J. Puhl

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THE TRACE OF FINITE AND NUCLEAR ELEMENTS IN BANACH ALGEBRAS

J. PUHL, Jena

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1. INTRODUCTION

M. FREUNDLICH [4] has introduced and studied finite and compact elements in a commutative normed algebra. Different definitions were given by K. VALA [14]. He called an element u of a normed algebra *finite* (*compact*) if the linear operator $x \to uxu$ is finite-dimensional (compact). In [15] it is shown that the set of all finite (compact) elements in the Banach algebra of bounded operators on a Banach space coincides with the class of all finite-dimensional (compact) operators.

The purpose of this paper is to introduce the trace of finite elements of a Banach algebra. This concept includes the notion of the trace of operators. Throughout this paper M denotes a complex, semi-prime Banach algebra.

An element $u \in \mathfrak{M}$ will be called *one-dimensional* if there exists a linear functional τ_u on $\mathfrak{M}$, such that

$$uxu = \langle \tau_u, x \rangle \, u \quad \text{for all} \quad x \in \mathfrak{M} .$$

Then there exists a unique complex number $\operatorname{tr}(u)$ such that $u^2 = \operatorname{tr}(u)\, u$, which will be called the *trace* of u. A finite element u in the sense of K. Vala has a representation $u = \sum_1^n u_i$, where u_i is one-dimensional. Moreover, by setting

$$\operatorname{tr}(u) = \sum_1^n \operatorname{tr}(u_i)$$

we get a well-defined trace of u. The trace properties are known to be valid. In a natural way the definition of nuclear elements is given. It will be shown if the algebra fulfills certain conditions then the trace admits an extension to the nuclear elements. Such a condition is the following. We say the Banach algebra $\mathfrak{M}$ possesses the quasi-approximation property (q.a.p.) if for each minimal idempotent $q \in \mathfrak{M}$ the Banach space $\mathfrak{M}q$ (resp. $q\mathfrak{M}$) has the approximation property. Commutative

Banach algebras and C^*-algebras have q.a.p.. We also prove that the trace of a nilpotent nuclear element is zero. Section 6 is devoted the study of trace formulas. The methods presented here admit the use of the well-known results concerning trace formulas of certain operator classes to obtain similar results in the general case of Banach algebras. Particularly we show the effectiveness of the trace formula for finite elements and the Lidskij trace formula for nuclear elements in C^*-algebras. Finally we obtain for C^*-algebras that the nuclear elements coincides with the trace class of $\mathfrak{M}$.

2. ONE-DIMENSIONAL ELEMENTS

Throughout this paper we suppose that $\mathfrak{M}$ is a complex, semi-prime Banach algebra. The following condition is equivalent for $\mathfrak{M}$ to be semi-prime.

2.1. If $uxu = 0$ for all $x \in \mathfrak{M}$ then $u = 0$.

In the next definition we generalize the concept of a one-dimensional operator to algebras.

Definition 2.2. A non-zero element $u \in \mathfrak{M}$ is called *one-dimensional*, if there exists a linear functional τ_u on $\mathfrak{M}$ such that

$$uxu = \langle \tau_u, x \rangle \, u \quad \text{for all} \quad x \in \mathfrak{M} .$$

The *trace* $\mathrm{tr}(u)$ of u is defined by

$$u^2 = \mathrm{tr}(u) \, u .$$

For any $x_0 \in \mathfrak{M}$ with $ux_0u \neq 0$ we obtain from

$$\langle \tau_u, x_0 u \rangle \, u = ux_0uu = \mathrm{tr}(u) \, ux_0u = \mathrm{tr}(u) \, \langle \tau_u, x_0 \rangle \, u$$

that

$$\mathrm{tr}(u) = \frac{\langle \tau_u, x_0 u \rangle}{\langle \tau_u, x_0 \rangle} .$$

If $\mathfrak{M}$ has an identity then $\mathrm{tr}(u) = \langle \tau_u, 1 \rangle$. We denote the set of all one-dimensional elements of $\mathfrak{M}$ by $\mathfrak{F}_1$.

Remark 2.3. Let $\mathfrak{M}$ be a commutative algebra. Then an element $u \in \mathfrak{M}$ is one-dimensional if and only if there exists a linear functional σ_u on $\mathfrak{M}$ such that

$$ux = \langle \sigma_u, x \rangle \, u \quad \text{for all} \quad x \in \mathfrak{M} .$$

Because of

$$\langle \tau_u, x \rangle \, u = uxu = u^2 x = \mathrm{tr}(u) \, ux = \mathrm{tr}(u) \, \langle \sigma_u, x \rangle \, u$$

we have the identity

$$\langle \tau_u, x \rangle = \mathrm{tr}(u) \, \langle \sigma_u, x \rangle \quad \text{for all} \quad x \in \mathfrak{M} .$$

Remark 2.4. Minimal idempotents of $\mathfrak{M}$ are one-dimensional elements ([3], p. 157).

Remark 2.5. The notion of a one-dimensional element is related to that of a minimal ideal. RICKART [11] has observed that every minimal left (right) ideal $\mathfrak{J}$ contains a minimal idempotent p such that $\mathfrak{J} = \mathfrak{M}p$. Conversely, let u be a one-dimensional element and put $\mathfrak{J} := \mathfrak{M}u$. Then $\mathfrak{J}$ is a minimal left ideal. In order to prove this statement let us suppose $\mathfrak{I}$ to be a left ideal contained in $\mathfrak{J}$. Each non-zero element $z \in \mathfrak{I}$ has the form $z = z_0 u$. Choose $y_0 \in \mathfrak{M}$ such that $zy_0 z \neq 0$. Since $\langle \tau_u, y_0 z_0 \rangle z_0 u = zy_0 z \neq 0$ we obtain $\langle \tau_u, y_0 z_0 \rangle \neq 0$. For an arbitrary element $x \in \mathfrak{M}$ it follows

$$xu = \frac{1}{\langle \tau_u, y_0 z_0 \rangle} xuy_0 z_0 \in \mathfrak{I} .$$

This yields $\mathfrak{M}u = \mathfrak{I}$.

We have a first simple proposition.

Proposition 2.6. *Let $\mathfrak{M} = \mathfrak{L}(E)$ be the algebra of all bounded linear operators on a Banach space E. Then the one-dimensional elements of $\mathfrak{M}$ are exactly the one-dimensional operators of $\mathfrak{L}(E)$. Moreover* $\operatorname{tr}(a \otimes x) = \langle a, x \rangle$.

We now give further examples of one-dimensional elements.

1. Let G be a compact abelian group. The one-dimensional elements of $L_1(G)$ are of the form $\alpha\chi$, where α is a complex number and χ is a character of G.

2. Let K be a complete regular Hausdorff space. By $C_b(K)$ we denote the Banach algebra of all complex valued bounded continuous functions on K with the supremum norm. Then the one-dimensional elements are of the form

$$\delta_{t_0}(t) = \begin{cases} \alpha & \text{for} \quad t = t_0 \\ 0 & \text{otherwise} \end{cases}$$

where $\alpha \in C$ is fixed and $t_0 \in K$ is an isolated point of K.

