K. P. Bhaskara Rao; M. Bhaskara Rao
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ON THE LATTICE OF SUBALGEBRAS OF A BOOLEAN ALGEBRA

K. P. S. BHASKARA RAO (Calcutta) and M. BHASKARA RAO, Sheffield

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1. Introduction. Let B be a Boolean algebra. We denote L_B the collection of all subalgebras of B. On L_B we define a partial order $\leqq$ as follows. Let C and $D \in L_B$. We say $C \leqq D$ if $C \subset D$. With respect to this partial order L_B is a complete lattice with the first element $\{0, 1\}$ and the last element B itself. (For notation in Boolean algebras we follow HALMOS [2].) For any family $\{C_\alpha\} \subset L_B$, $\operatorname{Sup}_\alpha C_\alpha =$ the subalgebra of B generated by the family $\{C_\alpha\}$ and is denoted by $\bigvee_\alpha C_\alpha$. Similarly, $\operatorname{Inf}_\alpha C_\alpha = \bigcap_\alpha C_\alpha$ and is denoted by $\bigwedge_\alpha C_\alpha$. The symbols $\vee$, $\wedge$ as applied to families of subalgebras should not be confused with the same symbols used for the elements of some fixed Boolean algebra. In the context it will be clear in what sense these symbols are used. The primary object of this paper is to study the lattice structure of L_B.

Some of the natural questions that arise in the study of the lattice L_B are the following.

(i) Is L_B distributive? i.e., is it true that $C \wedge (D \vee E) = (C \wedge D) \vee (C \wedge E)$ for every C, D and E in L_B?

(ii) Is L_B complemented? i.e., given any C in L_B does there exist a D in L_B satisfying $C V D = B$ and $C \wedge D = \{0, 1\}$? (We say that D is a complement of C.)

We remark that the lattice L_B is distributive if and only if B consists of four elements or $B = \{0, 1\}$. The if part is easy to see. If B consists of more than four elements, take three nonzero disjoint elements a, b, c from B satisfying $a \vee b \vee c = 1$. Let $C = \{0, a \vee b, c. 1\}$, $D = \{0, a \vee c, b, 1\}$ and $E = \{0, b \vee c, a, 1\}$. Note that $C \wedge (DVE) \neq (C \wedge D) \vee (C \wedge E)$.

The study of the second question is the central theme of this paper.

2. Preliminary results. We need the following definitions.

Definition 2.1. Let B be a Boolean algebra and I an ideal in B. We have the natural homomorphism $h : B \to B/I$ defined $h(b) = [b]$, where B/I is the quotient Boolean algebra and $[b]$ is the equivalence class containing b. We say that h admits a lifting if there is a subalgebra C of B such that h restricted to C is one to one and takes C onto B/I. We call C *a lifting of* h. (Actually h becomes an isomorphism between C and B/I).

In the more familiar language, B/I is a retract of B. See Halmos [2, p. 130]. We prefer the term 'lifting' to retract which is more suggestive in that one has to pick an element from each equivalence class in B/I so that the resultant collection of points becomes a subalgebra of B. This notion of 'lifting' is also in conformity with the one adopted by IONESCU TULCEA and IONESCU TULCEA [3].

Definition 2.2. Let X be a topological space. Let Y be a closed subset of X. Y is said to be *a retract of* X if there exists a continuous map $f: X \to Y$ which is an identity on Y. See SIKORSKI [14, p. 46].

Let I be an ideal in a Boolean algebra B. Let X be the Stone space of B. Then the Stone space of the quotient Boolean algebra B/I can be identified as a closed subset Y of X. See Sikorski [14, p. 31 – last paragraph]. The following theorem connects lifting and retract.

Theorem 2.3. *The following are equivalent.*

(i) *The natural homomorphism* $h: B \to B/I$ *admits a lifting.*

(ii) *The closed subset* Y *of* X *is a retract of* X.

Proof is easy and we omit it.

As a first result on complementation we have the following result.

Theorem 2.4. *Let* B *be a Boolean algebra. Any finite subalgebra* A *of* B *has a complement in* L_B.

Proof. Let $a_1, a_2, \ldots, a_n$ be all the atoms of A. So $\bigvee_1^n a_i = 1$. We shall straightaway construct a complement of A. Let $F_1, F_2, \ldots, F_n$ be any maximal filters in B containing $a_1, a_2, \ldots, a_n$ respectively. Let $F = F_1 \cap F_2 \cap \ldots \cap F_n$. Then F is a filter in B. Let C be the Boolean algebra generated by F in B. In fact, $C = \{b \in B : b \text{ or } b' \in F\}$. We claim that

(1) $$A \wedge C = \{0, 1\},$$

and

(2) $$A \vee C = B.$$

Proof of (1). Let $b \in A \wedge C$ be such that $b \neq 0$ and $b \neq 1$. Since $b \in C$, we can assume without loss of generality that $b \in F$. Since $b \in A$, $b = a_{i_1} \vee a_{i_2} \vee \ldots \vee a_{i_k}$ for some atoms $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ of A. Take a j such that $j \neq i_1, i_2, \ldots, i_k$ and $1 \leqq j \leqq n$. As $a_j \in F_j$ and $b \in F_j$, we note that $0 = b \wedge a_j \in F_j$ which is a contradiction. Hence $b = 0$ or 1.

Proof of (2). It is sufficient to show that every element of B which is $\leqq a_1$ belongs to $A \vee C$. Then, by a similar argument, it follows that every element of B which is $\leqq a_j$ belongs to $A \vee C$ for every j. Consequently, we observe that for any $b \in B$, $b = (b \wedge a_1) \vee (b \wedge a_2) \vee \ldots \vee (b \wedge a_n) \in A \vee C$.

Let $c \in B$ be such that $c \leqq a_1$. Then either $c \in F_1$ or $a_1 - c \in F_1$. If $c \in F_1$, $c \vee a_2 \vee \ldots \vee a_n \in F$ and hence $a_1 \wedge (c \vee a_2 \vee a_3 \vee \ldots \vee a_n) = c \in A \vee C$. If $a_1 - c \in F_1$, by a similar argument, it follows that $a_1 - c \in A \vee C$. But $a_1 \in A \vee C$. Hence $c \in A \vee C$. This completes the proof.

Our next theorem characterises complements of certain subalgebras of B. We need the following notation and Lemmas.

For an ideal I in a Boolean algebra B, let $B(I) = \{c \in B : c \text{ or } c' \in I\}$, i.e., $B(I)$ is the subalgebra of B generated by I.

Lemma 2.5. *Let B be a Boolean algebra and I an ideal in B. Let C be a subalgebra of B. Then*

$$B(I) \vee C = \{b \in B : b \,\Delta\, c \in I \text{ for some } c \in C\}.$$

(Δ *denotes the operation of symmetric difference.*)

Proof. Let $D = \{b \in B : b \,\Delta\, c \in I \text{ for some } c \in C\}$. First we show that D is a Boolean algebra. Let $b_1, b_2 \in D$. There exist $c_1, c_2 \in C$ such that $b_1 \,\Delta\, c_1$ and $b_2 \,\Delta\, c_2 \in I$. Note that $(b_1 \vee b_2) \,\Delta\, (c_1 \vee c_2) \leqq (b_1 \,\Delta\, c_1) \vee (b_2 \,\Delta\, c_2)$. Since I is an ideal in B, $(b_1 \vee b_2) \,\Delta\, (c_1 \vee c_2) \in I$. Since C is a subalgebra of B, $c_1 \vee c_2 \in C$. Hence $b_1 \vee b_2 \in D$. Now, let $b \in D$. There exists a $c \in C$ such that $b \,\Delta\, c \in I$. Note that $b' \,\Delta\, c' = b \,\Delta\, c \in I$ and $c' \in C$. Hence $b' \in D$. Obviously, $0 \in D$. Hence D is a subalgebra of B.

