction by (III). Evidently $p_3 \prec u_l \prec \ldots \prec u_1$. Now it is easy to see that if $s_0 = q_2$ then $L \simeq F_{2l+1}$ and if $s_0 < q_2$ then $L \simeq F_{2l+2}$; in the first case we have $l \neq 0$, since $q_2 \wedge q_1 \nleq b_2$.

Now let $k \neq 0$. Suppose $l \neq 0$. Denote by f the only element with $e_1 \prec f$ and $f \neq d_1$ and by u the only element with $u \prec t_1$ and $u \neq s_1$. We have $b_2 \vee u = t_1 > e_1 = q_2 \wedge f$, a contradiction by (III). Hence $l = 0$ and thus the situation is dual to $k = 0$, so that L is isomorphic to F_n for some $n \geqq 2$. ∎

10. WEAKLY PRIMITIVE LATTICES: THE PRESENCE OF STAR ELEMENTS

In this section let L be a weakly primitive lattice and let z be its star element; define elements $b_1, b_2, b_3, a_1, a_2, a_3, u, q_1, q_2, q_3$ in the same way as in Section 8. We know already that z is the only star element of L and u is the only costar element of L.

10.1. Lemma. *If $L \setminus (z, u)$ is a sublattice of L then $[z, u]$ is a weakly primitive lattice and $L \setminus (z, u)$ is a weakly primitive lattice without star and costar elements. Moreover, $z \prec u$ in $L \setminus (z, u)$, z is both $\vee$- and $\wedge$-irreducible in $L \setminus (z, u)$ and u is both $\vee$- and $\wedge$-irreducible in $L \setminus (z, u)$.*

Proof. It is easy to see that $[z, u]^2 \cup \mathrm{id}_L$ is a congruence of L. It follows from 1.4 that $[z, u]$ is a weakly primitive lattice. Let α be a non-trivial congruence of $L \setminus (z, u)$. If it were $\langle z, u\rangle \notin \alpha$ then evidently $\alpha \cup \mathrm{id}_L$ would be a congruence of L, a contradiction, since $\omega_L \subseteq [z, u]^2 \cup \mathrm{id}_L$. Hence $\langle z, u\rangle \in \alpha$ for every non-trivial congruence α of $L \setminus (z, u)$ and $L \setminus (z, u)$ is subdirectly irreducible. The rest is obvious. ∎

10.2. Lemma. *Let K be a weakly primitive lattice without star and costar elements; let $z, u \in K$, $z \prec u$ and let both z and u be both $\vee$- and $\wedge$-irreducible in K. Then there exists a finite sequence $K_0, K_1, \ldots, K_n$ $(n \geqq 0)$ of lattices and a finite sequence $j_1, \ldots, j_n$ of elements $j_i \in K_i$ such that $K \simeq K_n$, $K_0 = \{z, u\}$, $K_1 = R(K_0)$, $j_1 = c_{K_0}$ and such that for every $i \in \{2, \ldots, n\}$ one of the following five cases takes place:*

(i) *$K_i = R(K_{i-1})$ and $j_i = c_{K_{i-1}}$.*

(ii) *j_{i-1} is not a coatom of K_{i-1}, $K_i = P(K_{i-1}, j_{i-1})$ and $j_i = c_{K_{i-1}}$.*

(iii) j_{i-1} *is a coatom of* K_{i-1}, $K_i = Q(K_{i-1}, j_{i-1})$ *and* $j_i = d_{K_{i-1}}$.
(iv) j_{i-1} *is not an atom of* K_{i-1}, $K_i = P^*(K_{i-1}, j_{i-1})$ *and* $j_i = c_{K_{i-1}}$.
(v) j_{i-1} *is an atom of* K_{i-1}, $K_i = Q^*(K_{i-1}, j_{i-1})$ *and* $j_i = d_{K_{i-1}}$.