Lemma 2.7. *Let $u \in \mathfrak{F}_1$ and $x, y \in \mathfrak{M}$ such that $xuy \neq 0$. Then $xuy \in \mathfrak{F}_1$ and* $\langle \tau_{xuy}, t \rangle = \langle \tau_u, ytx \rangle$.

Proof. Let $xuy \neq 0$. Then

$$\langle \tau_{xuy}, t \rangle xuy = (xuy)\, t(xuy) = \langle \tau_u, ytx \rangle xuy .$$

Thus the assertion is proved.

The spectrum of an element $u \in \mathfrak{M}$ is denoted by $\operatorname{Sp}_{\mathfrak{M}}(u)$. When $u, v \in \mathfrak{M}$ we put $u \circ v := uv - u - v$.

Lemma 2.8. *Let $u \in \mathfrak{F}_1$. Then* $\operatorname{Sp}_{\mathfrak{M}}(u) = \{0, \operatorname{tr}(u)\}$.

Proof. Suppose $\lambda \notin \mathrm{Sp}_{\mathfrak{M}}(u)$, then there exists a quasi-inverse $y \in \mathfrak{M}$ and $0 = u((1/\lambda)\, u \circ y) = ((\mathrm{tr}(u)/\lambda) - 1)\, uy - (\mathrm{tr}(u)/\lambda)\, u$ implies $\lambda \neq \mathrm{tr}(u)$. On the other hand if $\lambda \neq \mathrm{tr}(u)$, $\lambda \neq 0$, then $(1/(\mathrm{tr}(u) - \lambda))\, u$ is a quasi-inverse for $(1/\lambda)\, u$.

3. FINITE ELEMENTS

By $\mathfrak{F}$ we denote the set of all elements $u \in \mathfrak{M}$ of the form

$$u = \sum_{i=1}^{n} u_i \quad \text{with} \quad u_i \in \mathfrak{F}_1 .$$

By convention, $0 \in \mathfrak{F}$. The elements of $\mathfrak{F}$ are called *finite*.

Clearly, $\mathfrak{F}$ is a two-sided ideal and by Remark 2.5 $\mathfrak{F} = \mathrm{soc}\,(\mathfrak{M})$, where the socle of $\mathfrak{M}$ is denoted by $\mathrm{soc}\,(\mathfrak{M})$. Denote by Y_u the centralizer of $u \in \mathfrak{M}$, i.e.

$$Y_u := \{x : ux = xu\} .$$

Then Y_u is a closed subalgebra of $\mathfrak{M}$. Put $D_u x := uxu$ for $x \in Y_u$. By Theorem 1.6.9 and 1.6.10 in [18] we have

$$\mathrm{Sp}_{L(Y_u)}(D_u) = (\mathrm{Sp}_{\mathfrak{M}}(u))^2 . \tag{3.1}$$

If $u \in \mathfrak{F}$, then by Lemma 4.2 D_u is a finite-dimensional operator and therefore, $\mathrm{Sp}_{\mathfrak{M}}(u)$ is finite.

Lemma 3.2. *If $u \in \mathfrak{M}$ is a given non-zero element such that*

$$\dim u\mathfrak{M}u < \infty ,$$

then there exists a minimal idempotent $p \in \mathfrak{M}u$ (resp. $p \in u\mathfrak{M}$).

Proof. Let $v \in u\mathfrak{M}u$ be a non-zero element such that $\dim\,(v\mathfrak{M}v)$ is as small as possible. Then for an arbitrary element $y \in \mathfrak{M}$, with $vyv \neq 0$, we have

$$(vyv)\,\mathfrak{M}(vyv) = v\mathfrak{M}v.$$

Consequently, there exists an element $z \in \mathfrak{M}$ such that

$$vyv = vyvzvyv .$$

It is

$$(vzvyv - v)\,\mathfrak{M}(vzvyv - v) \subseteq v\mathfrak{M}v .$$

Equality is impossible, since it would imply

$$0 = vy(vzvyv - v)\,\mathfrak{M}(vzvyv - v)\,yv = vyv\mathfrak{M}vyv = v\mathfrak{M}v .$$

Hence

$$(vzvyv - v)\,\mathfrak{M}(vzvyv - v) = 0\,.$$

By condition (2.1) it follows $v = vzvyv$.

Now we show that $\mathfrak{M}v$ is a minimal left ideal. Let $\mathfrak{J} \subset \mathfrak{M}v$ be a non-zero left ideal. Then there exist $y_0 \in \mathfrak{J}$, $x_0 \in \mathfrak{M}$ with $y_0 x_0 y_0 \neq 0$ and $y_0 = yv$. We can find an element $z \in \mathfrak{M}$ such that

$$v = vzvx_0yv\,.$$

Consequently,

$$\mathfrak{M}v \subseteq \mathfrak{M}vx_0yv \subseteq \mathfrak{J}\,.$$

It follows that $\mathfrak{M}v$ is a minimal left ideal. Thus, there is a minimal idempotent $p \in \mathfrak{M}v \subseteq \mathfrak{M}u$ and the assertion is proved.

Corollary 3.3. *Let u be a non-zero element of $\mathfrak{M}$. Then $u \in \mathfrak{F}_1$ if and only if* $\dim u\mathfrak{M}u = 1$.

Proof. Clearly, if $u \in \mathfrak{F}_1$ then $\dim u\mathfrak{M}u = 1$.

Conversely, let $\dim u\mathfrak{M}u = 1$. Then by the preceding Lemma there exists a minimal idempotent $p \in \mathfrak{M}u$. We have

$$(u - up)\,\mathfrak{M}(u - up) \subseteq u\mathfrak{M}u\,.$$

Equality would imply the false assertion $u\mathfrak{M}up = (0)$. Hence,

$$(u - up)\,\mathfrak{M}(u - up) = (0)$$

and by condition (2.1) we obtain $u = up \in \mathfrak{F}_1$.

Theorem 3.4. *Let $\mathfrak{M}$ be a complex, semi-prime Banach algebra and let $u \in \mathfrak{M}$ be a non-zero element such that*

$$\dim u\mathfrak{M}u < \infty\,.$$

Then there exists an idempotent $p \in \mathfrak{F} \cap u\mathfrak{M}$ and $pu = u$ (resp. $up = u$).

Proof. First we show that every subset of orthogonal idempotents of $u\mathfrak{M}$ is finite, Suppose $(p_n)_{n=1}^{\infty}$ to be an infinite set of non-zero orthogonal idempotents in $u\mathfrak{M}$. Then, $p_n = ux_n$ for some $x_n \in \mathfrak{M}$. Choose a sequence of distinct points $\lambda_n \in C$ such that $|\lambda_n| \leqq 2^{-n}\|x_n\|^{-1}$ for all natural numbers n. Then $x := \sum_n \lambda_n x_n \in \mathfrak{M}$ and $ux = \sum_n \lambda_n p_n$. Consequently, $\lambda_n \in \mathrm{Sp}(ux)$. This is impossible, since by (3.1) the spectrum has a finite number of points. By Lemma 3.2 there exists a non-empty set of orthogonal minimal idempotents of $u\mathfrak{M}$. Choose a maximal orthogonal set $(p_i)_{i=1}^{n}$ of minimal idempotents. Put $p := \sum_{i=1}^{n} p_i$. Suppose $pu - u \neq 0$, then

$$(pu - u)\,\mathfrak{M}(pu - u) \subseteq u\mathfrak{M}u\,.$$

By Lemma 3.2 there is a minimal idempotent $q \in (pu - u)\,\mathfrak{M}$. Clearly, $p_i q = 0$. Put $w := q - \sum_{i=1}^{n} qp_i$. Since $wq = q \neq 0$, it follows that $w \neq 0$. It can easily be checked that $p_i w = wp_i = 0$ for all i, $1 \leqq i \leqq n$, $w^2 = w$, and $w \in \mathfrak{F}_1 \cap u\mathfrak{M}$. This is a contradiction to the maximality of the system (p_i). Therefore, $pu = u$.