Now, we observe that $C \subset D$ and $I \subset D$. Hence, $B(I) \vee C \subset D$. Let $d \in D$. There exists a $c \in C$ such that $d \,\Delta\, c \in I$. Let $b = d \,\Delta\, c$. Then $d = b \,\Delta\, c$, since $b \in I$ and $c \in C$, $d \in B(I) \vee C$. Thus, we see that $D \subset B(I) \vee C$.

Lemma 2.6. *Let B be a Boolean algebra and I an ideal in B. Let C be a subalgebra of B such that $B(I) \wedge C = \{0, 1\}$. Then given $b \in B(I) \vee C$, there exists a unique $c \in C$ such that $b \,\Delta\, c \in I$.*

Proof. Existence of at least one $c \in C$ such that $b \,\Delta\, c \in I$ is guaranteed by the previous lemma. Suppose c_1 and $c_2 \in C$ and $b \,\Delta\, c_1 \in I$ and $b \,\Delta\, c_2 \in I$. Then $c_1 \,\Delta\, c_2 \in I$ and $c_1 \,\Delta\, c_2 \in C$. Hence $c_1 \,\Delta\, c_2 \in B(I) \wedge C$. Therefore, $c_1 \,\Delta\, c_2 = 0$ or 1. Since I is a proper ideal, $c_1 \,\Delta\, c_2 = 0$, i.e., $c_1 = c_2$.

Theorem 2.7. *Let B be a Boolean algebra and I an ideal in B. Let $B(I)$ be the subalgebra of B generated by I. Let X and Y be the Stone spaces of B and B/I respectively with $Y \subset X$. Then the following statements are equivalent.*

(i) *$B(I)$ has a complement in L_B.*
(ii) *The natural homomorphism of B onto B/I admits a lifting.*
(iii) *Y is a retract of X.*

Proof. (i) $\to$ (ii). Let $h : B \to B/I$ be the natural homomorphism, i.e., $h(b) = [b]$ for $b \in B$. Let C be a complement of $B(I)$ in L_B. We shall show that C is a lifting of h. Equivalently, the map h restricted to C is one-one and onto B/I. Since $B(I) \vee C = B$, by Lemma 2.5, for any $b \in B$. There is a $c \in C$ such that $b \,\Delta\, c \in I$. So, $h(c) = h(b) = [b]$. Hence h restricted to C is onto. Since $B(I) \wedge C = \{0, 1\}$, by Lemma 2.6, there is a unique $c \in C$ such that $b \,\Delta\, c \in I$. Hence h restricted to C is one-one.

(ii) $\to$ (i). Let C be a lifting of the natural homomorphism $h : B \to B/I$. We show that C is a complement of $B(I)$ in L_B. Let $b \in B(I) \wedge C$. Either $b \in I$ or $b' \in I$. Suppose $b \in I$. So, $h(b) = [b] = 0$. Since h restricted to C is one-one and since $h(b) = h(0)$, we have $b = 0$. The other case is similarly disposed. Hence $B(I) \wedge C = \{0, 1\}$. Now, we prove that $B(I) \vee C = B$. Let $b \in B$. Since the map $h : C \to B/I$ is onto, there is a $c \in C$ such that $h(c) = [b]$. Consequently, $c \,\Delta\, b \in I$. By Lemma 2.5, $b \in B(I) \vee C$. Hence $B(I) \vee C = B$.

The equivalence of (ii) and (iii) was stated in Theorem 2.3.

Remarks. If $B(I)$ has a complement in L_B, any complement of $B(I)$ is isomorphic to B/I. Consequently, any two complements of $B(I)$ are isomorphic. This statement is not true for any subalgebra of B. As an example, we have

$X = \{1, 2, 3, 4\}$; $B =$ The collection of all subsets of X; $C = \{\emptyset, X, \{1, 2\}, \{3, 4\}\}$; $D = \{\emptyset, X, \{1, 3\}, \{2, 4\}\}$ and $E = \{\emptyset, X, \{1\}, \{3\}, \{2, 4\}, \{1, 3\}, \{1, 2, 4\}, \{2, 3, 4\}\}$. D and E are both complements of C. Clearly, D and E are not isomorphic.

Now we give an example of a Boolean algebra B and a subalgebra A of B such that A has no complement in L_B. For any set X, we shall denote the power set of X by $P(X)$, i.e., $P(X)$ is the class of all subsets of X.

Theorem 2.8. *Let N be the set of all natural numbers. Let C be the field of all finite-cofinite subsets of N. Then C has no complement in $L_{P(N)}$.*

Proof. Observe that C is the field generated by the ideal I of all finite subsets of N. It is well known that the Stone-Čech compactification of N, βN is the Stone space of $P(N)$ and $\beta N - N$ that of the Boolean algebra $P(N)/I$. If C were to admit a complement in $L_{P(N)}$, then, by Theorem 2.7, $\beta N - N$ would be a retract of N. But this is not true. See Gillman and Jerison [1, 6Q, *p.* 97].

In the next two sections, we study certain classes of Boolean algebras in the light of complementation problem in Boolean algebras.

3. C_1-Boolean algebras. In this section we introduce a new class of Boolean algebras.

Definition 3.1. A Boolean algebra B is said to be a *C_1-Boolean algebra* if every subalgebra of B has a complement in L_B.

Theorem 3.2. *Every finite Boolean algebra is a C_1-Boolean algebra.*

Proof. This follows from Theorem 2.4.

Theorem 3.3. *Let B and D be two Boolean algebras such that D is a homomorphic image of B. If B is a C_1-Boolean algebra so is also D.*

Proof. Let $h : B \to D$ be a homomorphism mapping B onto D. Let E be a subalgebra of D. Let $F = \{b \in B : h(b) \in E\}$. It is easy to verify that F is a subalgebra of B. Let G be a complement of F in L_B. Now, we claim that $h(G)$ is a complement of E in L_D. Let $d \in E \wedge h(G)$. There exist $b_1 \in F$ and $b_2 \in G$ such that $h(b_1) = d = h(b_2)$. Clearly, $b_2 \in F$. Since $F \wedge G = \{0, 1\}$, $b_2 = 0$ or 1. Hence $d = 0$ or 1. So, $E \wedge h(G) = \{0, 1\}$. Since F and G generate B, $h(F)$ and $h(G)$ generate $h(B)$. But $h(F) = E$ and $h(B) = D$. Hence $E \vee h(G) = D$.

