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LATTICES OF CONVEX EQUIVALENCES*)

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This paper is a continuation of [1] and [4–6]. In the first part of the present paper ther is proved that the lattice of all convex equivalences of an ordered set can be interpreted as a lattice of all congruences of an object of a special category. The second part contains a characterization of compact elements of this lattice. In the final part, we shall study an equivalence on the set of all orderings of a given set. I want to express my great thanks to prof. MIROSLAV NOVOTNÝ for his numerous suggestions that had deep influence to my work.

CONVEX EQUIVALENCES AS $\mathcal{T}$-CONGRUENCES

1. Notation. a) Let $\mathscr{X} = (X; \leqq)$ be an ordered set. Let us define for every $x, y \in X$

$$(x\rangle_{\mathscr{X}} =_{\mathrm{Df}} \{z \in X \mid z \leqq x\}\,, \quad \langle x)_{\mathscr{X}} =_{\mathrm{Df}} \{z \in X \mid x \leqq z\}\,,$$

$$\langle x, y\rangle_{\mathscr{X}} =_{\mathrm{Df}} \langle x)_{\mathscr{X}} \cap (y\rangle_{\mathscr{X}}\,, \quad [x, y]_{\mathscr{X}} =_{\mathrm{Df}} \langle x, y\rangle_{\mathscr{X}} \cup \langle y, x\rangle_{\mathscr{X}} \cup \{x, y\}\,.$$

Y is called *convex subset* of $\mathscr{X}$ if for every $x, y \in Y$, there is $[x, y]_{\mathscr{X}} \subseteq Y$. An equivalence σ on X is called *convex* in $\mathscr{X}$, if every element of the corresponding factor-set X/σ is a convex subset of $\mathscr{X}$. The set of all convex equivalences on $\mathscr{X}$ will be denoted by $c(\mathscr{X})$ or by $c(X; \leqq)$. The system $c(\mathscr{X})$ was studied in [4], Sections 35–45 and in [5], Section 5. (See also [8].)

Let r be a binary relation. By the way of the following definition, we obtain a ternary relation ξ_r:

$$(x, y, z) \in \xi_r \Leftrightarrow_{\mathrm{Df}} (x, y) \in r \quad \text{et} \quad (y, z) \in r\,.$$

If $\mathscr{Y} = (Y; r)$, then we put $\mathscr{Y}^* =_{\mathrm{Df}} (Y; \xi_r)$.

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If Z is a system of sets, then the ordered set $(Z; \subseteq)$ will often be denoted by Z, as well.

b) In this paper we shall use the notation of [6] (Section 1). Especially recall: $E(X)$ is the set of all equivalences on X. Let r be a relation of arity n. For the sets $Y_1, \ldots, Y_n$ put $(Y_1, \ldots, Y_n) \in \dot{r}$, if either $Y_i = \emptyset$ for every $i = 1, \ldots, n$, or there exist $y_1 \in Y_1, \ldots, y_n \in Y_n$ such that $(y_1, \ldots, y_n) \in r$. The symbol $(X; r)$ denotes a structure with the support X and an n-ary relation $r \cap X^n$.

c) In the whole paper, $\mathscr{A} = (A; \leqq)$ will denote a given ordered set. If $x, y \in A$, then we shall write $(x\rangle, \langle x), \langle x, y\rangle, [x, y]$ instead of $(x\rangle_{\mathscr{A}}, \langle x)_{\mathscr{A}}, \langle x, y\rangle_{\mathscr{A}}, [x, y]_{\mathscr{A}}$.

2. Remark. Section 1 of paper [1] contains the definition of the class $\mathrm{Con}_{\mathscr{K}}(X)$ of all $\mathscr{K}$-congruences for every $\mathscr{K}$-object X of a given category $\mathscr{K}$. Section 1 of paper [3] contains the following result: If $\mathscr{K}$ is the category of algebraic structures of a given type, then $\mathrm{Con}_{\mathscr{K}}(X)$ is a set and it holds

$$\mathrm{Con}_{\mathscr{K}}(X) = \{\ker f \mid \mathrm{dom}\, f = X,\ f \text{ is a } \mathscr{K}\text{-morphism}\}.$$

In sections 3–7 of the present paper, we consider a problem in a certain sens converse: Let us associate, to every ordered set $\mathscr{X}$, the set $c(\mathscr{X})$. A shown in the following, there exists a quasivariety $\mathscr{T}$ of relational systems with a ternary relation and a mapping $\mathscr{X} \mapsto \mathscr{X}^o$ (where $\mathscr{X}^o \in \mathscr{T}^{\mathrm{Ob}}$ such, that $c(\mathscr{X}) = \mathrm{Con}_{\mathscr{T}}(\mathscr{X}^o)$.