The following corollary was also obtained by J. C. Alexander [1].

Corollary 3.5. *Let u be a non-zero element of a complex, semi-prime Banach algebra $\mathfrak{M}$. Then the operator $x \to uxu$ has a finite rank if and only if $u \in \mathfrak{F}$.*

Proof. Let $u \in \mathfrak{F}$. By Lemma 4.2 it immediately follows that the operator $x \to uxu$ has a finite rank. Conversely, suppose that the operator $x \to uxu$ has a finite rank, then by the preceding theorem there is an idempotent $p \in \mathfrak{F}$ such that $u = pu \in \mathfrak{F}$.

4. THE TRACE OF FINITE ELEMENTS

In this section the notion of a trace of finite elements is introduced. The results we get are analogous to well-known results of the classical operator theory.

Two elements $u, v \in \mathfrak{F}_1$ are called *equivalent* $(u \sim v)$ if there is some $x_0 \in \mathfrak{M}$ such that $ux_0v \neq 0$.

Lemma 4.1. *The relation $\sim$ is an equivalence relation on $\mathfrak{F}_1$.*

Proof. We show the transitivity only. Let $u, v, w \in \mathfrak{F}_1$, $u \sim v$, and $v \sim w$. There exist x_0 and x_1 such that $ux_0v \neq 0$ and $vx_1w \neq 0$. By (2.1) there is $y_0 \in \mathfrak{M}$ such that

$$0 \neq (ux_0v)\,y_0(ux_0v) = \langle \tau_v, y_0ux_0 \rangle\, ux_0v\,.$$

Consequently,

$$0 \neq \langle \tau_v, y_0ux_0 \rangle\, vx_1w = vy_0ux_0vx_1w\,.$$

Thus, $ux_0vx_1w \neq 0$. This yields $u \sim w$.

Lemma 4.2. *Let $u, v \in \mathfrak{F}_1$ and $u \sim v$. Then the operator*

$$D_{u,v}x := uxv = \frac{\langle \tau_u, xvy_0 \rangle}{\langle \tau_v, y_0ux_0 \rangle}\, ux_0v$$

is one-dimensional, $\|D_{u,v}\| \leqq \|u\|\,\|v\|$, and

$$\operatorname{trace} D_{u,v} = \operatorname{tr}(u)\operatorname{tr}(v)\,.$$

Proof. Let $ux_0v \neq 0$ for some $x_0 \in \mathfrak{M}$. Then there is $y_0 \in \mathfrak{M}$ such that

$$0 \neq (ux_0v)\, y_0(ux_0v) = \langle \tau_v, y_0ux_0 \rangle\, ux_0v\,.$$

Consequently,

$$D_{u,v}x = \frac{1}{\langle \tau_v, y_0ux_0 \rangle}\, uxvy_0ux_0v = \frac{\langle \tau_u, xvy_0 \rangle}{\langle \tau_v, y_0ux_0 \rangle}\, ux_0v\,.$$

Therefore, $D_{u,v}$ is one-dimensional. It follows from

$$\frac{\langle \tau_u, ux_0vvy_0 \rangle}{\langle \tau_v, y_0ux_0 \rangle}\, ux_0v = uux_0vv = \mathrm{tr}(u)\,\mathrm{tr}(v)\, ux_0v$$

that

$$\mathrm{trace}\; D_{u,v} = \mathrm{tr}(u)\,\mathrm{tr}(v)\,.$$

Lemma 4.3. *Let $u_i \in \mathfrak{F}_1$ such that $\sum_{i=1}^{n} u_i = 0$. Then*

$$\sum_{i=1}^{n} \mathrm{tr}(u_i) = 0\,.$$

Proof. The equivalence relation $\sim$ induces a disjoint decomposition A_k of $\{1, \ldots, n\}$. For fixed k we get

$$0 = \Big(\sum_{i=1}^{n} u_i\Big)\, x\Big(\sum_{j\in A_k} u_j\Big) = \sum_{i,j\in A_k} u_ixu_j \quad \text{for all} \quad x \in \mathfrak{M}\,.$$

By Lemma 4.2 it follows that

$$0 = \sum_{i,j\in A_k} \mathrm{tr}(u_i)\,\mathrm{tr}(u_j) = \Big(\sum_{i\in A_k} \mathrm{tr}(u_i)\Big)^2\,.$$

Therefore,

$$\sum_{i=1}^{n} \mathrm{tr}(u_i) = \sum_{k} \sum_{i\in A_k} \mathrm{tr}(u_i) = 0\,.$$

Because of the preceding lemma we can introduce a well-defined trace for finite elements.

Definition 4.4. Let $u = \sum_{i=1}^{n} u_i$, $u_i \in \mathfrak{F}_1$, be any representation of $u \in \mathfrak{F}$. Then

$$\mathrm{tr}(u) := \sum_{i=1}^{n} \mathrm{tr}(u_i)$$

is called the *trace* of u.

Theorem 4.5. *Suppose that $\mathfrak{M}$ is a semi-prime Banach algebra. Then the trace has following properties:*

(i) *The trace is a linear functional on $\mathfrak{F}$.*

(ii) *If $u \in \mathfrak{F}$, and $x \in \mathfrak{M}$, then* $\operatorname{tr}(ux) = \operatorname{tr}(xu)$.

(iii) *If* $u \in \mathfrak{F}$ is nilpotent, then $\operatorname{tr}(u) = 0$.

(iv) *Let $\mathfrak{M}$ be an algebra with involution and let $u \in \mathfrak{F}$. Then*

$$\operatorname{tr}(u^*) = \overline{\operatorname{tr}(u)}\,.$$

(v) *Let $\mathfrak{M}$ be a C^*-algebra. Then τ_{vv^*} is a positive functional on $\mathfrak{M}$ for all $v \in \mathfrak{F}_1$.*

Proof. (i) is obvious.