Proof. It follows by induction on Card (K) from the results of Sections 3, 4, 5, 6, 7. ∎

10.3. Lemma. *If* $L \setminus (z, u)$ *is a sublattice of* L *then* $[z, u]$ *is a weakly primitive lattice and there exists a finite sequence* $L_0, L_1, \ldots, L_n$ $(n \geqq 0)$ *of lattices and a finite sequence* $j_1, \ldots, j_n$ *of elements* $j_i \in L_i$ *such that* $L \simeq L_n$, $L_0 = [z, u]$, $L_1 = R(L_0)$, $j_1 = c_{L_0}$ *and such that for every* $i \in \{2, \ldots, n\}$ *one of the following five cases takes place*:

(i) $L_i = R(L_{i-1})$ *and* $j_i = c_{L_{i-1}}$.
(ii) j_{i-1} *is not a coatom of* L_{i-1}, $L_i = P(L_{i-1}, j_{i-1})$ *and* $j_i = c_{L_{i-1}}$.
(iii) j_{i-1} *is a coatom of* L_{i-1}, $L_i = Q(L_{i-1}, j_{i-1})$ *and* $j_i = d_{L_{i-1}}$.
(iv) j_{i-1} *is not an atom of* L_{i-1}, $L_i = P^*(L_{i-1}, j_{i-1})$ *and* $j_i = c_{L_{i-1}}$.
(v) j_{i-1} *is an atom of* L_{i-1}, $L_i = Q^*(L_{i-1}, j_{i-1})$ *and* $j_i = d_{L_{i-1}}$.

Proof. It follows from 10.1 and 10.2. ∎

Now we shall investigate the case of $L \setminus (z, u)$ being not a sublattice. By the results of Section 8, it is enough to consider the case of q_1 being $\wedge$-reducible; the remaining five possibilities are either symmetrical or dual. If q_1 is $\wedge$-reducible then by 8.5 the join of any two elements from $L \setminus (z, u)$ belongs to $L \setminus (z, u)$.

10.4. Lemma. *Let* q_1 *be* $\wedge$*-reducible. Denote by* $[f, g]$ *the fine interval of* L. *Then* $b_1 \leqq f \prec g = q_1$.

Proof. Evidently $b_1 \leqq f \prec g \leqq q_1$. Suppose $g < q_1$. Since $[g, q_1]$ contains no fine interval, there exists an l with $g \prec l$ and $l \parallel q_1$. Evidently $l \parallel u$. There exists exactly one j with $u \prec j$. We have $l \leqq j$, since otherwise we should have $j \wedge l = g < u = q_2 \vee q_3$, a contradiction by (III). Hence $j = u \vee l$. By (I) there exists a greatest element h with the property $u \wedge h = q_1$. Since q_1 is $\wedge$-reducible, $q_1 < h$, $h \parallel u$. We have $l \leqq h$, since otherwise we should have $h \wedge l = g < u = q_2 \vee q_3$, a contradiction by (III). Now evidently $q_1 \vee l \leqq h \wedge j$; since $q_1 \parallel l$ and $[q_1 \vee l, h \wedge j]$ can not contain a fine interval, we obtain $h \nparallel j$ and so $h < j$. Evidently $h \prec j$. The following implications are true:

(i) If $x \geqq u$ then $x \nparallel j$. (This is clear.)
(ii) If $x \leqq j$ and $x \not\leqq h$ then $x \nparallel u$. (This is clear.)
(iii) If $x \geqq g$ and $x \ngeqq u$ then $x \leqq h$. (Indeed, $p_1 \wedge x = p_1 \wedge u \wedge x = p_1 \wedge q_1 \wedge x = z$ and $p_1 \wedge q_1 = z$, so that $p_1 \wedge (x \vee q_1) = z$ by (I) and so $x \vee q_1 \ngeqq u$; hence $u \wedge (x \vee q_1) = q_1$, $x \vee q_1 \leqq h$ and so $x \leqq h$.)

Now, since $[g, h]^2 \cup [u, j]^2 \cup \mathrm{id}_L$ can not be a congruence, by 1.2 there exists a d with $d \leqq h$ and $d \parallel g$. Evidently $d \nleqq u$ and so $u \vee d = j$; since $u \vee l = j$ as well, we get $u \vee (d \wedge l) = j$ by (II), so that $d \wedge l \nleqq u$ and consequently $d \wedge l \nleqq g$. This, together with $d \parallel g$, implies that l is $\vee$-reducible.

