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METHOD OF ROTHE AND NONLINEAR PARABOLIC BOUNDARY VALUE PROBLEMS OF ARBITRARY ORDER

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Introduction. In this paper we consider the first initial-boundary value problem for parabolic equations with a nonlinear elliptic (monotone) operator of order $2k$ of the form

$$\frac{\partial u}{\partial t} + \sum_{|i| \leqq k} (-1)^{|i|} D^i a_i(x, Du) = f(x, t)$$

in a domain $Q = \Omega \times (0, T)$, where Ω is a bounded domain $x \in \Omega \subset E^N$ (N-dimensional Euclidean space), $t \in \langle 0, T \rangle$ ($T < \infty$), i is a multiindex and

$$D^i = \frac{\partial^{|i|}}{\partial x_1^{i_1} \dots \partial x_N^{i_N}} \quad \text{with} \quad |i| = \sum_{p=1}^{N} i_p .$$

Du is the vector function $Du = (D^i u, |i| \leqq k)$.

The functions $a_i(x, \xi)$, $\xi \in E^d$ ($d = \operatorname{card}\{i, |i| \leqq k\}$) satisfy the assumptions of Carathéodory and sufficiently general growth conditions (see applications).

Initial-boundary conditions are given by a sufficiently smooth function $u_0(x)$:

$$u(x, 0) = u_0(x), \quad D_\nu^l u(x, t)\,|_{\partial\Omega \times (0,T)} = D_\nu^l u_0(x)|_{\partial\Omega}, \quad l = 0, 1, \dots, k-1$$

where D_ν^l is the normal derivative of order l.

We obtain the solution of our problem and some of its properties by a suitable application of Rothe's method which is called also the method of lines. In [6], E. ROTHE solved by this method a linear parabolic equation of the second order with one space variable. Later on this method has been used in the papers [7–9], where linear (also quasilinear) equations have been solved. A priori estimates of Schauder from the theory of linear elliptic equations have been used there. K. REKTORYS in [3] solved linear parabolic equations by the same method using a priori estimates of the type L_2 only.

In solving our problem we apply the idea of Rothe in the following way:

Let $\{t_i\}_{i=1}^n$ be a uniform partition of $\langle 0, T\rangle$, $h = T/n$ and $t_j = jh$. We solve the nonlinear elliptic equations

$$\frac{u_j - u_{j-1}}{h} + \sum_{|i| \leq k} (-1)^{|i|} D^i a_i(x, Du_j) = f(x, t_j)$$

successively for $j = 1, 2, \ldots, n$ with the Dirichlet boundary conditions

$$D_\nu^l u_j(x)|_{\partial\Omega} = D_\nu^l u_0(x)|_{\partial\Omega} \quad l = 0, 1, \ldots, k-1$$

where $u_0 = u_0(x)$. Then we construct Rothe's function

$$u^n(x, t) = u_{j-1}(x) + (t - t_j) h^{-1}(u_j(x) - u_{j-1}(x)) \quad \text{for} \quad t_{j-1} \leqq t \leqq t_j \, ;$$
$$j = 1, 2, \ldots, n \, .$$

By a simple technique we obtain sufficiently strong a priori estimates for $u^n(x, t)$ and then, using results from the theory on monotone operators [11–13], we prove by the limiting process that $u^n(x, t)$ converges to the (weak) solution $u(x, t)$ of our problem. We can easily prove the estimate

$$\max_{0 \leqq t \leqq T} \|u^n(x, t) - u(x, t)\|^2_{L_2(\Omega)} = \frac{\text{const.}}{n} \, .$$

which is interesting also from the numerical point of view. The derivative (in the classical sense) $\partial u(x, t)/\partial t \in L_2(\Omega)$ exists for a.e. $t \in (0, T)$ and $x \in \Omega$ and we have $u(x, t) \to u_0(x)$ in $L_2(\Omega)$ for $t \to 0$. Owing to a priori estimates of $\partial u^n(t)/\partial t$ we do not work in the space of distributions with values in Banach spaces. In solving our problem by this method we use direct variational methods for the parabolic initial-boundary value problems. The attention to this fact has been called by many authors, e.g., by K. Rektorys [3] and P. P. Mosolov [10]. Thus, some properties of solutions of elliptic boundary value problems can be transferred to parabolic initial-boundary value problems.