(ii) For $v \in \mathfrak{F}_1$ and $x \in \mathfrak{M}$ we have

$$\langle \tau_v, x \rangle\, xv = xvxv = \operatorname{tr}(xv)\, xv$$

and

$$\langle \tau_v, x \rangle\, vx = vxvx = \operatorname{tr}(vx)\, vx\,.$$

Thus, $\operatorname{tr}(xv) = \operatorname{tr}(vx)$. Now it follows that

$$\operatorname{tr}(xu) = \sum_1^n \operatorname{tr}(xu_i) = \sum_1^n \operatorname{tr}(u_i x) = \operatorname{tr}(ux)\,.$$

(iii) Let A_k have the same meaning as in the proof of Lemma 4.3. Put

$$D_k x := ux\Big(\sum_{i \in A_k} u_i\Big) = \sum_{i,j \in A_k} u_i x u_j\,.$$

Since $u \in \mathfrak{F}$ is nilpotent the finite-dimensional operator D_k is nilpotent, therefore trace $D_k = 0$. On the other hand by using Lemma 4.2 we get

$$\operatorname{trace} D_k = \sum_{i,j \in A_k} \operatorname{tr}(u_i) \operatorname{tr}(u_j) = \Big(\sum_{i \in A_k} \operatorname{tr}(u_i)\Big)^2\,.$$

Consequently,

$$\operatorname{tr}(u) = \sum_k \sum_{i \in A_k} \operatorname{tr}(u_i) = 0\,..$$

(iv) It can easily be checked that $v \in \mathfrak{F}_1$ implies $v^* \in \mathfrak{F}_1$ and $\operatorname{tr}(v^*) = \overline{\operatorname{tr} v}$. Therefore,

$$\operatorname{tr}(u^*) = \sum_i \operatorname{tr}(u_i^*) = \sum_i \overline{\operatorname{tr}(u_i)} = \overline{\operatorname{tr}(u)}$$

for $u \in \mathfrak{F}$.

(v) The assumption is an easy consequence of $\langle \tau_{vv^*}, x^*x \rangle = \operatorname{tr}(vv^*x^*x) = \operatorname{tr}(xv(xv)^*)$ and $\operatorname{tr}(xv(xv)^*) \in \operatorname{Sp}(xv(xv)^*)$ by Lemma 2.8.

Let $\mathfrak{M} = \mathfrak{L}(E)$ be the algebra of all bounded linear operators on the Banach space E. An operator $U \in \mathfrak{L}(E)$ is called *nuclear* if $U = \sum_{i=1}^{\infty} a_i \otimes y_i$, where $a_i \in E'$, $y_i \in E$, and $\sum_{i=1}^{\infty} \|a_i\| \, \|y_i\| < \infty$. The two-sided ideal of all nuclear operators is denoted by $\mathfrak{N}(E)$. Note that $a_i \otimes y_i$ are one-dimensional elements. As you know a nuclear operator $U \in \mathfrak{N}(E)$ has a well-defined trace, when the underlying Banach space E or its dual E' possesses the approximation property. For more detailed informations the reader is referred to [8].

The definition of nuclear operators gave rise to the following

Definition 5.1. An element $u \in \mathfrak{M}$ is called *nuclear* if $u = \sum_{i=1}^{\infty} u_i$, where $u_i \in \mathfrak{F}_1$ and $\sum_{i=1}^{\infty} \|u_i\| < \infty$. We put $\nu(u) = \inf \sum_{i=1}^{\infty} \|u_i\|$, where the infimum is taken over all representations. The class of all nuclear elements is denoted by $\mathfrak{N}$.

Theorem 5.2. *$\mathfrak{N}$ is a two-sided ideal of $\mathfrak{M}$ with $\mathfrak{F} \subset \mathfrak{N}$ and ν is norm on $\mathfrak{N}$ such that*

$$\nu(xuy) \leqq \|x\| \, \nu(u) \, \|y\| \quad \textit{for all} \quad x, y \in \mathfrak{M}, \quad u \in \mathfrak{N}.$$

Moreover, $\mathfrak{N}$ is complete with respect to this norm.

Examples 5.3. (i) Let $\mathfrak{M}$ be the algebra $\mathfrak{L}(E)$. Then $u \in \mathfrak{M}$ is nuclear if and only if u is a nuclear operator.

(ii) Let $\mathfrak{M}$ be the algebra l_∞. Then $\mathfrak{N} = l_1$.

(iii) Let $\mathfrak{M}$ be the algebra of all absolutely summable complex valued functions with period 2π. Then

$$\mathfrak{N} = \{ \sum_{-\infty}^{+\infty} \alpha_k e^{ikt} : \sum_{-\infty}^{+\infty} |\alpha_k| < \infty \} \quad \text{(Wiener algebra)}.$$

Theorem 5.4. *If $u \in \mathfrak{N}$, then the operator $x \to uxu$ is nuclear.*

Proof. Choose a representation $u = \sum_{i=1}^{\infty} u_i$, $\sum_{i=1}^{\infty} \|u_i\| < \infty$, $u_i \in \mathfrak{F}_1$. Since

$$D_u x = uxu = \sum_{i,j=1}^{\infty} u_i x u_j = \sum_{i,j=1}^{\infty} \langle a_{ij}, x \rangle \, u_{ij} \quad \text{and} \quad \|a_{ij}\| \, \|u_{ij}\| \leqq \|u_i\| \, \|u_j\|$$

it follows that

$$\nu(D_u) \leqq \sum_{i,j=1}^{\infty} \|u_i\| \, \|u_j\| = (\sum_{i=1}^{\infty} \|u_i\|)^2 < \infty.$$

Consequently, D_u is a nuclear operator.

Remark 5.5. Even for C^*-algebras the converse is not true. Put $\mathfrak{M} = l_\infty$. Then $u = (1/n) \in l_\infty$ and $u \notin \mathfrak{N}$ (example (ii)) but the operator $x \to uxu = ((1/n^2)\,\xi_n)$, where $x = (\xi_n)$, is nuclear.

Remark 5.6. By (3.1) for every $u \in \mathfrak{N}$ the set of all eigenvalues is either finite or countable and has no non-zero point of accumulation.

Now we inverstigate certain conditions under which the trace admits an extension to the nuclear elements.

Theorem 5.7. *Let $\mathfrak{M}$ be a semi-prime Banach algebra having the approximation property. Then every $u \in \mathfrak{N}$ has a well-defined trace.*

Proof. Let $0 = \sum_{i=1}^{\infty} u_i$, $u_i \in \mathfrak{F}_1$ and $\sum_{i=1}^{\infty} \|u_i\| < \infty$. Let A_k have the same meaning as in the proof of Lemma 4.3. Put

$$D_k x := \sum_{i,j \in A_k} u_i x u_j = (\sum_{i=1}^{\infty} u_i)\, x (\sum_{j \in A_k} u_j) = 0 \, .$$

Because of $\nu(D_k) \leqq (\sum_{i \in A_k} \|u_i\|)^2$ and the approximation property it follows that

$$0 = \text{trace } D_k = (\sum_{i \in A_k} \operatorname{tr}(u_i))^2 \, .$$

Therefore,

$$\sum_{i=1}^{\infty} \operatorname{tr}(u_i) = 0 \, .$$

Let $\mathfrak{M}$ be a semi-prime Banach algebra possessing the following property:

Given $u \in \mathfrak{F}$ and $\varepsilon > 0$, then there exists $x \in \mathfrak{F}$, $\|x\| \leqq 1 + \varepsilon$ such that $xu = u$ or $ux = u$.

Remark 5.8. If $\mathfrak{M} = \mathfrak{L}(E)$, then this property is fulfilled if E or E' has the metric approximation property (see [8]).