3. Remark. a) Let us define the category $\mathscr{T}$. The class $\mathscr{T}^{\mathrm{Ob}}$ is a quasivariety of all relational systems $Y = (Y'; \zeta_Y)$ satisfying the following quasiidentity

(1) $$(\forall x, y \in Y')\left((x, y, x) \in \zeta_Y \Rightarrow x = y\right).$$

If $Y, Z \in \mathscr{T}^{\mathrm{Ob}}$, then $f \in \mathrm{Hom}_{\mathscr{T}}(Y, Z)$ iff $f: Y' \to Z'$ is the usual homomorphism of Y to Z *). Following Section 5 of paper [3], $\mathrm{Con}_{\mathscr{T}}(Y)$ is an algebraic closure system of the lattice $E(Y')$ ($\mathscr{T}^{\mathrm{Ob}}$ being a quasivariety). Hence $\mathrm{Con}_{\mathscr{T}}(Y)$ is an algebraic lattice (see Section 8/a or [2], Section 5).

b) Theorem 2.c of paper [3] yields the following characterization of $\mathscr{T}$-congruences:

Let $Y \in \mathscr{T}^{\mathrm{Ob}}$ and let $\sigma \in E(Y')$. Then $\sigma \in \mathrm{Con}_{\mathscr{T}}(Y)$ iff $(Y'/\sigma; (\zeta_Y)^{\cdot}) \in \mathscr{T}^{\mathrm{Ob}}$.

(A quasivariety is closed under isomorphisms and homomorphic images; the set of operational symbols is empty in our case, hence every equivalence on Y' is an absolute congruence on Y. It is easy, after all, to prove this result directly.)

c) Let $\mathscr{B} = (B; \leqq)$ be an ordered set. The ordering being an antisymetric relation, $\mathscr{B}^* = (B; \xi_{\leqq})$ is a $\mathscr{T}$-object. Given $f: A \to B$, it is obvious that $f \in \mathrm{Hom}_{\mathscr{T}}(\mathscr{A}^*, \mathscr{B}^*)$ iff f is an isotonic mapping from $\mathscr{A}$ to $\mathscr{B}$.

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*) i.e. it holds

$$(\forall x, y, z \in Y')\left((x, y, z) \in \zeta_Y \Rightarrow (f(x), f(y), f(z)) \in \zeta_Z\right).$$

4. Theorem. *There is* $c(\mathscr{A}) = \mathrm{Con}_{\mathscr{T}}(\mathscr{A}^*)$.

Proof. Let $\sigma \in c(\mathscr{A})$. All the elements of A/σ arė convex subsets of $\mathscr{A}$, hence

$$(\forall X, Y \in A/\sigma)\left((X, Y, X) \in (\xi_{\leqq})^{\cdot} \Rightarrow X = Y\right),$$

i.e. $(A/\sigma; (\xi_{\leqq})^{\cdot}) \in \mathscr{T}^{\mathrm{Ob}}$. This yield, by Section 3/b, $\sigma \in \mathrm{Con}_{\mathscr{T}}(\mathscr{A}^*)$.

Now, let $\sigma \in \mathrm{Con}_{\mathscr{T}}(\mathscr{A}^*)$. Then there exists $h \in \mathrm{Hom}_{\mathscr{T}}(\mathscr{A}^*, Y)$ (where $Y \in \mathscr{T}^{\mathrm{Ob}}$) such that $\sigma = \ker h$. Let $x_1, x_2 \in X \in A/\sigma$ and $y \in A$ be such that $x_1 \leqq y \leqq x_2$. Then $(x_1, y, x_2) \in \xi_{\leq}$ and thus, h being a $\mathscr{T}$-morphism, there is $(h(x_1), h(y), h(x_2)) \in \in \zeta_Y$. Further $(x_1, x_2) \in \sigma = \ker h$, i.e. $h(x_1) = h(x_2)$. Hence, by (1), there is $h(x_1) = = h(y)$, i.e. $y \in X$. Thus $\sigma \in c(\mathscr{A})$.