Denote by k the only element with $l \prec k$ and by $\bar{g}$ the only element with $g \prec \bar{g} \leqq \leqq q_1$. We have $\bar{g} < k$, since otherwise we should have $\bar{g} \wedge k = g < l = f \vee \vee (l \wedge d)$, a contradiction by (III). Since $\bar{g}$ is $\wedge$-reducible, there exists an x with $\bar{g} \prec x$ and $x \nleqq u$. If it were $x \nleqq k$ then $x \wedge k = \bar{g} < u = q_2 \vee q_3$, a contradiction by (III). Hence $x \leqq k$. If it were $x < k$ then $x \wedge l = g < u = q_2 \vee q_3$, a contradiction by (III). Hence $x = k$ and so $\bar{g} \prec k$. Now evidently $[g, \bar{g}]^2 \cup [l, k]^2 \cup \mathrm{id}_L$ is a congruence, a contradiction. ∎

10.5. Lemma. *Let q_1 be $\wedge$-reducible and let $a \in L \setminus (z, u)$ be such that $u \wedge a \in \in (z, u)$. Then $u \wedge a = q_1$.*

Proof. Of course, $a \parallel u$. Since L has only one fine interval, it follows from 10.4 that if $u \wedge a \in [b_1, q_1] \cup [b_2, q_2] \cup [b_3, q_3]$ then $u \wedge a = q_1$.

Suppose $u \wedge a \in [p_1, q_2 \wedge q_3]$. We have $a \vee p_2 \geqq p_1 \vee p_2 = b_3$ and $a \vee \vee p_2 \nleq u$, so that $a \vee p_2 \geqq u$ and thus $a \vee p_2 = a \vee u$. Quite analogously $a \vee p_3 = a \vee u$. By (II) we get $a \vee u = a \vee (p_2 \wedge p_3) = a \vee z = a$, so that $a \geqq u$, a contradiction.

Suppose $u \wedge a \in [p_3, q_1 \wedge q_2]$. We have $a \vee p_2 \geqq p_3 \vee p_2 = b_1$ and $a \vee p_2 \nleq \nleq u$, so that $a \vee p_2 > q_1$ and thus $a \vee p_2 = a \vee q_1$. On the other hand, it is easy to see that $a \vee (q_1 \wedge q_2) > q_1$, so that $a \vee (q_1 \wedge q_2) = a \vee q_1$, too. By (II) we get $a \vee q_1 = a \vee (p_2 \wedge q_1 \wedge q_2) = a \vee z = a$, so that $a \geqq q_1$, a contradiction.

The case $u \wedge a \in [p_2, q_1 \wedge q_3]$ is symmetrical to the preceding case and thus impossible, too. ∎

10.6. Lemma. *Let q_1 be $\wedge$-reducible. Then $[z, u] \simeq D_0$ and $[z, u] = \{z, u, p_1, p_2, p_3, b_1, b_2, b_3, q_1\}$.*

Proof. Put $\alpha = [p_1, q_2 \wedge q_3]^2 \cup [p_2, q_1 \wedge q_3]^2 \cup [p_3, q_1 \wedge q_2]^2 \cup [b_1, q_1]^2 \cup \cup [b_2, q_2]^2 \cup [b_3, q_3]^2 \cup \{z\}^2 \cup \{u\}^2$. Evidently, α is a congruence of $[z, u]$. Since $z \vee b \in L \setminus (z, u)$ for any $b \in L \setminus (z, u)$, using 10.5 it is easy to see that $\alpha \cup \mathrm{id}_L$ is a congruence of L. It is easy to see that if $x, y \in [z, u]$ and $\langle x, y \rangle \notin \alpha$ then the congruence of $[z, u]$ generated by $\langle x, y \rangle$ contains $\langle f, q_1 \rangle$ (where $[f, g] = [f, q_1]$ is the fine interval of L). By 1.4, $[z, u]$ is subdirectly irreducible and consequently it is a weakly primitive lattice. Since q_1 is $\vee$-irreducible, it follows from the results of Section 9 that $[z, u] \simeq D_0$ and so $[z, u] = \{z, u, p_1, p_2, p_3, b_1, b_2, b_3, q_1\}$. ∎

10.7. Lemma. *Let q_1 be $\wedge$-reducible. Then $L \setminus \{p_2, p_3, b_2, b_3\}$ is a sublattice of L; it is a weakly primitive lattice without star and costar elements, the interval $[z, u]_{L \setminus \{p_2, p_3, b_2, b_3\}}$ is isomorphic to the pentagon, p_1 is both $\vee$- and $\wedge$-irreducible* *in* $L\setminus\{p_2, p_3, b_2, b_3\}$, b_1 *is both* $\vee$- *and* $\wedge$-*irreducible in* $L\setminus\{p_2, p_3, b_2, b_3\}$ *and* q_1 *is* $\wedge$-*reducible in* $L\setminus\{p_2, p_3, b_2, b_3\}$.