In the first place (for technical simplicity) we prove existence of the solution for an operator equation in a Banach space and then we apply this result to a sufficiently arge class of nonlinear parabolic equations.

NOTATION AND DEFINITIONS

Let us consider the problem

$$\text{(1)} \qquad \frac{\mathrm{d}u(t)}{\mathrm{d}t} + A\,u(t) = f(t)\,, \quad u(0) = u_0\,, \quad t \in (0, T)$$

where $0 < T < \infty$, with a nonlinear operator A from a separable reflexive Banach space V into V' (V' is the dual space to V). We denote the norms by $\|\cdot\|_V$ and $\|\cdot\|_{V'}$, respectively. Let H be a separable Hilbert space with a norm $\|\cdot\|$ and a scalar product $(\cdot, \cdot)$. We suppose that the space $V \cap H$ (with the norm $\|\cdot\|_{V\cap H} \doteq \|\cdot\|_V + \|\cdot\|$) is a dense set in both V and H with the corresponding norms. Duality between V and V' is denoted by $[f, v]$ for $f \in V'$ and $v \in V$.

We shall assume that

(2) $A : V \to V'$ is demicontinuous and bounded, i.e., it is continuous from the strong topology in V into V' with the weak topology and transforms bounded sets into bounded sets,

$$[Au - Av, u - v] \geqq 0 \quad \text{for all} \quad u, v \in V, \tag{3}$$

$$[Au, u] \geqq [u]\, r([u]) \quad \text{where} \quad r(t) \to \infty \quad \text{for} \quad t \to \infty \tag{4}$$

and $[\cdot]$ is a seminorm in V such that there exist $\lambda_0 > 0$ and $c_0 > 0$ such that

$$[u] + \lambda_0 \|u\| \geqq c_0 \|u\|_V \quad \text{for all} \quad u \in V \cap H\,,$$

$$u_0 \in V \cap H \quad \text{and} \quad Au_0 \in H\,, \tag{5}$$

(6) $f(t)$ is Lipschitz continuous: $I \equiv \langle 0, T\rangle \to H$, i.e.,

$$\|f(t') - f(t)\| \leqq L|t - t'| \quad (L > 0 \text{ is a constant})\,.$$

Let X be a Banach space with a norm $\|\cdot\|_X$.

Definition 1. We denote by $L_p(I, X)$ $(1 \leqq p \leqq \infty)$ the set of all measurable abstract functions $v(t)$ from I into X (see [15]) such that

$$\|v\|_{L_p}^p = \int_I \|v(t)\|_X^p \,\mathrm{d}t < \infty \quad \text{for} \quad 1 \leqq p < \infty \quad \text{and}$$

$$\|v\|_{L_\infty(I,X)} = \sup_{t\in I} \operatorname{ess} \|v(t)\|_X < \infty \quad \text{for} \quad p = \infty\,.$$

Let $C(I, H)$ be the set of all continuous functions $u(t) : I \to H$ with $\|u\|_{C(I,H)} = \max_{t\in I} \|u(t)\| < \infty$. Denote by $C^1(I, H)$ the set of all continuously differentiable functions $u(t) : I \to H$ with $\|u\|_{C^1(I,H)} = \|u\|_{C(I,H)} + \|u'\|_{C(I,H)} < \infty$. Let M be a linear dense set in $L_2(I, H)$ of functions $v(t) \in C^1(I, H)$ such that $\operatorname{supp} v(t) \subset (0, T)$.

Definition 2. We say that $u(t) \in L_2(I, H)$ is *weakly differentiable*, $u \in W_2^1(I, H)$, if

$$\sup_{\substack{v\in M \\ \|v\|_{L_2(I,H)}\leqq 1}} \left| \int_I (u(t), v'(t))\,\mathrm{d}t \right| < \infty\,.$$

In this case (Riesz Theorem) there exists a uniquely determined $g(t) \in L_2(I, H)$ such that

$$\int_I (u(t), v'(t))\, dt = -\int_I (g(t), v(t))\, dt \quad \text{for all} \quad v \in M \tag{8}$$

and we denote by $du(t)/dt = g(t)$ the weak derivative of $u(t)$. $W_2^1(I, H)$ is a Hilbert space with the scalar product

$$(u, v)_{W_2^1} = \int_I (u(t), v(t))\, dt + \int_I \left(\frac{du}{dt}, \frac{dv}{dt}\right) dt\,.$$

Definition 3. Under *the solution* of the problem (1) we understand $u(t) \in W_2^1(I, H) \cap$ $\cap L_\infty(I, V \cap H)$ such that $u(0) = u_0$ and

$$\int_I \left(\frac{du(t)}{dt}, v(t)\right) dt + \int_I [A\, u(t), v(t)]\, dt = \int_I (f(t), v(t))\, dt \tag{9}$$

holds for each $v \in L_1(I, V \cap H)$.