Theorem 3.4. *Let B be any infinite Boolean σ-algebra. Let N be the set of all natural numbers and $P(N)$ the power set of N. Then $P(N)$ is a homomorphic image of B.*

Proof. Let $b_1, b_2, \ldots$ be a sequence of nonzero, pairwise disjoint elements in B such that $b_1 \vee b_2 \vee \ldots = 1$. Let $F_1, F_2, \ldots$ be a sequence of maximal filters in B containing $b_1, b_2, \ldots$ respectively. Define $h : B \to P(N)$ as follows. $h(b) = \{n \geqq 1 : b \in F_n\}$. Clearly, $h(0) = \emptyset$ and $h(1) = N$. $h(a_1 \vee a_2) = \{n : a_1 \vee a_2 \in F_n\} = \{n : a_1 \in F_n\} \cup \{n : a_2 \in F_n\}$. For, let $n \in$ Left hand side expression. $a_1 \vee a_2 \in F_n$. Then, either $a_1 \in F_n$ or $a_2 \in F_n$. If not, $a_1' \in F_n$ and $a_2' \in F_n$. Since F_n is a filter, $a_1' \wedge a_2' \in F_n$. This together with $a_1 \vee a_2 \in F_n$ implies $(a_1 \vee a_2) \wedge (a_1' \wedge a_2') = 0 \in F_n$. This is a contradiction. Hence $n \in$ Right hand side expression. If $n \in$ Right hand side expression, it is obvious that $n \in$ Left hand side expression. Next, we claim that h is onto. Let $N_1 = \{n_1, n_2, \ldots\}$ be a subset of N. Then, $h(b_{n_1} \vee b_{n_2} \vee \ldots) = N_1$.

Corollary 3.5. *Let B be any infinite Boolean σ-algebra. Then B is not a C_1-Boolean algebra.*

Proof. If B were to be a C_1-Boolean algebra, then, in view of Theorems 3.3 and 3.4, $P(N)$ would be a C_1-Boolean algebra. But this is not the case with $P(N)$. See Theorem 2.8.

The following is a non-trivial example of a C_1-Boolean algebra.

Theorem 3.6. *Let X be any set and $\mathbf{C}$ the field of all finite-cofinite subsets of X. Then $\mathbf{C}$ is a C_1-Boolean algebra.*

Proof. Since $\boldsymbol{C}$ is a superatomic Boolean algebra, every subalgebra $\boldsymbol{D}$ of $\boldsymbol{C}$ is atomic. See Sikorski [14, example D, p. 35].

Case (i). One of the atoms of $\boldsymbol{D}$ is cofinite. Then $\boldsymbol{D}$ is a finite algebra. Hence $\boldsymbol{D}$ has a complement in $L_{\boldsymbol{C}}$, by Theorem 2.5.

Case (ii). Every atom of $\boldsymbol{D}$ is finite. Let $\{D_\alpha : \alpha \in J\}$ be the collection of all atoms of $\boldsymbol{D}$. Choose and fix one element $x_\alpha \in D_\alpha$. Let $\boldsymbol{E}$ be the subfield of $\boldsymbol{C}$ generated by $\{\{x\} : x \in X$ and $x \neq x_\alpha$ for any $\alpha\}$. If $M = \{x_\alpha : \alpha \in J\}$, then $\boldsymbol{E} = \{A \subset X : A \cap \cap M = \emptyset$ and A is finite or $A \supset M$ and A is cofinite$\}$. Now, we claim that $\boldsymbol{E}$ is a complement of $\boldsymbol{D}$ in $L_{\boldsymbol{C}}$. Let $H \in \boldsymbol{D} \wedge \boldsymbol{E}$. H is either finite or cofinite. Assume H is finite. Since $H \in \boldsymbol{D}$, $H = D_{\alpha_1} \cup D_{\alpha_2} \cup \ldots \cup D_{\alpha_n}$ for some $\alpha_1, \alpha_2, \ldots, \alpha_n$ in J. We can write $H = (D_{\alpha_1} - \{x_{\alpha_1}\}) \cup (D_{\alpha_2} - \{x_{\alpha_2}\}) \cup \ldots \cup (D_{\alpha_n} - \{x_{\alpha_n}\}) \cup \{x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_n}\}$. Each $D_{\alpha_i} - \{x_{\alpha_i}\} \in \boldsymbol{E}$. Since $H \in \boldsymbol{E}$, we find that $\{x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_n}\} \in \boldsymbol{E}$. This shows that $H = \emptyset$. In the case when H is cofinite, we can prove that $H' = \emptyset$. Hence $H = X$. Thus, we have $\boldsymbol{E} \wedge \boldsymbol{D} = \{\emptyset, X\}$. Next, we prove that $\boldsymbol{D} \vee \boldsymbol{E} = \boldsymbol{C}$. For this, it is sufficient to prove that $\{x\} \in \boldsymbol{D} \vee \boldsymbol{E}$ for every $x \in X$. Each $\{x_\alpha\} \in \boldsymbol{D} \vee \boldsymbol{E}$. For, $D_\alpha \in \boldsymbol{D}$ and $D_\alpha - \{x_\alpha\} \in \boldsymbol{E}$. Hence $D_\alpha \cap (D_\alpha - \{x_\alpha\})' = \{x_\alpha\} \in \boldsymbol{D} \vee \boldsymbol{E}$. If $x \neq x_\alpha$ for any α, then $\{x\} \in \boldsymbol{E} \subset \boldsymbol{D} \vee \boldsymbol{E}$. Hence $\boldsymbol{D} \vee \boldsymbol{E} = \boldsymbol{C}$.

Remark 1. A similar proof can be given to show that $L_{\boldsymbol{C}}$ for the above $\boldsymbol{C}$ is a relatively complemented lattice, i.e., $\boldsymbol{B}, \boldsymbol{D}, \boldsymbol{E} \in L_{\boldsymbol{C}}$, $\boldsymbol{B} \subset \boldsymbol{D} \subset \boldsymbol{E}$ implies there exists $\boldsymbol{F} \in L_{\boldsymbol{C}}$ such that $\boldsymbol{D} \wedge \boldsymbol{F} = \boldsymbol{B}$ and $\boldsymbol{D} \vee \boldsymbol{F} = \boldsymbol{E}$.

Remark 2. We do not know if every superatomic Boolean algebra is a C_1-Boolean algebra.

4. C_2-Boolean algebras. Definition 4.1. A Boolean algebra B is said to be a C_2-*Boolean algebra* if for every ideal I in B, $B(I)$ has a complement in L_B.

Theorem 4.2. *Let B be a Boolean algebra and X its Stone space. Then B is a C_2-Boolean algebra if and only if every closed subset of X is a retract of X.*

Proof. This follows from Theorem 2.7.

Corollary 4.3. *Every countable Boolean algebra B is a C_2-Boolean algebra.*

Proof. The Stone space X of B is a compact totally disconnected metric space and for such a space X, every closed subset of X is a retract of X. See the last paragraph in Sikorski [14, p. 46] or Kelley [4, 0, p. 165]. This Corollary can also be obtained from von Neumann-Stone's Theorem 17 of [7, p. 369] which is stated as Theorem 6.3 below.