5. Corollary. *$c(\mathscr{A})$ is an algebraic closure system of the algebraic lattice $E(A)$. Especially: $c(\mathscr{A})$ is an algebraic lattice.*

Proof. This assertion follows immediately from Theorem 4 and the results mentioned in Section 3/a and 8/a. This assertion can also be proved directly.

6. Remark. Hence lattices of convex equivalences can be interpreted as lattices of $\mathscr{T}$-congruences. Of course, it is possible that there exist some other categories with the same property. From this point of view it is interesting to consider the following theorem, showing that a relatively natural category $\mathscr{K}$ is not convenient for that.

7. Theorem. *Let $\mathscr{K}$ be a category satisfying the following three conditions:*

Every $\mathscr{K}$-object X is a relational system $(X'; r_X)$ with a binary relation r_X.
Every ordered set is a $\mathscr{K}$-object.
There is, for every $X, Y \in \mathscr{K}^{\mathrm{Ob}}$,

$$\mathrm{Hom}_{\mathscr{K}}(X, Y) = \{f : X' \to Y' \mid (\forall x, y \in X')(x r_X y \Rightarrow f(x)\, r_Y f(y))\}. \text{ *)}$$

Then there exists an ordered set B with

$$c(B) \neq \mathrm{Con}_{\mathscr{K}}(B).$$

Proof. Let $C = (C'; r_C)$ be the ordered set characterized by Fig. 1.

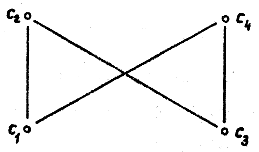

Fig. 1.

*) I.e. the class of all $\mathscr{K}$-morphisms is the class of all usual homomorphisms of $\mathscr{K}$-objects.

Then

$$\sigma =_{\mathrm{Df}} \{c_1, c_2\}^2 \cup \{c_3, c_4\}^2$$

is a convex equivalence on C. If $\sigma \notin \mathrm{Con}_{\mathscr{K}}(C)$, then the theorem holds. If we suppose that $\sigma \in \mathrm{Con}_{\mathscr{K}}(C)$, then there exist $X \in \mathscr{K}^{\mathrm{Ob}}$ and $h \in \mathrm{Hom}_{\mathscr{K}}(C, X)$ such that $\sigma = \ker h$. This yields the existence of two elements $x, y \in X'$ with

$$h(c_1) = h(c_2) = x \neq y = h(c_3) = h(c_4)\,.$$

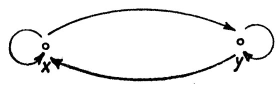

Fig. 2.

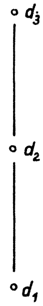

Fig. 3

Since h is a $\mathscr{K}$-morphism and $(c_1, c_4) \in r_C$, $(c_3, c_2) \in r_C$, then $(x, y) \in r_X$, $(y, x) \in r_X$, as well. (See Fig. 2.) Let D be an ordered set characterized by Fig. 3. There is, by assumption, $D \in \mathscr{K}^{\mathrm{Ob}}$. Defining

$$g(d_1) = g(d_3) = x\,, \quad g(d_2) = y\,,$$

there is $g \in \mathrm{Hom}_{\mathscr{K}}(D, X)$. Further,

$$\ker g = \{d_1, d_3\}^2 \cup \{(d_2, d_2)\} \notin c(D)\,,$$

i.e. $c(D) \neq \mathrm{Con}_{\mathscr{K}}(D)$.

COMPACT ELEMENTS OF THE LATTICE $(c(\mathscr{A}); \subseteq)$

8. Remark. a) (*Construction of the closure operator induced by* $c(\mathscr{A})$). $c(\mathscr{A})$ is an algebraic closure system of the algebraic lattice $E(A)$, following Section 5.