Theorem 5.9. *Let $\mathfrak{M}$ be an algebra with the above property. Then every $u \in \mathfrak{N}$ has a well-defined trace.*

Proof. For given $v \in \mathfrak{F}$ and $\varepsilon > 0$, there is $x \in \mathfrak{F}$, $x = \sum_{i=1}^{n} x_i$, $x_i \in \mathfrak{F}_1$, $\|x\| \leqq 1 + \varepsilon$, such that $xu = u$. If $v = \sum_{i=1}^{\infty} v_i$ is a nuclear representation, with

$$\nu(v) + \varepsilon \geqq \sum_{i=1}^{\infty} \|v_i\| \, ,$$

then

$$|\mathrm{tr}(v)| = |\mathrm{tr}(xv)| = |\sum_{i=1}^{n} \langle \tau_{x_i}, v \rangle| =$$

$$= |\sum_{i=1}^{n} \sum_{j=1}^{\infty} \langle \tau_{x_i}, v_j \rangle| \leqq \sum_{j=1}^{\infty} |\mathrm{tr}(xv_j)| \leqq (\sum_{j=1}^{\infty} \|v_j\|) \|x\| \leqq (\nu(v) + \varepsilon)(1 + \varepsilon) .$$

Thus, $|\mathrm{tr}(v)| \leqq \nu(v)$ for $v \in \mathfrak{F}$.

Therefore, the functional $v \to \mathrm{tr}(v)$ admits a unique extension on $\mathfrak{N}$. Furthermore, if $u = \sum_i u_i$ is any nuclear representation, then

$$\mathrm{tr}(u) = \lim_n \mathrm{tr}(\sum_{i=1}^{n} u_i) = \sum_{i=1}^{\infty} \mathrm{tr}(u_i) .$$

The most important condition seems to be the following.

Definition 5.10. A semi-prime Banach algebra $\mathfrak{M}$ possesses the *quasi-approximation property* (q.a.p.) if for each minimal idempotent $q \in \mathfrak{M}$ the Banach space $\mathfrak{M}q$ has the approximation property (resp. $q\mathfrak{M}$).

Theorem 5.11. (i) *Let* $\mathfrak{M} = \mathfrak{L}(E)$. *Then* $\mathfrak{M}$ *possesses* q.a.p. *if and only if* E *or* E' *has the approximation property.*

(ii) *A commutative Banach algebra* $\mathfrak{M}$ *possesses* q.a.p..

(iii) *A* C^**-algebra* $\mathfrak{M}$ *possesses* q.a.p..

Proof. Suppose $q \in \mathfrak{M}$ to be a minimal idempotent.

(i) Clearly by $\mathfrak{M}q \cong E$ (resp. $q\mathfrak{M} \cong E'$).

(ii) $\mathfrak{M}q$ is isomorphic to the complex numbers by Remark 2.3.

(iii) By using Theorem 4.5 (v) it is easy to check that by

$$(x, y) := \langle \tau_{qq^*}, y^*x \rangle \quad \text{for} \quad x, y \in \mathfrak{M}q$$

is defined an inner product on $\mathfrak{M}q$.

Moreover, from

$$\|xqq^*\|^2 = \|qq^*x^*xqq^*\| = \langle \tau_{qq^*}, x^*x \rangle \, \|qq^*\|$$

and

$$\frac{1}{\|q\|} \|xqq^*\| \leqq \|xq\| = \frac{1}{|\langle \tau_q, q^* \rangle|} \|xqq^*q\| \leqq \frac{1}{\|q\|} \|xqq^*\|$$

it follows that

$$(x, x) = \|x\|^2 \quad \text{for all} \quad x \in \mathfrak{M}q .$$

Thus, $\mathfrak{M}q$ is a Hilbert space.

Next we prove the main result of this section.

Theorem 5.12. *Suppose* $\mathfrak{M}$ *to be with* q.a.p.

(i) *Then every* $u \in \mathfrak{N}$ *has a well-defined trace.*

(ii) *If* $u \in \mathfrak{N}$ *is nilpotent, then* $\operatorname{tr}(u) = 0$.

Proof. Without loss of generality we can assume $\mathfrak{M}q$ having the approximation property for each minimal idempotent $q \in \mathfrak{M}$.

(i) For given $u \in \mathfrak{N}$ we choose a nuclear representation

$$u = \sum_{i=1}^{\infty} u_i, \quad \text{with} \quad \sum_{i=1}^{\infty} \|u_i\| < \infty .$$

Hence, the series $\sum_i \operatorname{tr}(u_i)$ must be convergent. Furthermore, it is enough to show that from $\sum_{i=1}^{\infty} v_i = 0$, where $v_i \in \mathfrak{F}_1$ and $\sum_{i=1}^{\infty} \|v_i\| < \infty$ it follows that

$$\sum_{i=1}^{\infty} \operatorname{tr}(v_i) = 0 .$$

The equivalence relation $\sim$ induces a disjoint decomposition A_k of the natural numbers, such that $v_i \sim v_j$ for $i, j \in A_k$. Choose minimal idempotents q_k with $q_k \sim v_i$ for $i \in A_k$. Put $w_k := \sum_{i \in A_k} v_i$ and define an operator $L_k \in \mathfrak{L}(\mathfrak{M}q_k)$ by

$$L_k x := w_k x .$$

Note that $w_k \in \mathfrak{N}$. Applying Lemma 4.2 we obtain $L_k \in \mathfrak{N}(\mathfrak{M}q_k)$ and

$$\operatorname{trace} L_k = \sum_{i \in A_k} \operatorname{tr}(v_i) .$$

Since $0 = \left(\sum_{i=1}^{\infty} v_i\right) x w_k = w_k x w_k$ we have $w_k = 0$. This yields

$$\sum_{i \in A_k} \operatorname{tr}(v_i) = 0$$

and therefore we have

$$0 = \sum_k \operatorname{tr}(w_k) = \sum_{i=1}^{\infty} \operatorname{tr}(v_i) .$$

(ii) If $u \in \mathfrak{N}$ is nilpotent, then $L_k \in \mathfrak{N}(\mathfrak{M}q_k)$, too. Consequently,

$$0 = \operatorname{trace} L_k \quad \text{and} \quad \operatorname{tr}(u) = 0 .$$

In order to prove the trace formula for finite elements in Banach algebras and to prove the Lidskij trace formula for nuclear elements in C^*-algebras we need some preparations.

Let $\mathfrak{A}$ be a left (right) ideal of $\mathfrak{M}$ contained in $\mathfrak{F}$. Any maximal orthogonal set of minimal idempotents in $\mathfrak{A}$ has the same cardinality denoted by $\theta(\mathfrak{A})$ and is called *the order of* $\mathfrak{A}$. Some basic results concerning ideals of finite order are proved in [2].

For any bounded operator T on $\mathfrak{M}$ the null space $N(T)$ is defined by

$$N(T) = \{y \in \mathfrak{M} : Ty = 0\} .$$

The smallest nonnegative integer n such that

$$N(T^{n+1}) = N(T^n)$$

or $+\infty$ if no such n exists, is called *ascent* of T and denoted by $\alpha(T)$. We do not assume that $\mathfrak{M}$ has an identity 1. If $\mathfrak{M}$ does not have an identity, then 1 is symbolic, but make sense when multiplied by an element of $\mathfrak{M}$. We denote $\lambda 1$ simply by λ. The left (right) multiplication operator on $\mathfrak{M}$ determined by $\lambda - u$ is the operator which takes $x \in \mathfrak{M}$ into $\lambda x - ux \in \mathfrak{M}$ (resp. $\lambda x - xu \in \mathfrak{M}$) and is denoted by $L_{\lambda-u}$ (resp. $R_{\lambda-u}$).