Proof. The subdirect irreducibility of $L\setminus\{p_2, p_3, b_2, b_3\}$ is the only assertion that needs a proof. It is enough to derive a contradiction from the assumption that there exists a non-trivial congruence α of $L\setminus\{p_2, p_3, b_2, b_3\}$ such that $\langle b_1, q_1\rangle \notin \alpha$. We shall get a contradiction with the subdirect irreducibility of L if we prove that $\alpha \cup \mathrm{id}_L$ is a congruence of L. Let $\langle x, y\rangle \in \alpha$ and $x \neq y$. If it were $x = p_1$, then we should have $\langle p_1, w\rangle \in \alpha$ for some $w \in L\setminus\{p_2, p_3, b_2, b_3\}$ with either $w < p_1$ or $w > p_1$, so that either $\langle z, p_1\rangle \in \alpha$ or $\langle p_1, u\rangle \in \alpha$, so that $\langle b_1, q_1\rangle \in \alpha$, a contradiction. Hence $x \neq p_1$. Similarly $y \neq p_1$, $x \neq b_1$, $y \neq b_1$.

Let $x \leqq z$. Then $y \leqq z$, since otherwise $\langle z \vee x, z \vee y\rangle \in \alpha$ would imply $\langle b_1, q_1\rangle \in \alpha$. Hence $\langle w \vee x, w \vee y\rangle \in \alpha \cup \mathrm{id}_L$ and $\langle w \wedge x, w \wedge y\rangle \in \alpha \cup \mathrm{id}_L$ for all $w \in \{p_2, p_3, b_2, b_3\}$ evidently.

Let $x \geqq u$. Then $y \geqq u$, since otherwise $\langle u \wedge x, u \wedge y\rangle \in \alpha$ would imply $\langle b_1, q_1\rangle \in \alpha$. Hence $\langle w \vee x, w \vee y\rangle \in \alpha \cup \mathrm{id}_L$ and $\langle w \wedge x, w \wedge y\rangle \in \alpha \cup \mathrm{id}_L$ for all $w \in \{p_2, p_3, b_2, b_3\}$ evidently.

Now let $x \nleqq z$ and $x \ngeqq u$. Then $y \nleqq z$ and $y \ngeqq u$. Evidently $x \vee z \neq p_1$, so that $x \vee z \geqq b_1$; analogously $y \vee z \geqq b_1$ and so $\langle p_2 \vee x, p_2 \vee y\rangle = \langle b_1 \vee x, b_1 \vee y\rangle \in \alpha$, $\langle p_3 \vee x, p_3 \vee y\rangle = \langle b_1 \vee x, b_1 \vee y\rangle \in \alpha$, $\langle b_3 \vee x, b_3 \vee y\rangle = \langle u \vee x, u \vee y\rangle \in \alpha$, $\langle b_2 \vee x, b_2 \vee y\rangle = \langle u \vee x, u \vee y\rangle \in \alpha$. If $x \geqq q_1$ then $y \geqq q_1$ (since otherwise $\langle q_1 \wedge x, q_1 \wedge y\rangle \in \alpha$ would imply $\langle b_1, q_1\rangle \in \alpha$), so that $w \wedge x = w \wedge y$ for all $w \in \{p_2, p_3, b_2, b_3\}$. If $x \ngeqq q_1$ then $y \ngeqq q_1$ and evidently $u \wedge x,\ u \wedge y \leqq z$, so that $\langle w \wedge x, w \wedge y\rangle = \langle z \wedge x, z \wedge y\rangle \in \alpha$ for all $w \in \{p_2, p_3, b_2, b_3\}$.

We have proved $\langle w \wedge x, w \wedge y\rangle \in \alpha \cup \mathrm{id}_L$ and $\langle w \vee x, w \vee y\rangle \in \alpha \cup \mathrm{id}_L$ for all $w \in \{p_2, p_3, b_2, b_3\}$; if $w \notin \{p_2, p_3, b_2, b_3\}$ then of course $\langle w \wedge x, w \wedge y\rangle \in \alpha$ and $\langle w \vee x, w \vee y\rangle \in \alpha$. ■

10.8. Lemma. *Let K be a weakly primitive lattice without star and costar elements and let* $\{z, u, p_1, b_1, q_1\}$ *be its interval isomorphic to the pentagon; let* $z \prec p_1 \prec u$ *and* $z \prec b_1 \prec q_1 \prec u$. *Moreover, let both* p_1 *and* b_1 *be both* $\vee$- *and* $\wedge$-*irreducible in K and let* q_1 *be* $\wedge$-*reducible in K. Then there exists a finite sequence* $K_0, K_1, \ldots, K_n$ $(n \geqq 1)$ *of lattices and a finite sequence* $j_0, j_1, \ldots, j_n$ *of elements* $j_i \in K_i$ *such that* $K \simeq K_n$, K_0 *is the pentagon* $\{z, u, p_1, b_1, q_1\}$, $j_0 = q_1$, $K_1 = Q(K_0, j_0)$, $j_1 = d_{K_0}$ *and such that for every* $i \in \{2, \ldots, n\}$ *one of the five cases* (i), (ii), (iii), (iv), (v) *from* 10.2 *takes place.*