Theorem 9 of paper [2] characterizes the compact elements of $(c(\mathscr{A}); \subseteq)$:

Let $\mathscr{L} = (L; \lesssim)$ *be an algebraic lattice, let* S *be an algebraic closure system of* $\mathscr{L}$ *and let* $u_S : L \to L$ *be the closure operator which corresponds to* S *). *Then an element* c *is compact in* $(S; \lesssim)$ *iff it is a* u_S*-image of a compact element in* $\mathscr{L}$; *moreover,* $(S; \lesssim)$ *is an algebraic lattice, too.* (See also [7]).

*) Recall: S is an *algebraic closure system of* $\mathscr{L}$ if the following two requirements hold:
There is, for every $X \subseteq S$, $\inf_{(S;\lesssim)} X = \inf_{\mathscr{L}} X$.
There is, for every nonvoid chain X of $(S; \lesssim)$, $\sup_{(S;\lesssim)} X = \sup_{\mathscr{L}} X$.
We put, for $x \in L$, $u_S(x) =_{\mathrm{Df}} \inf_{\mathscr{L}} \{y \in S \mid x \lesssim y\}$.

First, let us find the algebraic closure operator $u_{c(\mathscr{A})}$ of $E(A)$. Take a binary relation r and $n \in N =_{\mathrm{Df}} \{0, 1, 2, \ldots\}$, and put

$$\psi_0(r) =_{\mathrm{Df}} (r \cap A^2) - \mathrm{id}_A ,$$
$$\psi_{n+1}(r) =_{\mathrm{Df}} \bigcup\{[x, y]^2 . (\mathrm{id}_A \cup \psi_n(r)) \mid (x, y) \in \psi_n(r)\} .$$

Then there is, if $m, n \in N$, $m \leqq n$,

$$\psi_m(r) \subseteq \psi_n(r) \subseteq A^2 .$$

Put

$$\varkappa(r) =_{\mathrm{Df}} \mathrm{id}_A \cup \bigcup_{N \ni n} \psi_n(r) .$$

There is, for every $n \in N$, $\psi_n(r) \subseteq \varkappa(r)$, $r \cap A^2 \subseteq \varkappa(r)$ and $\mathrm{id}_A \subseteq \varkappa(r) \subseteq A^2$. It is easy to prove, that $\varkappa(r)$ is an equivalence on A. Let $(x, y) \in \varkappa(r)$, $z \in A$, $x \leqq z \leqq y$. If $x \neq y$, then there exists $n \in N$ such that $(x, y) \in \psi_n(r)$. Then $[x, y]^2 \subseteq \psi_{n+1}(r) \subseteq \varkappa(r)$; thus, $(x, z) \in \varkappa(r)$. If $x = y$, then $x = z$ and obviously $(x, z) \in \varkappa(r)$, as well. Therefore $\varkappa(r) \in c(\mathscr{A})$.

Let $\sigma \in c(\mathscr{A})$, $r \cap A^2 \subseteq \sigma$. Using the induction over n, we get $\psi_n(r) \subseteq \sigma$, hence $\varkappa(r) \subseteq \sigma$, as well. Then

$$\varkappa(r) = \bigcap\{\sigma \in c(\mathscr{A}) \mid r \cap A^2 \subseteq \sigma\} ;$$

especially

$$\varkappa \mid E(A) = u_{c(\mathscr{A})} ,$$

where $\varkappa \mid E(A)$ denotes the restriction of $\varkappa$ to $E(A)$. (See also [9], Section 2.)

b) (*Function m*). Let r be a binary relation and let $x \in A$. If there exists $y \in A$ such that for some $n \in N$ it holds

$$(x, y) \in \psi_n(r) \cup (\psi_n(r))^{-1} ,$$

then we put $m(x)$ equal to the smallest of those n. If such a natural number n does not exist, we put $m(x) =_{\mathrm{Df}} -1$. By this way we have defined a mapping $m : A \to N \cup \{-1\}$. (Function m depends also on r, but the binary relation r is always fixed.)