Lemma 6.1. (BARNES [2], p. 499). *Let $u \in \mathfrak{N}$ and $\lambda \in \mathrm{Sp}(u)$. Then $\alpha := \alpha(L_{\lambda-u}) = \alpha(R_{\lambda-u}) < \infty$ and $\theta(N(L^\alpha_{\lambda-u})) < \infty$.*

A proof for semi-prime algebras can be found in [10].

Lemma 6.2. *Let $\mathfrak{M} = \mathfrak{L}(E)$ be the algebra of all bounded linear operators on a Banach space E. Let be $S \in \mathfrak{M}$ and $\lambda \in C$ be an eigenvalue of S with finite algebraic multiplicity. Then $\theta(N(L^\alpha_{\lambda-S}))$ is equal to the algebraic multiplicity of the eigenvalue λ of the operator S.*

Proof. Suppose n to be the algebraic multiplicity of the eigenvalue λ, i.e.

$$n = \dim\{x \in E : (\lambda I - S)^\beta x = 0\}$$

for a certain natural number β.

Assume $x_1, \ldots, x_n$ to be any basis of the subspace $\{x \in E : (\lambda I - S)^\beta x = 0\}$. There exist functionals $a_1, \ldots, a_n \in E$ such that $\langle x_i, a_k\rangle = \delta_{ik}$. Put $P_i := a_i \otimes x_i$. Then $P_i \in N(L^\beta_{\lambda-S})$ are mutually orthogonal minimal idempotents of $\mathfrak{L}(E)$. Therefore,

$$n \leqq \theta(N(L^\beta_{\lambda-S})) \leqq \theta(N(L^\alpha_{\lambda-S})) .$$

On the other hand, if $Q_j \in N(L^\alpha_{\lambda-S})$ are a maximal system of mutually orthogonal minimal idempotents, then $Q_j = b_j \otimes y_j$, where $b_j \in E'$, $y_j \in E$, and $(\lambda I - S)\, y_j = 0$.

Since the elements y_j are linear independent it follows that

$$\theta(N(L^{\alpha}_{\lambda-s})) \leqq n .$$

The preceding lemma gives rize that $n(\lambda, u) := \theta(N(L^{\alpha}_{\lambda-u})$ is called the *algebraic multiplicity* of the eigenvalue $\lambda \in \mathrm{Sp}_{\mathfrak{M}}(u)$.

Lemma 6.3. *Let $u \in \mathfrak{N}$ and $\lambda \in \mathrm{Sp}(u)$, $\lambda \neq 0$. Then there exists an idempotent $p(\lambda) \in N(L^{\alpha}_{\lambda-u})$, $p(\lambda) = \sum_{i=1}^{n(\lambda,u)} p_i(\lambda)$, $p_i(\lambda)$ being mutually orthogonal minimal idempotents such that*

$$p(\lambda)\, x = x \quad \textit{for all} \quad x \in N(L^{\alpha}_{\lambda-u}) .$$

The proof can be given in the same way as for Theorem 3.4 (also see [10], Lemma 4.4).

Throughout this section let A_k, L_k, w_k and q_k have the same meaning as in the proof of Theorem 5.12. Recall that $n(\lambda, L_k)$ imply the algebraic multiplicity of the eigenvalue λ of the operator L_k. If λ does not belong to the spectrum of L_k, then we put $n(\lambda, L_k) = 0$.

Lemma 6.4. *Suppose $u \in \mathfrak{N}$ and $\lambda \in \mathrm{Sp}(u)$, $\lambda \neq 0$. Then*

$$n(\lambda, u) = \sum_k n(\lambda, L_k) .$$

Proof. Let $u \in \mathfrak{N}$ and choose any representation

$$u = \sum_{i=1}^{\infty} u_i \quad \text{with} \quad \sum_{i=1}^{\infty} \|u_i\| < \infty .$$

By Lemma 6.3 there is an idempotent $p(\lambda) \in N(L^{\alpha}_{\lambda-u})$,

$$p(\lambda) = \sum_{i=1}^{n(\lambda,u)} p_i(\lambda) , \quad \text{and} \quad p(\lambda)\, x = x \quad \text{for all} \quad x \in N(L^{\alpha}_{\lambda-u}) .$$

For fixed k we put $B_k := \{i : p_i \sim q_k\}$. Note that may be $B_k = \emptyset$. We obtain

$$p(\lambda) = \sum_k \sum_{i \in B_k} p_i(\lambda)$$

and

$$0 = (\lambda - u)^{\alpha}\, p_i(\lambda) = (\lambda - w_k)^{\alpha}\, p_i(\lambda) \quad \text{for} \quad i \in B_k .$$

There exist $x_i \in \mathfrak{M}$ such that $p_i(\lambda)\, x_i q_k \neq 0$. Since $p_i(\lambda)\, x_i q_k \in \mathfrak{M} q_k$ are linear independent it follows that

$$n(\lambda, L_k) \geqq \mathrm{card}\,(B_k) .$$

On the other hand, if for any natural number β there exists $z \in \mathfrak{M} q_k$, $z = x q_k$, such that

$$0 = (\lambda I - L_k)^\beta z = (\lambda - w_k)^\beta z = (\lambda - u)^\beta z ,$$

then by applying Lemma 4.2 it follows that

$$z = p(\lambda) z = \sum_{i \in B_k} p_i(\lambda) x q_k = \sum_{i \in B_k} \frac{\langle \tau_{p_i(\lambda)}, x q_k y_0 \rangle}{\langle \tau_{q_k}, y_0 \, p_i(\lambda) \, x_i \rangle} p_i(\lambda) \, x_i q_k .$$

Therefore, we have

$$n(\lambda, L_k) \leqq \text{card}\,(B_k)$$

and we obtain

$$n(\lambda, u) = \sum_k \text{card}\,(B_k) = \sum_k n(\lambda, L_k) .$$

Theorem 6.5. (Trace formula for finite elements.) *Let* $u \in \mathfrak{F}$. *Then*

$$\text{tr}(u) = \sum_i \lambda_i \, n(\lambda_i, u) .$$

Proof. Since $L_k \in \mathfrak{L}(\mathfrak{M} q_k)$ is a finite-dimensional operator we have

$$\text{trace}\, L_k = \sum_i \lambda_i \, n(\lambda_i, L_k) .$$

On the other hand Lemma 4.2 yields

$$\text{trace}\, L_k = \text{tr}(w_k) .$$

Using Lemma 6.4 we obtain

$$\text{tr}(u) = \sum_k \text{tr}(w_k) = \sum_k \sum_i \lambda_i \, n(\lambda_i, L_k) = \sum_i \lambda_i \, n(\lambda_i, u) .$$

Theorem 6.6 (Lidskij trace formula). *Suppose* $\mathfrak{M}$ *to be a C*-algebra and let* $u \in \mathfrak{N}$. *Then*

$$\text{tr}(u) = \sum_i \lambda_i \, n(\lambda_i, u) .$$

Proof. For fixed k $\mathfrak{M} q_k$ is a Hilbert space (Theorem 5.11 (iii)) and $L_k \in \mathfrak{N}(\mathfrak{M} q_k)$. Lidskij's trace formula yields

$$\text{trace}\, L_k = \sum_i \lambda_i \, n(\lambda_i, L_k) .$$

Since trace $L_k = \sum_{i \in A_k} \text{tr}(u_i)$ exists it follows from Lemma 4.2 that there also exists $\text{tr}(w_k)$ and trace $L_k = \text{tr}(w_k)$. Moreover, the representation $u = \sum_k w_k$ is unique. This is an easy consequence of the fact that $u x w_k = w_k x w_k$ for $x \in \mathfrak{M}$. Using Lemma

6.4 and that $\sum_k |\mathrm{tr}(w_k)| \leqq \sum_i \|u_i\| < \infty$ we obtain

$$\mathrm{tr}(u) = \sum_k \mathrm{tr}(w_k) = \sum_k \sum_i \lambda_i\, n(\lambda_i, L_k) = \sum_i \lambda_i\, n(\lambda_i, u) .$$

Finally we give a further trace formula for certain nuclear elements.