Corollary 4.4. *Let B and D be two Boolean algebras such that D is a homomorphic image of B. If B is a C_2-Boolean algebra so also is D.*

Proof. Let X and Y be the Stone spaces of B and D respectively. Then Y is a closed subspace of X. Since every closed subset of X is a retract of X so also every closed subset of Y is a retract of Y.

Let X and Y be the Stone spaces of the Boolean algebras B and D respectively. Then the Stone space of $B + D$ is the disjoint union $X \cup Y$ of X and Y equipped with union topology. See Sikorski [14, Section 16, p. 50].

Corollary 4.5. *If B and D are two C_2-Boolean algebras, then $B + D$ is also a C_2-Boolean algebra.*

Proof. Let X and Y be the Stone spaces of B and D respectively. If every closed subset of X is a retract of X and every closed subset of Y is a retract of Y, then every closed subset of $X \cup Y$ is a retract of $X \cup Y$.

Remark. The above corollary does not extend to countable direct union of Boolean algebras. For, $P(N)$ is a countable direct union of two element Boolean algebras and $P(N)$ is not a C_2-Boolean algebra.

Theorem 4.6. *No infinite Boolean σ-algebra B is a C_2-Boolean algebra.*

Proof. If B were to be a C_2-Boolean algebra, then $P(N)$ would be a C_2-Boolean algebra which is not the case. See Theorem 3.4 and Corollary 4.4.

Corollary 4.7. *Let X be any set. $P(X)$ is a C_2-Boolean algebra if and only if X is finite.*

Theorem 4.8. *Let α be any ordinal number.* Equip $[0, \alpha]$ *with order topology. Let $\mathbf{C}$ be the field of all clopen subsets of* $[0, \alpha]$. *Then $\mathbf{C}$ is a C_2-Boolean algebra.*

Proof. It is sufficient if we prove that every closed subset H of $[0, \alpha]$ is a retract of $[0, \alpha]$. The main idea in the proof is taken from some observations in the proof of Lemma 1 of Bhaskara Rao and Bhaskara Rao [9, p. 195].

We shall denote $[0, \alpha]$ by X. We have to define a map f from X onto H which is continuous and identity on H. Let α_0 be the first element in H. Since H is closed there exists a last element in H which we call α_1. For $\beta \in H$, let β' be the first succeeding element of β in H. Since H is closed, for any $x \in [\alpha_0, \alpha_1] - H$ there exists a $\beta \in H$ such that $\beta < x < \beta'$. (Such a β is unique.) We define f as follows.

$$\begin{aligned} f(x) &= \alpha_0 \quad \text{if} \quad 0 \leqq x < \alpha_0\,, \\ &= \alpha_1 \quad \text{if} \quad \alpha_1 < x \leqq \alpha\,, \\ &= x \quad \text{if} \quad x \in H\,, \\ &= \beta' \quad \text{if} \quad x \in [\alpha_0, \alpha_1] - H\,, \quad \text{where } \beta \text{ is the element of } H \text{ satisfying } \beta < x < \beta'\,. \end{aligned}$$

To conclude that f is continuous on X it is sufficient to show that for any well ordered transfinite sequence x_i increasing to x_0, $f(x_i)$ converges to $f(x_0)$.

This we prove as follows.

Case (i). $x_0 \leqq \alpha_0$. Then $f(x_i) = \alpha_0 = f(x_0)$. Hence $f(x_i)$ converges to $f(x_0)$.

Case (ii). $\alpha_1 < x_0$. x_i is eventually greater than α_1 and hence $f(x_i)$ is eventually equal to α_1 which is equal to $f(x_0)$.

Case (iii). $x_0 \in [\alpha_0, \alpha_1] - H$. There exists a $\beta \in H$ such that $\beta < x_0 < \beta'$. Then x_i is eventually in the interval (β, β') and hence $f(x_i)$ is eventually equal to $\beta' = f(x_0)$.

Case (iv). $x_0 \in H$ and $x_0 = \beta'$ for some β in H. Then $f(x_0) = \beta'$ and $(\beta, \beta']$ is an open neighbourhood of x_0. Hence eventually $x_i > \beta$. Consequently, $f(x_i) = \beta'$ eventually.

Case (v). $x_0 \in H$, $x_0 \neq \alpha_0$ and for no $\beta \in H$, $x_0 = \beta'$. Then $\beta \in H$, $\beta < x_0$ implies $\beta' < x_0$. Since H is closed, $x \in X - H$ and $x < x_0$ implies there exists a $\beta \in H$ such that $x < \beta < x_0$. So x_i is eventually in $(\beta, x_0]$. Consequently, $f(x_i)$ is eventually in $(\beta, x_0]$. Hence $f(x_i)$ is eventually in $(x, x_0]$. This implies that $f(x_i)$ converges to $f(x_0)$. The proof is complete.

5. Complementation in general fields. B. V. Rao [11] considered the following problem. Let X be any arbitrary set. Let $L^\sigma_{P(X)}$ be the collection of all sub σ-fields of $P(X)$. $L^\sigma_{P(X)}$ is a complete lattice with $\{\emptyset, X\}$ as the first element and $P(X)$ as the last element in the natural order of inclusion. For any family $\{\boldsymbol{C}_\alpha\}$ of sub σ-fields of $P(X)$, $\bigvee_\alpha \boldsymbol{C}_\alpha$ is the sub σ-field of $P(X)$ generated by the family $\{\boldsymbol{C}_\alpha\}$, and $\bigwedge_\alpha \boldsymbol{C}_\alpha = \bigcap_\alpha \boldsymbol{C}_\alpha$. $L^\sigma_{P(X)}$ is said to be complemented if for every element $\boldsymbol{B}$ in $L^\sigma_{P(X)}$ there is an element $\boldsymbol{D}$ in $L^\sigma_{P(X)}$ such that $\boldsymbol{B} \wedge \boldsymbol{D} = \{\emptyset, X\}$ and $\boldsymbol{B} \vee \boldsymbol{D} = P(X)$. B. V. Rao [11, p. 214] proved that if X is uncountable, then $L^\sigma_{P(X)}$ is not complemented. In effect he showed that the countable-cocountable σ-field on X has no complement in $L^\sigma_{P(X)}$. In this section, we give two simple proofs of this result, one based on measure theory and the other on set theory.

Let the cardinality of X be $\aleph_1$, Aleph-one. Let $\boldsymbol{C}$ be the countable-cocountable σ-field on X. Suppose $\boldsymbol{C}$ has a complement $\boldsymbol{D}$ in $L^\sigma_{P(X)}$. Then $\boldsymbol{C} \vee \boldsymbol{D} = P(X) = \{Y \subset X : Y \Delta D$ is countable for some D in $\boldsymbol{D}\}$. This may be proved along the same lines of the proof of Lemma 2.5. Moreover, for every $Y \subset X$, there exists unique $D \in \boldsymbol{D}$ such that $Y \Delta D$ is countable. The proof is similar to the one given in Lemma 2.6. Let μ be any measure on $\boldsymbol{D}$. We can define a measure λ on $P(X)$ which is an extension of μ by the following formula. For $Y \subset X$, let $\lambda(Y) = \mu(D)$, where D is the unique set in $\boldsymbol{D}$ such that $Y \Delta D$ is countable. Observe that λ is always a continuous measure, i.e., $\lambda(\{x\}) = 0$ for every x in X, whatever be the nature of the measure μ on $\boldsymbol{D}$. For, for x in X, $\{x\} \Delta \emptyset$ is countable. Hence, $\lambda(\{x\}) = \mu(\emptyset) = 0$.