Proof. It follows by induction on Card (K) from the results of Sections 3, 4, 5, 6, 7. ■

10.9. Lemma. *Let* q_1 *be* $\wedge$-*reducible. Then there exists a finite sequence* $L_0, L_1, \ldots, L_n$ $(n \geqq 1)$ *of lattices and a finite sequence* $j_0, j_1, \ldots, j_n$ *of elements* $j_i \in L_i$ *such* *that* $L \simeq L_n$, $L_0 = [z, u]$, $j_0 = q_1$, $L_1 = Q(L_0, j_0)$, $j_1 = d_{L_0}$ *and such that for every* $i \in \{2, \ldots, n\}$ *one of the five cases* (i), (ii), (iii), (iv), (v) *from* 10.3 *takes place.*

Proof. It follows from 10.7 and 10.8. ■

11. THE RESULTS

A lattice L is called primitive if the class of all lattices that do not contain a sublattice isomorphic to L is a variety.

11.1. Theorem. *Let L be a lattice. The following assertions are equivalent:*

(i) *L is primitive.*

(ii) *L is weakly primitive.*

(iii) *L is strongly primitive.*

(iv) *L is a finite subdirectly irreducible sublattice of a free lattice.*

(v) *L is finite, subdirectly irreducible and projective.*

(vi) *L is finite, subdirectly irreducible and the following holds: whenever there exists a homomorphism of some lattice K onto L, then K contains a sublattice isomorphic to L.*

(vii) *L belongs to the smallest class* $\mathscr{K}$ *of lattices with the following six properties:*

(1) *The lattices* A_1, A_2, A_3, A_4, B_n $(n \geqq 1)$, C_n $(n \geqq 1)$, D_n $(n \geqq 0)$, E_n $(n \geqq 0)$, F_n $(n \geqq 2)$, G_n $(n \geqq 2)$ *and their duals belong to* $\mathscr{K}$; *the class* $\mathscr{K}$ *is closed under isomorphic images.*

(2) *If* $S \in \mathscr{K}$ *then* $R(S) \in \mathscr{K}$.

(3) *If* $S \in \mathscr{K}$ *and a is a perfect element of S such that a is not a coatom of S then* $P(S, a) \in \mathscr{K}$.

(4) *If* $S \in \mathscr{K}$ *and a is a perfect element of S such that a is a coatom of S then* $Q(S, a) \in \mathscr{K}$.

(5) *If* $S \in \mathscr{K}$ *and a is a coperfect element of S such that a is not an atom of S then* $P^*(S, a) \in \mathscr{K}$.

(6) *If* $S \in \mathscr{K}$ *and a is a coperfect element of S such that a is an atom of S then* $Q^*(S, a) \in \mathscr{K}$.

Proof. (iii) ⇒ (i) follows from 2.1, (i) ⇒ (ii) is easy (a primitive lattice is a sublattice of a free lattice and thus satisfies (I), (II), (III); moreover, a primitive lattice is evidently finite and subdirectly irreducible) and (vii) ⇒ (iii) follows from 2.2, 2.3, 2.5, 2.6, 2.7, 2.8. The implication (ii) ⇒ (vii) follows from the results of Sections 3, 4, 5, 6, 7, 8, 9, 10. The equivalence of (i) with (iv) and (v) follows from [3] and (i) ⇔ (vi) is proved in [1]. ■

In addition to the lattices $A_1, \ldots, G_n$ introduced in Section 2 and the lattices H_n, I_n defined in Section 9 we shall define lattices A_5, J_n $(n \geqq 2)$ and K_n $(n \geqq 3)$ so that we indicate again their underlying sets and all the pairs x, y with $x \prec y$:

$A_5 = \{0, 2, 3, 4\}$; $0 \prec 2 \prec 3 \prec 1$, $0 \prec 4 \prec 1$. (A_5 is the pentagon; see Fig. 18.)