Let $m(x) \geqq 1$. Denote by $i = m(x)$. Then there exists $y \in A$ such that $(x, y) \in \psi_i(r) \cup (\psi_i(r))^{-1}$ and for every $z \in A$, there is $(x, z) \notin \psi_{i-1}(r) \cup (\psi_{i-1}(r))^{-1}$. This implies the existence of $x_1, x_2 \in A$, satisfying at least one of the following eventualities

(2) $$(x_1, x_2) \in \psi_{i-1}(r) , \quad x \in [x_1, x_2] ;$$

(3) $$(x_1, x_2) \in \psi_{i-1}(r) , \quad (x, y) \in [x_1, x_2]^2 . \psi_{i-1}(r) ;$$

(4) $$(x_1, x_2) \in \psi_{i-1}(r) , \quad (y, x) \in [x_1, x_2]^2 . \psi_{i-1}(r) .$$

Possibility (4) is excluded as shown by the following: if $(y, x) \in [x_1, x_2]^2 . \psi_{i-1}(r)$, then

$y \in [x_1, x_2]$ and there exists $z \in [x_1, x_2]$ such that $(z, x) \in \psi_{m(x)-1}(r)$ – which is in contradiction with the definition of $m(x)$. In (2) and (3), there is $x \in [x_1, x_2]$. Since $(x_1, x_2) \in \psi_{m(x)-1}(r)$, we have $x_1 \neq x \neq x_2$. This yields that there exist $x_1, x_2 \in \in A$ such that $(x_1, x_2) \in \psi_{m(x)-1}(r)$ and that $x_1 < x < x_2$ (we still suppose $m(x) \geqq 1$)

c) Lemma. *Let $\{X_i \mid i \in I\}$ be a system of nonvoid subsets of A. Put*

$$r =_{\mathrm{Df}} \bigcup\{X_i^2 \mid i \in I\} .$$

Let $Y \in A/\varkappa(r)$. If $X_i \cap Y \neq \emptyset$ for some $i \in I$, then $X_i \subseteq Y$. If $2 \leqq |Y|$, then there exists $j \in I$ such that $X_j \subseteq Y$ and $2 \leqq |X_j|$. (See also [9], Section 5/b.)

Proof. Let $a \in X_i \cap Y$ for a given $i \in I$. Then for every $x \in X_i$, there is $(x, a) \in \in X_i^2 \subseteq r \subseteq \varkappa(r)$, hence $(x, a) \in Y^2$, too; i.e. there is $x \in Y$.

Let $2 \leqq |Y|$. Suppose that $X_i \cap Y = \emptyset$, whenever X_i has at least two points. Put

$$s =_{\mathrm{Df}} (\varkappa(r) - Y^2) \cup \mathrm{id}_A .$$

Obviously, $s \in c(\mathscr{A})$ and s is a proper subset of $\varkappa(r)$, following the assumption $|Y| \geqq 2$. For all $i \in I$, for which $2 \leqq |X_i|$, there exist $Z_i \in A/\varkappa(r)$ with $X_i^2 \subseteq Z_i^2$ (since $r \subseteq \varkappa(r)$ and $Y \cap \bigcup\{X_i \mid i \in I,\ 2 \leqq |X_i|\} = \emptyset$, then we get $Y \neq Z_i$). Thus $Z_i \in A/s$ and, following, $r \subseteq s$. We have proved

$$r \subseteq s \subset \varkappa(r) , \quad s \in c(\mathscr{A}) ,$$

and that is a contradiction (see Section 8/a).

d) Let $\sigma \in E(A)$. Then σ is compact in the complete lattice $E(A)$ iff it satisfies the following two requirements:

$$|\{X \in A/\sigma \mid 2 \leqq |X|\}| < \aleph_0 ; \quad (\forall X \in A/\sigma)(|X| < \aleph_0) .$$

The proof of this statement follows immediately from the definition of a compact element and from the properties of $(E(A); \subseteq)$, and it is left to the reader.

9. Lemma. *Let $\sigma \in c(\mathscr{A})$. Then σ is a $\varkappa$-image of a compact element in $E(A)$ iff it satisfies the following requirements:*

(5) *The system of all classes of A/σ having at least 2 elements, is finite.*

(6) *For every $X \in A/\sigma$, the set of all maximal elements in $(X; \leqq)$ as well as the set of all minimal elements in $(X; \leqq)$, is finite.*

(7) *For every $X \in A/\sigma$, every maximal chain in $(X; \leqq)$ is bounded in $(X; \leqq)$.*