So, if we start with a $0-1$ valued measure μ on $\boldsymbol{D}$, we end up with a $0-1$ valued continuous measure λ on $P(X)$. This is a contradiction to Ulam's theorem which states that there is no continuous probability measure on $P(X)$. See, for example, Bhaskara Rao and Bhaskara Rao [9, p. 196]. In Section 6 we use the above argument to prove a more general theorem.

Set theoretic proof of the above result is included in the more general theorem 5.3 to be proved later in this section.

Let k, λ be any two cardinal numbers. Let X be any set of cardinality λ. Let L_λ^k denote the collection of all k-fields on X. A collection $\boldsymbol{D} \subset P(X)$ is said to be a k-field if it is nonempty, closed under complementation and $<k$ many unions. L_λ^k is complete under the following lattice operations. If $\{\boldsymbol{D}_\alpha\}$ is a family of k-fields contained in $P(X)$, $\bigvee_\alpha \boldsymbol{D}_\alpha$ is defined to be the smallest sub k-field on X generated by the family $\{\boldsymbol{D}_\alpha\}$ and $\bigwedge_\alpha \boldsymbol{D}_\alpha = \bigcap_\alpha \boldsymbol{D}_\alpha$. In this terminology, every σ-field is an $\aleph_1$-field, and every field is an $\aleph_0$-field. Further, $L_{P(X)}^\sigma = L_\lambda^{\aleph_1}$ and $L_{P(X)} = L_\lambda^{\aleph_0}$. Let Y be a set of cardinality k. For any cardinal number k, its succeeding cardinal number k^+ is regular. This observation and the following theorem reduces the problem of complementation in L_λ^k for any k to the problem of complementation for the case of regular cardinals k.

Theorem 5.1. *If k is not regular, then $L_\lambda^k = L_\lambda^{k^+}$.*

Proof. It is clear that $L_\lambda^{k^+} \subset L_\lambda^k$. Let $\boldsymbol{C}$ be any k-field on X. We will show that $\boldsymbol{C}$ is also a k^+-field. Let $\{C_j : j \in J\}$ be a family of sets in $\boldsymbol{C}$ such that the cardinality of J is $<k^+$. It is enough if we treat the case when the cardinality of J is k. Since k is not regular, we can write $J = \bigcup_{i \in I} J_i$, where the cardinality of each J_i and I is $<k$ and J_i's are disjoint. Note that $\bigcup_{j \in J} C_j = \bigcup_{i \in I} \bigcup_{j \in J_i} C_j$. Since $\boldsymbol{C}$ is a k-field, $\bigcup_{j \in J_i} C_j \in \boldsymbol{C}$ for every i in I and hence, $\bigcup_{i \in I} \bigcup_{j \in J_i} C_j \in \boldsymbol{C}$. This completes the proof.

Theorem 5.2. *If $\lambda < k$, then every element $\boldsymbol{C}$ in L_λ^k has a complement $\boldsymbol{D}$ in L_λ^k.*

Proof. First, observe that every element of L_λ^k is a complete field. Hence every element of L_λ^k is atomic. See Sikorski [14, p. 105]. Let $\{C_\alpha : \alpha \in J\}$ be the collection of all atoms $\boldsymbol{C} \in L_\lambda^k$. Choose and fix one element $x_\alpha \in C_\alpha$. Let $\boldsymbol{D}$ be the k-field on X generated by $\{\{x\} : x \neq x_\alpha \text{ for any } \alpha\}$. It is easy to verify that $\boldsymbol{D}$ is a complement of $\boldsymbol{C}$.

Theorem 5.3. *Let k be a regular cardinal number. If $\lambda = k$, then there exists an element $\boldsymbol{C}$ in L_λ^k such that $\boldsymbol{C}$ has no complement in $L_{P(X)}$.*

Proof. Let $\boldsymbol{C} = \{A \subset X : \text{cardinality of } A \text{ or } A' \text{ is } <k\}$. Since k is regular, $\boldsymbol{C} \in L_\lambda^k$. Suppose $\boldsymbol{D}$ is a complement of $\boldsymbol{C}$ in $L_{P(X)}$. Let $\boldsymbol{I}$ be the collection of all subsets of X of cardinality $<k$. Then $P(X)/\boldsymbol{I}$ is isomorphic to $\boldsymbol{D}$ by Theorem 2.7. Now, we use the following theorem of Sierpiński [13, Theorem 1, p. 448]. "Let X be a set of car-

dinality k. Then there exists a family $\{X_j : j \in J\}$ of subsets of X such that the cardinality of each X_j is k, the cardinality of J is $>k$ and the cardinality of each $X_i \cap X_j$ is $<k$ for $i \neq j$." In view of this result, we find that there are more than k disjoint nonzero elements in the Boolean algebra $P(X)/\boldsymbol{I}$. For example, one can take the family $\{[X_j] : j \in J\}$, where $[X_j] \in P(X)/\boldsymbol{I}$ is the equivalence class containing X_j. But, $\boldsymbol{D}$ being a sub field of $P(X)$, $\boldsymbol{D}$ cannot contain more than k $(=\lambda)$ pairwise disjoint nonempty sets. Hence $\boldsymbol{D}$ and $P(X)/\boldsymbol{I}$ cannot be isomorphic. This contradiction establishes the result.

Corollary 5.4. *Let k be a regular cardinal number. If $\lambda = k$, then there exists an element $\boldsymbol{C}$ in L_λ^k such that $\boldsymbol{C}$ has no complement in L_λ^k.*

Proof. Let $\boldsymbol{C}$ and $\boldsymbol{I}$ be as defined in the proof of the previous theorem. It is clear that the field generated by $\boldsymbol{I}$ on X, $P(X)(\boldsymbol{I}) = \boldsymbol{C}$. Suppose $\boldsymbol{C}$ has a complement $\boldsymbol{D}$ in L_λ^k, i.e., $\boldsymbol{D}$ is a k-field, $\boldsymbol{C} \wedge \boldsymbol{D} = \{\emptyset, X\}$ and $\boldsymbol{C} \vee \boldsymbol{D} =$ the smallest k-field containing both $\boldsymbol{C}$ and $\boldsymbol{D} = P(X)$. We shall show that the smallest field containing both $\boldsymbol{C}$ and $\boldsymbol{D}$ is $P(X)$. Since $\boldsymbol{C}$ is a field generated by the ideal $\boldsymbol{I}$, the smallest field containing both $\boldsymbol{C}$ and $\boldsymbol{D} = \boldsymbol{E} = \{A \subset P(X) : A \,\Delta\, B \in \boldsymbol{I}$ for some $B \in \boldsymbol{D}\}$, by Lemma 2.5. Since $\boldsymbol{I}$ is a k-ideal and $\boldsymbol{D}$ is a k-field, $\boldsymbol{E}$ is a k-field. Hence $\boldsymbol{E} = P(X)$. Consequently, we note that $\boldsymbol{D}$ is a complement of $\boldsymbol{C}$ in $L_{P(X)}$. This is a contradiction to the Theorem 5.3.