$J_n = \{0, 2, 3, 4, 5, a_1, \ldots, a_n, b_1, \ldots, b_n\}$; $0 \prec 2 \prec 5 \prec a_1 \prec \ldots \prec a_n$, $0 \prec 4 \prec 5$, $2 \prec 3 \prec b_1 \prec b_3 \prec b_5 \prec \ldots$, $4 \prec b_2 \prec b_4 \prec b_6 \prec \ldots$, $b_1 \prec a_1, \ldots, b_n \prec a_n$. (See Fig. 19 for $n = 4$.)

$K_n = \{0, 2, 3, 4, a_1, \ldots, a_n, b_1, \ldots, b_n\}$; $0 \prec 2 \prec 4 \prec a_1 \prec \ldots \prec a_n$, $0 \prec 3 \prec 4$, $2 \prec b_1 \prec b_3 \prec b_5 \prec \ldots$, $3 \prec b_2 \prec b_4 \prec b_6 \prec \ldots$, $b_1 \prec a_1, \ldots, b_n \prec a_n$. (See Fig. 20 for $n = 4$.)

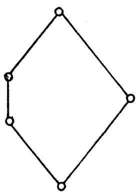

Fig. 18: A_5

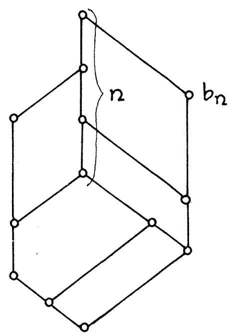

Fig. 19: J_n

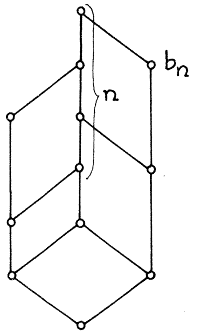

Fig. 20: K_n

Evidently, the following assertions are true:

$A_5 \simeq R(A_1)$.

$H_0 \simeq A_2$, $H_1 \simeq P(H_0, 5)$, $H_2 \simeq P(H_1, 7)$; if $n \geqq 3$ then $H_n \simeq P(H_{n-1}, b_{n-2})$.

$I_2 \simeq Q^*(A_5, 2)$; if $n \geqq 3$ then $I_n \simeq P(I_{n-1}, b_{n-2})$.

$J_2 \simeq Q^*(A_5, 4)$; if $n \geqq 3$ then $J_n \simeq P(J_{n-1}, b_{n-2})$.

$K_3 \simeq A_3$; if $n \geqq 4$ then $K_n \simeq P(K_{n-1}, b_{n-2})$.

Now it follows from the results of Section 2 that the lattices A_5, H_n $(n \geqq 0)$, I_n $(n \geqq 2)$, J_n $(n \geqq 2)$, K_n $(n \geqq 3)$ and their duals are primitive.

Let L be a finite lattice, $a \in L$ and $0 < a < 1$. For every finite sequence $e_1, \ldots, e_k$ $(k \geqq 0)$ of numbers from $\{1, 2, 3\}$ define a lattice $Z(L; a; e_1, \ldots, e_k)$ and its element $z(L; a; e_1, \ldots, e_k)$ as follows: If $k = 0$ then $Z(L; a; e_1, \ldots, e_k) = L$ and $z(L; a; e_1, \ldots, e_k) = a$; if $k \geqq 1$ and the lattice $Y = Z(L; a; e_1, \ldots, e_{k-1})$ and its element $y = z(L; a; e_1, \ldots, e_{k-1})$ are already defined, consider five cases:

(i) If $e_k = 1$, put $Z(L; a; e_1, \ldots, e_k) = R(Y)$ and $z(L; a; e_1, \ldots, e_k) = c_Y$.

(ii) If $e_k = 2$ and if y is not a coatom of Y, put $Z(L; a; e_1, \ldots, e_k) = P(Y, y)$ and $z(L; a; e_1, \ldots, e_k) = c_Y$.

(iii) If $e_k = 2$ and if y is a coatom of Y, put $Z(L; a; e_1, \ldots, e_k) = Q(Y, y)$ and $z(L; a; e_1, \ldots, e_k) = d_Y$.

(iv) If $e_k = 3$ and if y is not an atom of Y, put $Z(L; a; e_1, \ldots, e_k) = P^*(Y, y)$ and $z(L; a; e_1, \ldots, e_k) = c_Y$.

(v) If $e_k = 3$ and if y is an atom of Y, put $Z(L; a; e_1, \ldots, e_k) = Q^*(Y, y)$ and $z(L; a; e_1, \ldots, e_k) = d_Y$.

