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ON THE EXISTENCE OF 1-FACTORS IN PARTIAL SQUARES OF GRAPHS

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Although TUTTE's characterization [6] of graphs having 1-factors was published in 1947, the problem of existence of 1-factors is still one of the topical subjects of the contemporary graph theory. Obviously, a necessary condition for a graph G to have a 1-factor is that G have even order. CHARTRAND, POLIMENI and STEWART [2], and independently SUMNER [5] proved that if a connected graph G of even order is either a line graph or a square (i.e. the square of a graph), then G has a 1-factor. HOBBS' ideas in [4] concerning the need of common generalization of at least some of the concepts of the square, the cube, the total graph, and the line graph of a given graph inspired the present author to introduce the concept of a partial square which generalizes the concepts of a square and a line graph. In the present note it will be proved that if a connected graph of even order is a partial square, then it has a 1-factor.

In the present not graphs are considered in the sense of the books [1] and [3]. Let G be a graph. We denote by $V(G)$ and $E(G)$ the vertex set of G and the edge set of G, respectively. The number $|V(G)|$ is referred to as the order of G. If $u, v \in V(G)$, then we denote by $d_G(u, v)$ the distance between u and v in G. A set $W \subseteq V(G)$ is called a vertex cover of G if for every pair of adjacent vertices u and v of G, either $u \in W$ or $v \in W$. If W is a vertex cover of G, then we shall say that G is W-connected if there exists a component G' of G such that $W \subseteq V(G')$. We shall say that $w \in V(G)$ is a Y-vertex of G if there exists an induced subgraph F of G such that (a) F is isomorphic to the star $K_{1,3}$, (b) $w \in V(F)$, and (c) w has degree one in F. A vertex cover W of G will be called a Y-cover of G if every Y-vertex of G belongs to W, and $W \neq \emptyset$.

Let G be a graph. The graph G_1 with $V(G_1) = V(G)$ and such that for every pair $u, v \in V(G)$

$$uv \in E(G_1) \quad \text{if and only if} \quad 1 \leqq d_G(u, v) \leqq 2\,,$$

is called the *square* of G. If $E(G) \neq \emptyset$, then the graph G_2 with $V(G_2) = E(G)$ and such that for every pair $e, f \in E(G)$,

$$ef \in E(G_2) \quad \text{if and only if } e \text{ and } f \text{ are adjacent in } G\,,$$

is called the *line graph* of G. The graph G_3 with $V(G_3) = V(G) \cup E(G)$ and such that for every pair $x, y \in V(G) \cup E(G)$,

$$xy \in E(G_3) \text{ if and only if } x \text{ and } y \text{ are adjacent or incident in } G,$$

is called the *total graph* of G. Finally, the graph G_4 obtained from G by inserting precisely one new vertex (of degree two) into each edge of G is called the *subdivision graph* of G. We denote by G^2, $L(G)$, $T(G)$ and $S(G)$ the square of G, the line graph of G, the total graph of G and the subdivision graph of G, respectively.

Let G be a graph and let W be a Y-cover of G. The subgraph of G^2 induced by W will be called the *partial square* of G with respect to W and denoted by the symbol $\mathrm{psq}(G, W)$. Obviously, if $W = V(G)$, then $\mathrm{psq}(G, W) = G^2$. If G is the graph in Fig. 1 and W is the set of black vertices in Fig. 1, then $\mathrm{psq}(G, W)$ is the graph in Fig. 2.

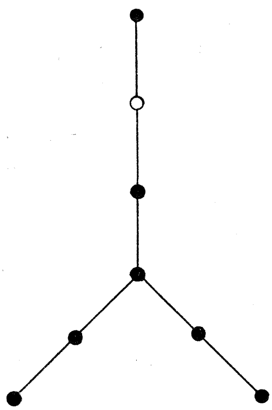

Fig. 1

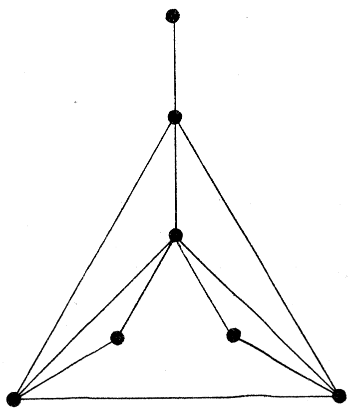

Fig. 2

Let G be a graph. It is well-known that $T(G)$ is isomorphic to $(S(G))^2$. If $E(G) \neq \emptyset$, then it is easy to see that $L(G)$ is isomorphic to $\mathrm{psq}(S(G), V(S(G) - V(G)))$.

Proof of the following proposition may be omitted:

Proposition. *Let G be a graph and let W be a Y-cover of G. Then $\mathrm{psq}(G, W)$ is connected if and only if G is W-connected.*

Let T be a tree and let $v \in V(T)$. Similarly as in [3], we mean by a branch at v (of the tree T) a subtree B of T which is maximal (by $\subseteq$ in $V(T)$) with respect to the property that it contains v as a vertex of degree one.

Lemma. *Let G be a graph and let W be a Y-cover of G. Assume that $|W| \geqq 3$ and that G is W-connected. Then there exist $w_1, w_2 \in W$ such that $1 \leqq d_G(w_1, w_2) \leqq 2$ and that $G - w_1 - w_2$ is $(W - \{w_1, w_2\})$-connected.*

Proof. There exists a component G' of G such that $W \subseteq V(G')$. Since G' is connected, there exists a tree S spanning the graph G'. Obviously, W is a vertex cover of S. We denote by T the tree obtained from S by deleting all the vertices u with the properties that u has degree one in S and $u \notin W$. Obviously, W is a vertex cover of T, and every vertex of degree one in T belongs to W. It is clear that no pair of vertices in $V(G) - V(T)$ is adjacent in G.