Theorem 5.5. *Let k be a regular cardinal number. If $\lambda > k$, then there exists an element $\boldsymbol{C}$ in L_λ^k which has no complement in L_λ^k.*

Proof. Let Y be a set of cardinality k and X a set of cardinality λ such that $Y \subset X$. By Corollary 5.4, $L_{P(Y)}^k$ contains an element which has no complement in $L_{P(Y)}^k$. It is not difficult to prove that $L_{P(X)}^k$ contains an element which has no complement in $L_{P(X)}^k$.

Combining the previous theorems, we have the following result.

Theorem 5.6. *Let k be a regular cardinal number and λ any cardinal number.*

(i) *If $\lambda < k$, then every element in L_λ^k has a complement in L_λ^k.*

(ii) *If $\lambda \geqq k$, then L_λ^k contains an element which has no complement in L_λ^k.*

Remarks. (i) If $\lambda = k = \aleph_1$, $L_\lambda^k = L_{P(X)}^\sigma$ contains an element which has no complement in $L_{P(X)}^\sigma$. This result of B. V. Rao [11, p. 214] is a special case of the above theorem. See also the second and third paragraphs of this section.

(ii) The above results strengthen B. V. Rao's [11] results in several directions.

(iii) The problem considered in the Theorem 5.6 was suggested by B. V. Rao.

Combining the theorems 5.1 and 5.6, we summarise the results of this section below.

Theorem 5.7. *Let λ and k be any cardinal numbers.*

(i) *If $\lambda < k$, then every element in L_λ^k has a complement in L_λ^k.*

(ii) *If $\lambda > k$, there exists an element in L_λ^k which has no complement in L_λ^k.*

(iii) *If $\lambda = k$ and k is not regular, then every element in L_λ^k has a complement in L_λ^k.*

(iv) *If $\lambda = k$ and k is regular, there exists an element in L_λ^k which has no complement in L_λ^k.*

6. Some complements to the complementation problem. In this section we consider the complementation problem for some special cases.

Firstly, the argument given in the second paragraph of section 5 gives the following theorem.

Theorem 6.1. *Let $\boldsymbol{A}$ be a σ-field of subsets of a set X containing all singletons and admitting no nonzero continuous $0-1$ valued measure on $\boldsymbol{A}$. Let $\boldsymbol{I}$ be any σ-ideal in $\boldsymbol{A}$ containing all singletons. Let $\boldsymbol{A}(\boldsymbol{I})$ be the σ-field generated by $\boldsymbol{I}$ on X. Then $\boldsymbol{A}(\boldsymbol{I})$ has no complement in $L_{\boldsymbol{A}}^\sigma$, where $L_{\boldsymbol{A}}^\sigma$ is the lattice of all sub-σ-fields of $\boldsymbol{A}$.*

Proof. Suppose $\boldsymbol{A}(\boldsymbol{I})$ has a complement $\boldsymbol{B}$ in $L_{\boldsymbol{A}}^\sigma$. Then the σ-field $\boldsymbol{B}$ is isomorphic to the Boolean σ-algebra $\boldsymbol{A}/\boldsymbol{I}$. The argument is similar to the one given in the proof of the Theorem 2.7. Since $\boldsymbol{B}$ is a σ-field of sets, $\boldsymbol{B}$ admits a $0-1$ valued measure. In fact, any degenerate measure on $\boldsymbol{B}$ would do. Consequently, there is a nonzero $0-1$ valued measure μ on the Boolean σ-algebra $\boldsymbol{A}/\boldsymbol{I}$. This measure μ can be lifted as a nonzero measure λ to $\boldsymbol{A}$. Since $\boldsymbol{I}$ contains all singletons, λ is a $0-1$ valued continuous measure on $\boldsymbol{A}$. This contradiction shows that $\boldsymbol{A}(\boldsymbol{I})$ has no complement in $L_{\boldsymbol{A}}^\sigma$.

Remarks. (i) In particular, Theorem 6.1 is applicable in the following cases.

(a) The cardinality of X is non-measurable, i.e., there is no nonzero $0-1$ valued continuous measure on $P(X)$, and $\boldsymbol{A} = P(X)$.

(b) X is any set and $\boldsymbol{A}$ is any separable σ-field on X containing all singletons.

(c) $X = R$, the real line; $\boldsymbol{A}$ = Borel σ-field; $\boldsymbol{I}$ = All Borel sets of μ-measure zero for any finite nonatomic measure μ on $\boldsymbol{A}$; or $\boldsymbol{I}$ = the collection of all Borel first category subsets of R.

(d) X is any set; $\boldsymbol{A}$ is any separable σ-field on $\boldsymbol{A}$ and $\boldsymbol{I}$ is any σ-ideal in $\boldsymbol{A}$ containing all the atoms of $\boldsymbol{A}$.

Now, we generalise the remark (i) (c) to general measure spaces and to general topological spaces.

Let $(X, \boldsymbol{A}, \mu)$ be a measure space where μ is σ-finite and $\boldsymbol{A}$ is complete with respect to μ, i.e., $A \in \boldsymbol{A}$, $\mu(A) = 0$, $B \subset A$ implies that $B \in \boldsymbol{A}$. Let $\boldsymbol{I}_\mu$ be the ideal of all sets in $\boldsymbol{A}$ with μ-measure zero.

Theorem 6.2. *$A(I\mu)$, the σ-field generated by $I\mu$ on X, has a complement in L_A^σ if and only if μ is completely atomic.*

Proof. Suppose μ is completely atomic. We can find a sequence $A_1, A_2, \ldots$ of sets in $\boldsymbol{A}$ with the following properties. (i) A_i's are pairwise disjoint. (ii) $\bigcup_{i \geqq 1} A_i = X$. (iii) Each A_i is a μ-atom. Since μ is σ-finite $\mu(A_i)$ is finite for each i. Given any $A \in \boldsymbol{A}$, we can find a unique subsequence $A_{i_1}, A_{i_2}, \ldots$ such that $\mu(A \,\Delta \bigcup_{j \geqq 1} A_{i_j}) = 0$. Let $\boldsymbol{B}$ be the sub-σ-field of $\boldsymbol{A}$ generated by $A_1, A_2, \ldots$. Then $\boldsymbol{B}$ is a lifting for the natural homomorphism $h : \boldsymbol{A} \to \boldsymbol{A}/\boldsymbol{I}_\mu$, i.e., $h(A) = [A]_\mu$. Hence $\boldsymbol{B}$ is a complement of $(\boldsymbol{A}\boldsymbol{I}_\mu)$.

Conversely, suppose that $\boldsymbol{A}(\boldsymbol{I}_\mu)$ has a complement $\boldsymbol{B}$ in L_A^σ. Consequently, $\boldsymbol{B}$ is isomorphic to the quotient Boolean σ-algebra $\boldsymbol{A}/\boldsymbol{I}_\mu$. Since μ is σ-finite, $\boldsymbol{A}/\boldsymbol{I}_\mu$ satisfies the countable chain condition. Hence the σ-field $\boldsymbol{B}$ satisfies the countable chain condition. Hence $\boldsymbol{B}$ is isomorphic to $P(N)$, where N is the set of all natural numbers. The σ-finite measure μ can be transferred, in a natural way, to $P(N)$ as a strictly positive measure. Any such measure on $P(N)$ is completely atomic. See Theorem 2 of [10, p. 352]. This proves the theorem.