For example, the lattice $Z(A_3; 7; 2, 1, 3, 3, 2, 2, 1)$ is pictured in Fig. 21.

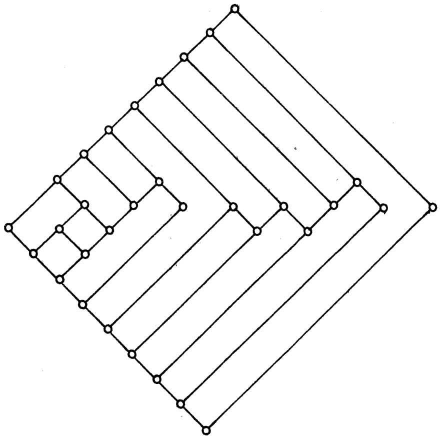

Fig. 21.

11.2. Theorem. *The following lattices are (up to isomorphism) just the only primitive lattices*:

(i) A_1.

(ii) A_5.

(iii) $Z(R(A_5); c_{A_5}; e_1, \ldots, e_k)$ *where* $k \geqq 0$, $e_i \in \{1, 2, 3\}$ *for all* i.

(iv) $Z(I_n; b_n; e_1, \ldots, e_k)$ *where* $n \geqq 2$, $k \geqq 0$, $e_i \in \{1, 2, 3\}$ *for all* i.

(v) $Z(I_n^*; b_n; e_1, \ldots, e_k)$ *where* $n \geqq 2$, $k \geqq 0$, $e_i \in \{1, 2, 3\}$ *for all* i.

(vi) $Z(J_n; b_n; e_1, \ldots, e_k)$ *where* $n \geqq 2$, $k \geqq 0$, $e_i \in \{1, 2, 3\}$ *for all* i.

(vii) $Z(J_n^*; b_n; e_1, \ldots, e_k)$ *where* $n \geqq 2$, $k \geqq 0$, $e_i \in \{1, 2, 3\}$ *for all* i.

(viii) A_2.

(ix) $Z(R(A_2); c_{A_2}; e_1, \ldots, e_k)$ *where* $k \geqq 0$, $e_i \in \{1, 2, 3\}$ *for all* i.

(x) $Z(H_n; b_n; e_1, \ldots, e_k)$ *where* $n \geqq 1$, $k \geqq 0$, $e_i \in \{1, 2, 3\}$ *for all* i.

(xi) $Z(H_n^*; b_n, e_1, \ldots, e_k)$ *where* $n \geqq 1$, $k \geqq 0$, $e_i \in \{1, 2, 3\}$ *for all* i.

(xii) $Z(K_n; b_n; e_1, \ldots, e_k)$ *where* $n \geqq 3$, $k \geqq 0$, $e_i \in \{1, 2, 3\}$ *for all* i.

(xiii) $Z(K_n^*; b_n; e_1, \ldots, e_k)$ *where* $n \geqq 3$, $k \geqq 0$, $e_i \in \{1, 2, 3\}$ *for all* i.

(xiv) $Z(A_4; 3; e_1, \ldots, e_k)$ *where* $k \geqq 0$, $e_i \in \{1, 2, 3\}$ *for all* i *and* $e_1 \neq 3$ *if* $k \neq 0$.

(xv) $Z(A_4; 6; e_1, \ldots, e_k)$ *where* $k \geqq 1$, $e_i \in \{1, 2, 3\}$ *for all i and* $e_1 = 3$.

(xvi) $Z(B_n; 3; e_1, \ldots, e_k)$ *where* $n \geqq 1$, $k \geqq 0$, $e_i \in \{1, 2, 3\}$ *for all i and* $e_1 \neq 3$ *if* $k \neq 0$.

(xvii) $Z(B_n; 10; e_1, \ldots, e_k)$ *where* $n \geqq 1$, $k \geqq 1$, $e_i \in \{1, 2, 3\}$ *for all i and* $e_1 = 3$.

(xviii) $Z(B_n^*; 3; e_1, \ldots, e_k)$ *where* $n \geqq 1$, $k \geqq 0$, $e_i \in \{1, 2, 3\}$ *for all i and* $e_1 \neq 2$ *if* $k \neq 0$.

(xix) $Z(B_n^*; 10; e_1, \ldots, e_k)$ *where* $n \geqq 1$, $k \geqq 1$, $e_i \in \{1, 2, 3\}$ *for all i and* $e_1 = 2$

(xx) $Z(C_n; d_n; e_1, \ldots, e_k)$ *where* $n \geqq 1$, $k \geqq 0$, $e_i \in \{1, 2, 3\}$ *for all i and* $e_1 \neq 3$ *if* $k \neq 0$.