For every $v \in V(T)$, we denote by $\mathscr{B}(v)$ the set of branches at v (of the tree T). It is obvious that $|V(B - w) \cap W| \geqq 1$ for every $w \in V(T)$ and every $B \in \mathscr{B}(w)$. We distinguish the following cases:

1. Assume that there exists $t \in V(T)$ such that $|V(B - t) \cap W| = 2$ for at least one $B \in \mathscr{B}(t)$. Let w_1 and w_2 be the elements of $V(B - t) \cap W$. Then $1 \leqq d_G(w_1, w_2) \leqq d_T(w_1, w_2) \leqq 2$. It is clear that $T - w_1 - w_2$ is $(W - \{w_1, w_2\})$-connected. Therefore, $G - w_1 - w_2$ is also $(W - \{w_1, w_2\})$-connected.

2. Assume that $|V(B - t) \cap W| \neq 2$ for every $t \in V(T)$ and every $B \in \mathscr{B}(t)$. It is not difficult to see that there exists $u \in V(T)$ such that u has degree at least three in T and there exists $B_0 \in \mathscr{B}(u)$ such that $|V(B' - u) \cap W| = 1$ for every $B' \in \mathscr{B}(u) - \{B_0\}$. For every $B \in \mathscr{B}(u)$, we denote by $v(B)$ the vertex of B adjacent to u in T. Denote $\mathscr{B}_0 = \mathscr{B}(u) - \{B_0\}$. Moreover, for every $B' \in \mathscr{B}_0$, we denote by $w(B')$ the vertex of B' which belongs to W.

2.1. Assume that for every $B' \in \mathscr{B}_0$, the vertices u and $w(B')$ are adjacent in T. Consider distinct branches $A_1, A_2 \in \mathscr{B}_0$. Then $d_G(w(A_1), w(A_2)) \leqq d_T(w(A_1), w(A_2)) = 2$. Since $T - w(A_1) - w(A_2)$ is $(W - \{w(A_1), w(A_2)\})$-connected, we conclude that also $G - w(A_1) - w(A_2)$ is.

2.2. Assume that there exists $B' \in \mathscr{B}_0$ such that u and $w(B')$ are not adjacent in T. Since W is a vertex cover of T, we have $u \in W$.

2.2.1. Assume that there exist distinct $B_1, B_2 \in \mathscr{B}_0$ such that $v(B_1)$ and $v(B_2)$ are adjacent in G. Since W is a vertex cover of G, we may assume without loss of generality that $v(B_1) \in W$. Hence $w(B_1) = v(B_1)$. This implies $d_G(w(B_1), w(B_2)) \leqq 2$. It is clear that $G - w(B_1) - w(B_2)$ is $(W - \{w(B_1), w(B_2)\})$-connected.

2.2.2. Assume that for no pair of distinct $B^*, B^{**} \in \mathscr{B}_0$, the vertices $v(B^*)$ and $v(B^{**})$ are adjacent in G. Since W is a Y-cover of G, we have $|\mathscr{B}_0| \leqq 2$. Since the degree of u in T is at least three, we have $|\mathscr{B}_0| = 2$. Let D_1 and D_2 be the elements of $\mathscr{B}_0$. Since W is a Y-cover of G, we may assume without loss of generality that $v(B_0)$ and $v(D_1)$ are adjacent in G. Clearly, $d_G(u, w(D_2)) \leqq d_T(u, w(D_2)) \leqq 2$. It is easy to see that $G - u - w(D_2)$ is $(W - \{u, w(D_2)\})$-connected.

Thus the proof of the lemma is complete.

Let G be a graph. We say that G is a square if there exists a graph G_1 such that G is isomorphic to $(G_1)^2$. We say that G is a line graph if there exists a graph G_2 with $E(G_2) \neq \emptyset$ such that G is isomorphic to $L(G)$. Finally, we shall say that G is a partial square if there exists a graph G' and a Y-cover W' of G' such that G is isomorphic to $\mathrm{psq}(G', W')$.

It is clear that the class of partial squares includes both the class of squares and the class of line graphs. The graph in Fig. 2 is an example of a partial square which is neither a square nor a line graph.

The following theorem is the main result of the present note:

Theorem. *Every connected partial square of even order has a 1-factor.*

Proof. Let G be a connected partial square of even order. Then there exist a graph G' and a Y-cover W' of G' such that G is isomorphic to $\mathrm{psq}(G', W')$, G' is W'-connected, and $|W'|$ is even. We shall prove that $\mathrm{psq}(G', W')$ has a 1-factor.

The case when $|W'| = 2$ is obvious. Let $|W'| = n \geqq 4$; assume that the assertion "$\mathrm{psq}(G'', W'')$ has a 1-factor" has been proved for every pair G'', W'' where W''' is a Y-cover of a W''-connected graph G'' and $|W''| = n - 2$. The lemma implies that there exist $w_1, w_2 \in W'$ such that $1 \leqq d_{G'}(w_1, w_2) \leqq 2$ and that $G' - w_1 - w_2$ is $(W' - \{w_1, w_2\})$-connected. Since $W' - \{w_1, w_2\}$ is a Y-cover of $G' - w_1 - w_2$, it follows from the induction hypothesis that

$$\mathrm{psq}(G' - w_1 - w_2, W' - \{w_1, w_2\})$$

has a 1-factor, say F. It is obvious that the graph obtained from F by adding the vertices w_1 and w_2 and the edge w_1w_2 is a 1-factor of $\mathrm{psq}(G', W')$, which completes the proof.

Corollary. (Chartrand, Polimeni, and Stewart [2]; Sumner [5]). *Let G be a connected graph of even order. If G is either a square or a line graph, then it has 1-factor.*

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