The above theorem raises the following natural question. Does $\boldsymbol{A}(\boldsymbol{I}_\mu)$ have a complement in the bigger lattice L_A, the lattice of all sub-fields (same as Boolean subalgebras) of $\boldsymbol{A}$? An affirmative answer to this question follows from a theorem of MAHARAM [6, Theorem 3, p. 992] which we quote below.

"For any measure space $(X, \boldsymbol{A}, \mu)$ with μ, a σ-finite complete measure, there exists a field of sets $\boldsymbol{C} \subset \boldsymbol{A}$ which is a lifting of the natural homomorphism from $\boldsymbol{A}$ onto $\boldsymbol{A}/\boldsymbol{I}_\mu$."

The following question remains open. Let $(X, \boldsymbol{A}, \mu)$ be a charge space, i.e., $\boldsymbol{A}$ is a fied of sets on X and μ is a charge (finitely additive) on $\boldsymbol{A}$. Does the field $\boldsymbol{A}(\boldsymbol{I}_\mu)$ generated by $\boldsymbol{I}_\mu$ have a complement in L_A?

Now, we examine the problem how far the assumption of the completeness of the measure μ is essential in Theorem 6.2. Below we shed some light on this aspect. For this we need the following theorem.

Theorem 6.3. (VON NEUMANN and STONE [7, Theorem 17, p. 369 and Theorem 15, p. 367]). *Let A be a Boolean algebra and let I be an ideal in A with the following property.*

(*) *For every $J \subset I$ with cardinality of $J <$ cardinality of A/I, there exists $a \in A$ which is the supremum of all elements in J.*

Then we can find a subalgebra B of A which is a lifting of the natural homomorphism from A onto A/I.

Remark. It is clear that the B given by Theorem 6.3 is a complement of $A(I)$, where $A(I)$ is the subalgebra of A generated by I.

Theorem 6.4. *Assume Continuum Hypothesis. Let $\boldsymbol{A}$ be any σ-field on X with cardinality $\leqq c$ (the cardinality of the continuum). Let $\boldsymbol{I}$ be any σ-ideal in $\boldsymbol{A}$. Then $\boldsymbol{A}(\boldsymbol{I})$ has a complement in $L_{\boldsymbol{A}}$.*

Proof. Note that the cardinality of $\boldsymbol{A}/\boldsymbol{I}$ is $\leqq c$. Since $\boldsymbol{I}$ is a σ-ideal, Continuum Hypothesis implies that $(*)$ of Theorem 6.3 is valid. This completes the proof.

Remarks. If $\boldsymbol{A}$ is a separable σ-field on X, then it is true that the cardinality of $\boldsymbol{A}$ is $\leqq c$. Theorem 6.4 covers the case of the Borel σ-field of the real line with the σ-ideal of all sets of μ-measure zero for some measure μ. Further, the Borel σ-field of the real line is never complete under any σ-finite measure.

Now, we turn our attention to the topological case. Let $(X, \boldsymbol{T})$ be a topological space. A subset B of X is said to have the property of Baire if we can write $B = U \,\Delta\, P$ for some open set $U \subset X$ and a first category set $P \subset X$. Let $\boldsymbol{B}$ be the collection of all subsets of X with the property of Baire. Then $\boldsymbol{B}$ is a σ-field on X. See Oxtoby [8, Theorem 4.3, p. 19]. Let $\boldsymbol{I}$ be the σ-ideal of all first category subsets of X. It is clear that $\boldsymbol{B}$ is complete with respect to $\boldsymbol{I}$, i.e., $B \in \boldsymbol{I}$, $C \subset B$ implies $C \in \boldsymbol{B}$. In this set-up the following two problems arise.

1. Does the sub σ-field of $\boldsymbol{B}$ generated by $\boldsymbol{I}$, i.e., $\boldsymbol{B}(\boldsymbol{I})$ have a complement in $L^{\sigma}_{\boldsymbol{B}}$? We answer this question in the negative.
2. Does $\boldsymbol{B}(\boldsymbol{I})$ have a complement in $L_{\boldsymbol{B}}$? We answer this question in the affirmative.

Proposition 6.5. *Let X be the real line equipped with the usual topology. Then $\boldsymbol{B}(\boldsymbol{I})$ has no complement in $L^{\sigma}_{\boldsymbol{B}}$.*

Proof. Note that $\boldsymbol{B}$ contains the Borel σ-field of the real line. Consequently, there is no nonzero $0-1$ valued continuous measure on $\boldsymbol{B}$. Further, $\boldsymbol{I}$ contains all singletons. An application of theorem 6.1 completes the proof.

To answer the second question we need the following theorem.

Theorem 6.6. (von Neumann and Stone [7, Theorem 18, p. 372 and Theorem 15, p. 367]). *Let A be a Boolean algebra and I an ideal in A satisfying the following property.*

(**) *For any two nonempty $J_1, J_2 \subset I$ such that the cardinalities of J_1 and $J_2 <$ the cardinality of A/I and $c \leqq d$ for every $c \in J_1$ and $d \in J_2$, there exists $a \in A$ such that $c \leqq a \leqq d$ for every $c \in J_1$ and $d \in J_2$.*

Suppose there is a function $F : A \to A$ satisfying

(i) *$F(a) \,\Delta\, a \in I$ for every $a \in A$.*
(ii) *$a, b \in A$ and $a \,\Delta\, b \in I$ implies $F(a) = F(b)$, and*
(iii) *$F(a \vee b) = F(a) \vee F(b)$ for $a, b \in A$.*

Then we can find a subalgebra B of A such that B is a lifting of the natural homomorphism from A onto A/I.

Using this result we prove the following theorem.

Theorem 6.7. *Let X be any topological space. Let $\boldsymbol{B}$ be the σ-field of all subsets of X having the property of Baire and $\boldsymbol{I}$ the σ-ideal of all first category subsets of X. Then $\boldsymbol{B}(\boldsymbol{I})$ has a complement in $L_{\boldsymbol{B}}$.*

Proof. Note that the ideal $\boldsymbol{I}$ satisfies the property $(**)$ of Theorem 6.6. Now, we define a function $F : \boldsymbol{B} \to \boldsymbol{B}$ as follows.

For $A \in \boldsymbol{B}$, $F(A) = \{x \in X :$ for every open set V containing x, $V \cap A$ is not of first category in $X\}$. Then F satisfies the following properties.

(i) $F(A) \in \boldsymbol{B}$, $A \in \boldsymbol{B}$.

(ii) $F(A) \,\Delta\, A \in \boldsymbol{I}$ for $A \in \boldsymbol{B}$.

(iii) $A, B \in \boldsymbol{B}$, $A \,\Delta\, B \in \boldsymbol{I}$ implies $F(A) = F(B)$.

(iv) $F(A \cup B) = F(A) \cup F(B)$ for $A, B \in \boldsymbol{B}$.

See Kuratowski [5, pp. 83–85].