(xxi) $Z(C_n; 3; e_1, \ldots, e_k)$ *where* $n \geqq 1$, $k \geqq 1$, $e_i \in \{1, 2, 3\}$ *for all i and* $e_1 = 3$.

(xxii) $Z(C_n^*; d_n; e_1, \ldots, e_k)$ *where* $n \geqq 1$, $k \geqq 0$, $e_i \in \{1, 2, 3\}$ *for all i and* $e_1 \neq 2$ *if* $k \neq 0$.

(xxiii) $Z(C_n^*; 3; e_1, \ldots, e_k)$ *where* $n \geqq 1$, $k \geqq 1$, $e_i \in \{1, 2, 3\}$ *for all i and* $e_1 = 2$.

(xxiv) $Z(D_n; 6; e_1, \ldots, e_k)$ *where* $n \geqq 0$, $k \geqq 0$, $e_i \in \{1, 2, 3\}$ *for all i and* $e_1 = 1$ *if* $k \neq 0$.

(xxv) $Z(D_0; 6; e_1, \ldots, e_k)$ *where* $k \geqq 1$, $e_i \in \{1, 2, 3\}$ *for all i and* $e_1 = 2$.

(xxvi) $Z(D_n^*; 6; e_1, \ldots, e_k)$ *where* $n \geqq 0$, $k \geqq 0$, $e_i \in \{1, 2, 3\}$ *for all i and* $e_1 = 1$ *if* $k \neq 0$.

(xxvii) $Z(D_0^*; 6; e_1, \ldots, e_k)$ *where* $k \geqq 1$, $e_i \in \{1, 2, 3\}$ *for all i and* $e_1 = 3$.

(xxviii) $Z(E_n; 2; e_1, \ldots, e_k)$ *where* $n \geqq 0$, $k \geqq 0$, $e_i \in \{1, 2, 3\}$ *for all i and* $e_1 = 1$ *if* $k \neq 0$.

(xxix) $Z(E_n^*; 2; e_1, \ldots, e_k)$ *where* $n \geqq 0$, $k \geqq 0$, $e_i \in \{1, 2, 3\}$ *for all i and* $e_1 = 1$ *if* $k \neq 0$.

(xxx) $Z(F_n; 2; e_1, \ldots, e_k)$ *where* $n \geqq 2$, $k \geqq 0$, $e_i \in \{1, 2, 3\}$ *for all i and* $e_1 = 1$ *if* $k \neq 0$.

(xxxi) $Z(F_n^*; 2; e_1, \ldots, e_k)$ *where* $n \geqq 3$, $k \geqq 0$, $e_i \in \{1, 2, 3\}$ *for all i and* $e_1 = 1$ *if* $k \neq 0$.

(xxxii) $Z(G_n; 2; e_1, \ldots, e_k)$ *where* $n \geqq 2$, $k \geqq 0$. $e_i \in \{1, 2, 3\}$ *for all i and* $e_1 = 1$ *if* $k \neq 0$.

(xxxiii) $Z(G_n^*; 2; e_1, \ldots, e_k)$ *where* $n \geqq 2$, $k \geqq 0$, $e_i \in \{1, 2, 3\}$ *for all i and* $e_1 = 1$ *if* $k \neq 0$.

Proof. All these lattices are primitive, as follows from the results of Section 2. It can be easily verified that any of the lattices A_1, A_2, A_3, A_4, B_n, C_n, D_n, E_n, F_n, G_n and their duals is isomorphic to a lattice belonging to the union of these 33 collections (we have $F_2^* \simeq F_2$) and that the union is closed under the constructions R, P, P^*, Q, Q^*. Now it follows from 11.1 that there are no other primitive lattices. ∎

Let us remark that for any pair j, h of different numbers from $\{1, \ldots, 33\}$ no lattice from the j-th collection is isomorphic to a lattice from the h-th collection.

Let us mention the following corollary of the description of all primitive lattices: If L is a primitive lattice, then either $\omega_L = [a, b]^2 \cup \mathrm{id}_L$ for some $a, b \in L$ with $a \prec b$ or $\omega_L = [a, b]^2 \cup [c, d]^2 \cup \mathrm{id}_L$ for some $a, b, c, d \in L$ with $a \prec b \prec d$, $a \prec c \prec d$ and $b \parallel c$.

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