Recall that for every operator $S \in \mathfrak{L}(E)$ the approximation numbers are defined by

$$a_n(S) = \inf \{\|S - A\| : A \in \mathfrak{L}(E),\ \mathrm{rank}\,(A) < n\} .$$

The $\mathfrak{S}_p$-classes are given by

$$\mathfrak{S}_p(E) := \{S \in \mathfrak{L}(E) : \sum_n a_n(S)^p < \infty\} \quad \text{for} \quad 0 < p < \infty$$

and

$$\mathfrak{S}_\infty(E) := \{S \in \mathfrak{L}(E) : \lim a_n(S) = 0\} .$$

By the definition

$$\sigma_p(S := \{\sum_n a_n(S)^p\}^{1/p}$$

we obtain a quasinorm on $\mathfrak{S}_p$.

Note that $\mathfrak{S}_1(E) \subseteq \mathfrak{N}(E)$ (see [9]).

Recently H. König [7] has proved that for the operators of $\mathfrak{S}_1(E)$ the trace formula is valid. We can give here a partially generalization to Banach algebras.

Definition 6.7. For every element $u \in \mathfrak{M}$ the n-th *approximation number* is defined by

$$a_n(u) = \inf \{\|u - \sum_{i=1}^{k} u_i\| : u_i \in \mathfrak{F}_1,\ k < n\} .$$

It is easy to check following properties:

(i) $\|u\| = a_1(u) \geqq a_2(u) \geqq \ldots \geqq 0$ for $u \in \mathfrak{M}$.

(ii) $a_n(u + v) \leqq a_n(u) + \|v\|$ for $u, v \in \mathfrak{M}$.

(iii) $a_n(tuv) \leqq \|t\|\, a_n(u)\, \|v\|$ for $t, u, v \in \mathfrak{M}$.

Let $\mathfrak{S}_p := \{u \in M : \sum_n a_n(u)^p < \infty\}$ and

$$\sigma_p(u) := \{\sum_n a_n(u)^p\}^{1/p} .$$

Then σ_p is a quasinorm on $\mathfrak{S}_p$.

Theorem 6.8. *Let* $u \in \mathfrak{S}_1 \cap \mathfrak{N}$. *Then*

$$\mathrm{tr}(u) = \sum_i \lambda_i\, n(\lambda_i, u) .$$

Proof. For given $\varepsilon > 0$ and n there exist $u_i \in F$ such that

$$\|u - \sum_i u_i\| \leqq a_n(u) + \varepsilon .$$

With any fixed k it follows from

$$\|(w_k - \sum_i u_i)\, x q_k\| = \|(u - \sum_i u_i)\, x q_k\| \leqq (a_n(u) + \varepsilon)\, \|x q_k\|$$

that

$$a_n(L_k) \leqq a_n(u) + \varepsilon .$$

Therefore for all n and k we obtain

$$a_n(L_k) \leqq a_n(u) .$$

$u \in \mathfrak{S}_1$ yields $L_k \in \mathfrak{S}_1(\mathfrak{M} q_k)$ for all natural numbers k. Applying the result of H. König [7] we have

$$\text{trace } L_k = \sum_i \lambda_i\, n(\lambda_i, L_k) .$$

Since trace L_k exists it follows from Lemma 4.2 that there also exists $\mathrm{tr}(w_k)$ and trace $L_k = \mathrm{tr}(w_k)$.

Since the decomposition $u = \sum_k w_k$ is unique and $\sum_k \mathrm{tr}(w_k)$ is convergent we obtain

$$\mathrm{tr}\, u = \sum_k \mathrm{tr}(w_k) = \sum_i \lambda_i\, n(\lambda_i, u) .$$

Motivated by the operator case arizes

Problem 6.9. $\mathfrak{S}_1 \subset \mathfrak{N}$?

Problem 6.10. Let $u \in \mathfrak{F}$. Exists there a representation $u = \sum_{i=1}^{n} u_i$, such that $\|u_i\| \leqq \leqq \|u\|$ for all $i = 1, \ldots, n$? If Problem 6.10 is positively solved the Problem 6.9, too. (see [9]). For commutative Banach algebras immediately follows from Remark 2.3 and the definition of $\mathfrak{S}_1$ that

$$\mathfrak{S}_1 = \mathfrak{N} \quad \text{and} \quad \nu = \sigma_1 .$$

For C^*-algebras the Problem 6.9 is treated in the next section.

Throughout this section $\mathfrak{M}$ is supposed to be a C^*-algebra. Our concern is devoted to prove the identity of $\mathfrak{N}$ and the trace-class $\mathfrak{S}_1$ of $\mathfrak{M}$.

An element $u \in \mathfrak{M}$ is called *compact* if the linear operator $x \to uxu$ is compact. The class of all compact elements is denoted by $\mathfrak{C}$. Ylinen shows that $\mathfrak{C}$ forms a two-sided ideal which is the closure of $\mathfrak{F}$ (see [18]). Let $u \in \mathfrak{C}$ and let (λ_n) be the eigenvalues of u^*u, arranged in decreasing order and repeated according to multiplicity. Put $s_n(u) := \sqrt{\lambda_n}$. This number is called *the n-th singular value* of the element u. Almost all results known from [5] about singular values of compact operators were proved for dual C^*-algebras by WONG [16]. The ideal $\mathfrak{C}$ is by Corollary 8.3 in [1] a dual C^*-algebra. Moreover, if $u \in \mathfrak{C}$ is one-dimensional with respect to $\mathfrak{C}$, then because of $uxu = uxux_0u = \tau(xux_0)\,u$ for all $x \in \mathfrak{M}$, where $x_0 \in \mathfrak{C}$ is choosen such that $\tau(x_0) = 1$ it follows that $u \in \mathfrak{F}_1$. Therefore, all results proved by Wong for singular values are valid for $\mathfrak{C}$. Particularly we obtain that

$$a_n(u) = s_n(u) \quad \text{for} \quad u \in \mathfrak{C}\,.$$

For more informations the reader is referred to [16, 17]. $\mathfrak{S}_1$ is called *the trace-class* of $\mathfrak{M}$.

In order to prove the next theorem we use the ideas of [18].