Invoking Theorems 6.6 and 2.1, we get the desired result.

Remark. This result can be viewed as a category analogue of Maharam's Theorem [6, p. 992].

Finally, we make a remark on the following problem raised by B. V. Rao [11, p. 215]. Characterise the sub σ-fields of the Borel σ-field $\boldsymbol{A}$ of the real line which have complements in $L^{\sigma}_{\boldsymbol{A}}$. Recently, Sarbadhikari and K. P. S. Bhaskara Rao [12] have shown that every separable sub σ-field of $\boldsymbol{A}$ has a complement in $L^{\sigma}_{\boldsymbol{A}}$. The problem still remains open for a complete solution.

7. Ultrastructures. Let B be a Boolean algebra with the associated lattice L_B of all subalgebras of B. With each element $C \neq B$, $C \in L_B$ we can associate an ideal $I_C = \{D \in L_B : D \leqq C\}$ in the lattice L_B. When is a maximal ideal (proper) in L_B is of the form I_C for some $C \in L_B$? To answer this question we introduce Ultrastructures.

An element $C \neq B$, $C \in L_B$ is said to be an ultrastructure in L_B if for any $D \in L_B$ such that $C \leqq D \leqq B$ implies either $D = C$ or $D = B$. (B. V. Rao [11, p. 215–216] defined ultrastructures in the lattice $L_{P(X)}$ and gave a characterisation of these structures.) It is easy to see that I_C is a maximal ideal in L_B if and only if C is an ultrastructure. So the problem of charactrisation of maximal ideals of the form I_C in L_B boils down to the characterisation of ultrastructures in L_B. We do this in this section.

The following characterisation of ultrastructures in L_B is similar to the one obtained by B. V. Rao [11, Theorem 4, p. 216].

Theorem 7.1. *Let I and J be two distinct maximal ideals in B. Let $A(I, J) = \{b \in B : b$ or $b' \in I \cap J\}$. Then $A(I, J)$ is an ultrastructure in L_B. Conversely, every ultrastructure in L_B is of this form.*

Proof. It is clear that $A(I, J)$ is a subalgebra of B. In fact, $A(I, J)$ is the subalgebra generated by the ideal $I \cap J$ in B. Since I and J are distinct, $A(I, J) \neq B$. Let D be an element in L_B which contains $A(I, J)$ properly. We will show that $D = B$. For this, it is sufficient to show that $I \subset D$. (Since I is maximal in B, the subalgebra generated by I is B itself). Let $d \in D$ be such that $d \notin A(I, J)$. Consequently, d and $d' \notin I \cap J$. Without loss of generality assume that $d \in I$ and $d' \in J$. Let $b \in I$. It follows that $b \wedge d' \in I \cap J$. Consequently, $b \wedge d' \in D$. Case (i). $b \in J$. We find that $b \wedge d \in I \cap J$. So, $b \wedge d \in D$. Consequently, $b = (b \wedge d') \vee (b \wedge d) \in D$. Case (ii). $b' \in J$. We observe that $b \Delta d = (b \vee d) \wedge (b' \vee d') \in I \cap J$. From this it follows that $b \Delta d \in D$. This together with $d \in D$ implies that $b \wedge d = d - (b \Delta d) \in D$. Hence $b \in D$. Thus we have proved that $I \subset D$.

To prove the converse, we need the following two lemmas.

Lemma 7.2. *Let $I_1, I_2, \ldots, I_k$ be k distinct maximal ideals in a Boolean algebra B. Let N_1 and N_2 be any arbitrary partition of the set $\{1, 2, \ldots, k\}$. Then there exists a $b \in B$ such that $b \in I_i$ for every $i \in N_1$ and $b \in I_j$ for every $j \in N_2$.*

Proof. We prove this lemma by induction. For two distinct maximal ideals the result is obvious. Assume the result to be true for any $n - 1$ $(n \geqq 3)$ distinct maximal ideals. Let $I_1, I_2, \ldots, I_n$ be n distinct maximal ideals. The case when one of the sets in the decomposition of $\{1, 2, \ldots, n\}$ is empty the result trivially follows. Assume, without loss of generality, $N_1 = \{1, 2, \ldots, m\}$ and $N_2 = \{m + 1, m + 2, \ldots, n\}$ and the cardinality of $N_2 > 1$. By induction hypothesis, there is an $a \in I_i$ for $1 \leqq i \leqq m$ and $a \in I_j$ for $m + j \leqq n - 1$, and a $b \in I_i$ for $1 \leqq i \leqq m$ and $b \in I_j$ for $m + 2 \leqq j \leqq n$. $a \vee b$ would do the trick.

Let B be a Boolean algebra and D a subalgebra of B. It is easy to verify that if I is a maximal ideal in B, then $I \cap D$ is a maximal ideal in D. It is also true that if I_1 is a maximal ideal in D, there exists a maximal ideal I in B containing I_1, which we call an extension of I_1.

Lemma 7.3. *Let B be a Boolean algebra and let D be a subalgebra of B. If every maximal ideal in D has a unique extension in B, then $D = B$.*

Proof. Let X and Y be the Stone spaces of B and D respectively. We identify the Stone spaces as the collection of maximal ideals. We define $f : X \to Y$ as follows. $f(I) = D \cap I$. From the hypothesis it follows that f is one-one. Hence the inclusion map $i : D \to B$ is onto. See Sikorski [14, first four paragraphs on p. 34]. Hence $D = B$.

Now, we prove the converse part of the theorem. Let C be an ultrastructure in L_B. We claim that there is no maximal ideal in C which admits more than two extensions in B. Suppose not. Let I_1 be a maximal ideal in C and let J_1, J_2, J_3 be three distinct extensions of I_1 in B. Now, $I_1 \subset J_1 \cap J_2 \cap J_3 \subset J_1 \cap J_2$. The latter inclusion is strict in view of Lemma 7.2. The subalgebra of B generated by $I_1 = B(I_1) = C$ is strictly contained in $B(J_1 \cap J_2) = A(J_1, J_2)$. Hence C is not an ultrastructure.

Now, we claim that there exists at least one maximal ideal I_1 in C which admits exactly two extensions I and J in B. This is obvious from Lemma 7.3.

Now, observe that $C = B(I_1) \subset B(I \cap J)$, since $I_1 \subset I \cap J$. Since $B(I \cap J) = A(I, J)$ is an ultrastructure in L_B, $C = B(I \cap J)$, i.e., C is of the form $A(I, J)$ for some maximal ideals I and J in B. This completes the proof of the theorem.

Finally, we make a remark about the complements of ultrastructures in L_B. Let C be an ultrastructure in L_B. Take any element $b \in B$ such that $b \notin C$. Then the Boolean subalgebra of B generated by b is a complement of C in L_B. Conversely, there exists a compelement of the Boolean subalgebra of B generated by a single element $b \in B$ such that $b \neq 0$ and $b \neq 1$ which is an ultrastructure in L_B. These statements are not difficult to prove and the proof is omitted.

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Authors' addresses: K. P. S. Bhaskara Rao, Research and Training School, Indian Statistical Institute, 203 B. T. Road, Calcutta 35 India; M. Bhaskara Rao, Department of Probability and Statistics, The University, Sheffield S3 7RH, U.K.