Proof. **1.** Suppose that there exists an element α, which is compact in $E(A)$ and such that $\sigma = \varkappa(\alpha)$. If $\alpha = \mathrm{id}_A$, then $\sigma = \varkappa(\alpha) = \mathrm{id}_A$ and requirements (5–7) are trivially satisfied. So, let $\alpha \neq \mathrm{id}_A$. Then there exist finitely many pairwise disjoint finite subsets $X_1, \ldots, X_n$ of A with at least two points and such that

$$\alpha = \mathrm{id}_A \cup \bigcup_{i=1}^{n} X_i^2$$

(see Section 8/d). Then $\sigma = \varkappa(\alpha)$ satisfies (5) following Section 8/c.

Let us show that σ satisfies (6). Take such an $X \in A/\sigma$, that $|X| \geqq 2$ (if $|X| = 1$, then (6) holds, of course). Let x be maximal in $(X; \leqq)$. From the definition of $m : A \to \to N \cup \{-1\}$, we get (for the considered α) $m(x) \geqq 0$, since $|X| \geqq 2$. We shall prove that $m(x) = 0$. If, to the contrary, $0 < m(x)$, then there exist $x_1, x_2 \in A$ such that

$$(x_1, x_2) \in \psi_{m(x)-1}(\alpha) \subseteq \varkappa(\alpha), \quad x_1 < x < x_2$$

following the statement of Section 8/b. Hence x cannot be maximal in $(X; \leqq)$; this contradiction yields $m(x) = 0$, thus $x \in \bigcup_{i=1}^{n} X_i$. Since $\bigcup_{i=1}^{n} X_i$ is finite, the set of all maximal elements is finite, too. By an analogical way can be proved that the set of all minimal elements is also finite.

We shall prove that σ satisfies (7). Let $X \in A/\sigma$, $|X| \geqq 2$ (if $|X| = 1$, then (7) holds trivially) and let $x \in X$. First, show that there exists $x' \in X \cap \bigcup_{i=1}^{n} X_i$ such that $x \leqq x'$. If $m(x) = 0$, then we can put $x' = x$. Let for a given $k \in N$ and for every $y \in X$, for which $m(y) \leqq k$, there exists $y' \in X \cap \bigcup_{i=1}^{n} X_i$ with $y \leqq y'(|X| \geqq 2$, $X \in A/\varkappa(\alpha)$, hence $m(y) \geqq 0$ for every $y \in X)$. Let x be a given element with $m(x) = k + 1$. Then, by Section 8/b, there exist $x_1, x_2 \in A$ such that

$$(x_1, x_2) \in \psi_k(\alpha) \subseteq \varkappa(\alpha), \quad x_1 < x < x_2 .$$

Then $m(x_2) \leqq k$, $x_2 \in X$, hence there exists $x_2' \in X \cap \bigcup_{i=1}^{n} X_i$ such that, by the assumption of induction, $x_2 \leqq x_2'$. Then $x < x_2'$, of course, and we can put $x' = x_2'$. Among those elements y of $X \cap \bigcup_{i=1}^{n} X_i$ for which $x \leqq y$, consider the maximal ones ($X \cap \bigcup_{i=1}^{n} X_i$ is a non-empty finite set and therefore it has at least one maximal element); chose one of them and denot it by x^{**}. If $x^{**} \leqq y$ for some $y \in X$, take $y' \in X \cap \bigcup_{i=1}^{n} X_i$ such that $y \leqq y'$. The element x^{**} is maximal in $X \cap \bigcup_{i=1}^{n} X_i$; on the other hand, $x^{**} \leqq y$ hence $x^{**} = y'$. We get $x^{**} = y$, thus there exists, for every $x \in X$, a maximal element x^{**} in $(X; \leqq)$, for which $x \leqq x^{**}$. Dually, there exists an element x^* which is minimal in $(X; \leqq)$ and such that $x^* \leqq x$.

Let R be a maximal chain in $(X; \leqq)$ *). If a is maximal in $(X; \leqq)$, put $M_a =_{\mathrm{Df}}$

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*) I want to thank to Professor Milan Sekanina, the reviewer of this paper, for his suggestion making this final part of the proof considerably simpler (included after the final version).