Theorem 7.1. *Every $s \in \mathfrak{C}$ admits a Schmidt representation*, i.e.

$$s = \sum_{i=1}^{\infty} \lambda_i u_i\,,$$

where $u_i \in \mathfrak{F}_1$, $\|u_i\| = 1$, and $u_i u_i^*$ (resp. $u_i^* u_i$) are mutually orthogonal idempotents. Moreover,

$$a_n(s) = \lambda_n\,.$$

Proof. As you know the C^*-algebra $\mathfrak{M}$ is $*$-isomorphic to a uniformly closed $*$-subalgebra of $\mathfrak{L}(H)$. Therefore by polar decomposition of $s \in \mathfrak{M}$ we obtain $s = U|s|$ and $|s| = U^*s$, where $U \in \mathfrak{L}(H)$ is partially isometric. Since $s \in \mathfrak{C}$, we have $|s| \in \mathfrak{C}$, and then there exists the following spectral decomposition [18] $|s| = \sum_{i=1}^{\infty} \lambda_i e_i$, where $\lambda_1 \geqq \lambda_2 \geqq \ldots \geqq 0$ and e_i are orthogonal minimal hermitian idempotents.

We obtain

$$s = \sum_{i=1}^{\infty} \lambda_i u_i\,,$$

with $u_i := Ue_i = (1/\lambda_i)\, se_i \in \mathfrak{F}_1$ and $\|u_i\| \leqq 1$. From $|s| = U^*U|s|$ it follows that

$e_i = U^*Ue_i = U^*u_i$ and $1 \leqq \|u_i\|$. Therefore,

$$(u_iu_i^*)(u_iu_i^*) = u_ie_iU^*u_iu_i^* = u_ie_iu_i^* = u_iu_i^*$$

and

$$(u_iu_i^*)(u_ju_j^*) = u_ie_iU^*Ue_ju_j^* = u_ie_ie_ju_j^* = 0 \quad \text{for} \quad i \neq j\,.$$

In the same way it can be shown that $u_i^*u_i$ are orthogonal minimal idempotents. By definition is $s_n(s) = \lambda_n$. Because of $a_n(s) = s_n(s)$ we have $a_n(s) = \lambda_n$.

Lemma 7.2. *Let $(u_i)_{i=1}^{\infty} \subset \mathfrak{F}_1$ such that $u_i^*u_i$ (resp. $u_iu_i^*$) are mutually orthogonal idempotents. Then*

$$\sum_{i=1}^{\infty} |\langle \tau_{u_i}, v\rangle| \leqq \|v\| \quad \textit{for all} \quad v \in \mathfrak{F}_1\,.$$

Proof. For $w \in \mathfrak{F}_1$ it follows from

$$\|w\|^4 = \|ww^*ww^*\| = |\langle \tau_w, w^*\rangle|\, \|ww^*\| = |\langle \tau_w, w^*\rangle|\, \|w\|^2$$

that $|\langle \tau_w, w^*\rangle| = \|w\|^2 \neq 0$. Therefore

$$\|wxw^*| = \left\| \frac{ww^*w}{\langle \tau_w, w^*\rangle} xw^* \right\| = \frac{|\langle \tau_{w^*}, wx\rangle|}{\|w\|^2} \|ww^*\|\,.$$

We obtain

$$|\langle \tau_{u_i}, v\rangle|^2 = \|u_ivu_i\|^2 = \|u_ivu_i(u_ivu_i)^*\| = \|u_ivu_iu_i^*v^*u_i^*\| =$$

$$= \frac{|\langle \tau_{v^*}, vu_iu_i^*\rangle|}{\|v\|^2} \|u_ivv^*u_i^*\| = \frac{1}{\|v\|^2} |\langle \tau_{v^*}, vu_iu_i^*\rangle|\, |\langle \tau_{u_i^*}, u_ivv^*\rangle|\,.$$

By Theorem 4.5 (v) we have $\langle \tau_{vv^*}, u_i^*u_i\rangle \geqq 0$.

Cauchy inequality yields

$$\sum_{i=1}^{m} |\langle \tau_{u_i}, v\rangle| = \frac{1}{\|v\|} \sum_{i=1}^{m} |\langle \tau_{v^*v}, u_iu_i^*\rangle|^{1/2}\, |\langle \tau_{vv^*}, u_i^*u_i\rangle|^{1/2} \leqq$$

$$\leqq \frac{1}{\|v\|} \{\sum_{i=1}^{m} \langle \tau_{v^*v}, u_iu_i^*\rangle\}^{1/2} \{\sum_{i=1}^{m} \langle \tau_{vv^*}, u_i^*u_i\rangle\}^{1/2} =$$

$$= \frac{1}{\|v\|} \{\langle \tau_{v^*v}, \sum_{i=1}^{m} u_iu_i^*\rangle\}^{1/2} \{\langle \tau_{vv^*}, \sum_{i=1}^{m} u_i^*u_i\rangle\}^{1/2} \leqq$$

$$\leqq \frac{1}{\|v\|} \{\|v\|^2 \|\sum_{i=1}^{m} u_iu_i^*\|\}^{1/2} \{\|v\|^2 \|\sum_{i=1}^{m} u_i^*u_i\|\}^{1/2} \leqq$$

$$\leqq \|v\| \text{ for all natural numbers } m\,.$$

Hence,

$$\sum_{i=1}^{\infty} |\langle \tau_{u_i}, v\rangle| \leqq \|v\| \quad \text{for each} \quad v \in \mathfrak{F}_1 .$$

Theorem 7.3. *$\mathfrak{S}_1 = \mathfrak{N}$ and $\sigma_1 = v$.*

Proof. Let $s \in \mathfrak{S}_1$. By Theorem 7.1 s admits a Schmidt representation

$$s = \sum_{i=1}^{\infty} \lambda_i u_i .$$

We obtain

$$v(s) \leqq \sum_i \|\lambda_i u_i\| \leqq \sum \lambda_i = \sum_i a_i(s) = \sigma_1(s) .$$

To prove the converse inclusion we consider $s \in \mathfrak{N}$. Because of Theorem 7.1 there exists a Schmidt representation

$$s = \sum_i \lambda_i u_i , \quad u_i \in \mathfrak{F}_1 \quad \text{and} \quad u_i u_i^* \quad (\text{resp. } u_i^* u_i) \quad \text{are}$$

orthogonal idempotents. Moreover, for given $\varepsilon > 0$, we can find a nuclear representation

$$s = \sum_i v_i , \quad v_i \in \mathfrak{F}_1 \quad \text{and} \quad \sum_i \|v_i\| \leqq (1 + \varepsilon)\, v(s) .$$

Since $u_i^* s u_i^* = e_i U^* s e_i U^* = e_i |s|\, e_i U^* = \lambda_i e_i U^* = \lambda_i u_i^*$ we have $\langle \tau_{u_i^*}, s\rangle = \lambda_i$. (For notations see proof of Theorem 7.1.) Lemma 7.2 yields

$$\sigma_1(s) = \sum_n a_n(s) = \sum_n \lambda_n = \sum_n \langle \tau u_n^*, \sum_i v_i\rangle \leqq$$

$$\leqq \sum_{i=1}^{\infty} \sum_n \langle \tau_{un^*}, v_i\rangle \leqq \sum_{i=1}^{\infty} \|v_i\| \leqq (1 + \varepsilon)\, v(s) .$$

This proves the assertion.

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Author's address: DDR-69 Jena, Universitäts-Hochhaus, Sektion Mathematik.