Lemma 1. *If* $u \in W_2^1(I, H)$ *then* $u \in C(I, H)$ *(after changing on a set of zero measure) and for a.e.* $t \in I$ *the strong derivative* $u'(t)$ *satisfying* $u'(t) = du(t)/dt$ *exists.*

Proof. Let us consider the Bochner integral (see [15])

$$v(t) = \int_0^t \frac{du(s)}{ds}\, ds\,.$$

From the properties of the Bochner integral we obtain that $v \in C(I, H)$, $v(t)$ is strongly differentiable for a.e. $t \in I$ and $v'(t) = du(t)/dt$. Easily we find that $v \in W_2^1(I, H)$ and $dv(t)/dt = du(t)/dt$. We prove $u(t) - v(t) = w(t) = z \in H$ for a.e. $t \in I$. From Definition 2 we obtain

$$\int_I (w(t), \varphi'(t)\, y)\, dt = 0 \quad \text{for all} \quad y \in H \quad \text{and} \quad \varphi(t) \in \mathscr{D}(I)\,.$$

$\mathscr{D}(I)$ is the set of all functions with support in $(0, T)$, having derivatives of all orders.) Every $\psi(t) \in \mathscr{D}(I)$ can be decomposed into the form

$$\psi(t) = \int_I \psi(t)\, dt\,.\, \chi(t) + \varphi'(t)$$

where $\chi(t) \in \mathscr{D}(I)$ is a fixed function with $\int_I \chi(t)\, dt = 1$ and $\varphi(t) \in \mathscr{D}(I)$ is chosen with respect to $\psi(t)$. Let us denote $\int_I w(t)\, \chi(t)\, dt = z \in H$ (Bochner integral). Then we have

$$\int_I (w(t), y)\, \psi(t)\, dt = \int_I \psi(t)\, dt\,.\int_I (w(t), \chi(t)\, y)\, dt = \int_I (z, y)\, \psi(t)\, dt\,,$$

hence $w(t) = z$ for a.e. $t \in I$ and Lemma 1 is proved.

Remark 1. From Definition 3 and Lemma 1 it follows that the solution $u(t)$ of the problem (1) satisfies

$$u'(t) + A\,u(t) = f(t)$$

for a.e. $t \in (0, T)$ in the space H.

Indeed, we put $v(t) = \psi(t)\,w$, where $\psi(t) \in L_1(I)$ and $w \in V \cap H$, into the identity (9). Since $f(t) \in H$ and $u'(t) = \mathrm{d}u(t)/\mathrm{d}t \in H$, we have $A\,u(t) \in H$.

Positive constants will be denoted by C and the dependence of C on the parameter ε by $C(\varepsilon)$. Constants C and $C(\varepsilon)$ may denote also various constants in the same discussion.

1. EXISTENCE OF THE SOLUTION

Following the idea of Rothe, we solve the equations

$$\frac{u_i - u_{i-1}}{h} + Au_i = f(t_i) \tag{10}$$

successively for $i = 1, 2, \ldots, n$, where $h = T/n$, $t_i = ih$ and u_0 is from (1).

In the sequel we shall suppose (2)–(6). The assumption (5) is used in Lemma 3 only.

Lemma 2. *For an arbitrary n and $1 \leqq j \leqq n$ there exists a unique solution $u_j \in V \cap H$ of the equation* (10).

Proof. Let us define an operator $\mathscr{A}_\lambda : V \cap H \to (V \cap H)'$ by the duality

$$(\mathscr{A}_\lambda u, v)_* = [Au, v] + \lambda(u, v) \quad \text{where} \quad \lambda > 0\,,$$

$(\cdot, \cdot)_*$ is duality between $V \cap H$ and $(V \cap H)'$.