$=_{\mathrm{Df}} \{x \in X \mid x^{**} = a\}$. $(X; \leqq)$ has only finitely many maximal elements; hence, there exists, among them, an element b such that M_b is confinal in $(R; \leqq)$. Since R is a maximal chain, then $b \in R$, especially, R has an upper bound. Dually can be proved that R has also a lower bound. Hence σ satisfies (7).

2. Now, suppose that $\sigma \in c(\mathscr{A})$ satisfies conditions (5–7). For $X \in A/\sigma$ put

$$\hat{X} =_{\mathrm{Df}} \{x \in X \mid x \text{ is either minimal or maximal in } (X; \leqq)\}$$

and

$$\alpha =_{\mathrm{Df}} \mathrm{id}_A \cup \bigcup\{(\hat{X})^2 \mid X \in A/\sigma\} .$$

We get a symmetrie relation α with $\mathrm{id}_A \subseteq \alpha \subseteq \sigma \subseteq A^2$. Let us prove that σ is transitive. Let $(x, y), (y, z) \in \alpha$. Then there exist $X, Y \in A/\sigma$ such that $x, y \in \hat{X}$, $y, z \in \hat{Y}$. This yields $X = Y$, hence $(x, z) \in (\hat{X})^2$, showing that $\alpha \in E(A)$.

By (6), $1 \leqq |\hat{X}| < \aleph_0$ for every $X \in A/\sigma$ and, by (5), there exist only finitely many $X \in A/\sigma$ having at least two elements. Hence, by Section 8/d, α is compact in $E(A)$.

There is $\sigma \in c(\mathscr{A})$ and $\alpha \subseteq \sigma$. This imply that $\varkappa(\alpha) \subseteq \varkappa(\sigma) = \sigma$ (see Section 8/a). Let $(x, y) \in \sigma$. Then there exists $X \in A/\sigma$ such that $(x, y) \in X^2$ and, by (7), $x^*, x^{**}, y^*, y^{**} \in \hat{X}$ such that $x^* \leqq x \leqq x^{**}$ $y^* \leqq y \leqq y^{**}$ (every element of X being an element of a maximal chain in $(X; \leqq)$). Then, of course

$$(x, y) \in [x^*, x^{**}]^2 \,.\, [x^{**}, y^{**}]^2 \,.\, [y^*, y^{**}]^2 \subseteq \varkappa(\alpha) .$$

Hence $\sigma = \varkappa(\alpha)$, where α is a compact element of $E(A)$.

10. Remark. For $\sigma \in E(A)$, formulate the following requirement:

(7′) $(\forall X \in A/\sigma)\,(\forall x \in X)\,(\exists x^*, x^{**} \in X)\,(x^* \leqq x \leqq x^{**}$ and x^* is minimal and x^{**} is maximal in $(X; \leqq))$.

Conditions (6) and (7′) imply (7), as shown at the end of part 1 of the proof of Lemma 9. It is easy to prove that (7) implies (7′): every element of X belongs to a maximal chain of $(X; \leqq)$.

Hence, in Lemma 9, (7′) can take the place of (7).

11. Theorem. *Let $\sigma \in c(\mathscr{A})$. Then σ is compact in the algebraic lattice $c(\mathscr{A})$ iff it satisfies conditions* (5–7).

Proof. $c(\mathscr{A})$ is an algebraic closure system of the algebraic lattice $E(A)$ by Section 5 and, by Section 8/a. $u_{c(\mathscr{A})} = \varkappa \mid E(A)$. The assertion of this theorem follows then immediately from Lemma 9 and the assertion mentioned in Section 8/a (Theorem 9 of paper [2]).

12. Remark. Let $\mathcal{O}rd$ be the category of ordered sets, morphisms are isotonic mappings. In paper [6], Section 28, there is shown that $\sigma \in \mathrm{Con}_{\mathcal{O}rd}(\mathscr{A})$ is compact in the algebraic lattice $\mathrm{Con}_{\mathcal{O}rd}(\mathscr{A})$ iff it satisfies conditions (5–7).

Moreover, there is $\mathrm{Con}_{\mathcal{O}rd}(\mathscr{A}) \subseteq c(\mathscr{A})$, as shown in [4], Section 36.
Those facts imply the following statement (see also [9], Section 11/c):

Let $\sigma \in \mathrm{Con}_{\mathcal{O}rd}(\mathscr{A})$. *Then* σ *is compact in* $\mathrm{Con}_{\mathcal{O}rd}(\mathscr{A})$ *iff it is compact in* $c(\mathscr{A})$.

AN EQUIVALENCE ON THE SET OF ALL ORDERINGS ON A

13. Remark. a) Let $\mathscr{U}(A)$ denote the set of all orderings on A, i.e.

$$\mathscr{U}(A) =_{\mathrm{Df}} \{u \in \exp A^2 \mid u \cap u^{-1} = \mathrm{id}_A,\ uu \subseteq u\}\,.$$

If $u \in \mathscr{U}(A)$, then by putting

$$(x, y, z) \in \zeta_u \Leftrightarrow_{\mathrm{Df}} (x, y), (y, z) \in u - \mathrm{id}_A \quad \text{or}$$
$$(z, y), (y, x) \in u - \mathrm{id}_A\,,$$

we get a ternary relation ζ_u. An equivalent definition of ζ_u is the following one: If $x, y, z \in A$, then

$$(x, y, z) \in \zeta_u \quad \text{iff} \quad |\{x, y, z\}| = 3 \quad \text{and} \quad y \in [x, z]_{(A;u)}\,.$$

b) We shall define an equivalence A_G on $\mathscr{U}(A)$ as follows (the motivation of this definition is given in [6], Section 22): if $u, v \in \mathscr{U}(A)$, put $(u, v) \in A_G$ iff $\mathrm{Con}_{\mathcal{O}rd}(A; u) = \mathrm{Con}_{\mathcal{O}rd}(A; v)$. The characterization of this equivalence A_G is not simple. However, equivalence A_G suggests the following problem, characterize the equivalence on $\mathscr{U}(A)$ defined by

$$u \sim v \Leftrightarrow_{\mathrm{Df}} c(A; u) = c(A; v) \ ^{*})\,.$$

The characterization of $\sim$ is essentially easier than this one of A_G. First, we shall prove the following theorem:

14. Theorem. *Let* $u, v \in \mathscr{U}(A)$. *Then* $c(A; u) \subseteq c(A; v)$ *iff* $\zeta_v \subseteq \zeta_u$.

Proof. First, let us suppose that the inclusion $\zeta_v \subseteq \zeta_u$ does not hold. Then there exists $(x, y, z) \in \zeta_v - \zeta_u$, hence $y \in [x, z]_{(A;v)} - [x, z]_{(A;u)}$ by Section 13/a. Put

$$\sigma =_{\mathrm{Df}} [x, z]^2_{(A;u)} \cup \mathrm{id}_A\,;$$

then $\sigma \in c(A; u) - c(A; v)$ and the inclusion $c(A; u) \subseteq c(A; v)$ does not hold.

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*) It is easy to formulate these problems for some other categories of algebraic structures: Let $\mathscr{G}r$ denote the category of all groupoids, and $\mathscr{N}(A)$ the set of all binary operations on A, e.g. Put, for $\circ, + \in \mathscr{N}(A)$,

$$\circ \approx + \Leftrightarrow_{\mathrm{Df}} \mathrm{Con}_{\mathscr{G}r}(A; \circ) = \mathrm{Con}_{\mathscr{G}r}(A; +)\,.$$

Then $\approx$ is an equivalence on $\mathscr{N}(A)$. It is possible to define similary equivalences between operations (or sets of operations) in other categories of algebraic structures. As far as I know, those — relatively natural — equivalences were not yet studied.

Suppose now that $\zeta_v \subseteq \zeta_u$. Then every u-convex subset of A is also v-convex, hence $c(A; u) \subseteq c(A; v)$.

15. Corollary. *Let $u, v \in \mathscr{U}(A)$. Then $u \sim v$ if and only if $\zeta_u = \zeta_v$.*

Proof. The assertion follows immediately from Theorem 14.

16. Remark. (included after the final version). The Theorem mentioned in Section 8.a was yet generated in paper [7], Section 2.1. The characterization of $\mathfrak{m}$-compact elements in the lattice of all convex equivalences for any infinite cardinal $\mathfrak{m}$ is given in paper [8], Section 2.7 and 3.4. See also [9].

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*) Included after the final version.