$\mathscr{A}_\lambda$ is a bounded demicontinuous and strictly monotone operator. From the estimate

$$\|u\|_{V \cap H} \leqq \frac{1}{c_0}\left([u] + (\lambda_0 + 1)\,\|u\|\right)$$

and the assumption (4) we deduce that from each sequence $\{u_n\}$ with $\|u_n\|_{V \cap H} \to \infty$ a subsequence $\{u_{n_k}\}$ can be chosen in such a way that

$$(\mathscr{A}_\lambda u_{n_k}, u_{n_k})_* \cdot \left(\|u_{n_k}\|_{V \cap H}\right)^{-1} \to \infty \quad \text{for} \quad k \to \infty\,.$$

This fact implies the coerciveness of $\mathscr{A}_\lambda$. Thus, the theory of monotone operators (see [12]) yields the existence of a unique solution of the equation

$$\mathscr{A}_\lambda u = f \in H \subset (V \cap H)' = V' + H$$

and hence Lemma 2 is proved.

Now we prove some a priori estimates for u_i, $i = 1, 2, \ldots, n$.

Lemma 3. *There exists $C(u_0, f)$ such that*

$$\left\| \frac{u_i - u_{i-1}}{h} \right\| \leqq C(u_0, f)$$

holds for each n and $i = 1, 2, \ldots, n$.

Proof. Let us subtract the identity

$$\left(\frac{u_j - u_{j-1}}{h}, v \right) + [Au_j, v] = (f(t_j), v), \quad v \in V \cap H \tag{11}$$

for $j = i$ and $j = i - 1$, where $v = u_i - u_{i-1}$. Then we obtain

$$\left(\frac{u_i - u_{i-1}}{h}, u_i - u_{i-1} \right) + [Au_i - Au_{i-1}, u_i - u_{i-1}] =$$

$$= \left(\frac{u_{i-1} - u_{i-2}}{h}, u_i - u_{i-1} \right) + (f(t_i) - f(t_{i-1}), u_i - u_{i-1})$$

from which, owing to (3), we deduce

$$\left\| \frac{u_i - u_{i-1}}{h} \right\| \leqq \left\| \frac{u_{i-1} - u_{i-2}}{h} \right\| + \|f(t_i) - f(t_{i-1})\| \leqq \left\| \frac{u_{i-1} - u_{i-2}}{h} \right\| + Lh\,.$$

Thus, we obtain successively

$$\left\| \frac{u_i - u_{i-1}}{h} \right\| \leqq \left\| \frac{u_1 - u_0}{h} \right\| + LT\,. \tag{12}$$

Now, we estimate $\|(u_1 - u_0)/h\|$. From (11) for $j = 1$ and $v = u_1 - u_0$ we have

$$\left\| \frac{u_1 - u_0}{h} \right\|^2 + \frac{1}{h} [Au_1 - Au_0, u_1 - u_0] = \tag{13}$$

$$= \left(f(t_1), \frac{u_1 - u_0}{h} \right) - \left[Au_0, \frac{u_1 - u_0}{h} \right].$$

Taking the assumption (5) into account we estimate

$$\left\| \left[Au_0, \frac{u_1 - u_0}{h} \right] \right\| \leqq \|Au_0\| \cdot \left\| \frac{u_1 - u_0}{h} \right\|$$

and hence (3) and (13) yields the estimate

$$\left\| \frac{u_1 - u_0}{h} \right\| \leqq C(u_0, f) . \tag{14}$$

Thus, from (12) and (14) we obtain the required result.

Lemma 4. *There exists* $C(u_0, f)$ *such that*

$$\|u_i\|_{V \cap H} \leqq C(u_0, f)$$

for all n *and* $i = 1, 2, \ldots, n$.

Proof. From the triangle inequality and Lemma 3 we deduce

$$\|u_i\| \leqq \sum_{j=1}^{i} \left\| \frac{u_j - u_{j-1}}{h} \right\| . h + \|u_0\| = C(u_0, f) \tag{15}$$

for all $i = 1, 2, \ldots, n$.

Let us put $v = u_i$ in (11). Then due to Lemma 3, (15) and (4) we have

$$[u_i]\, r([u_i]) \leqq C(u_0, f)$$

and hence there exists $C(u_0, f)$ such that

$$[u_i] \leqq C(u_0, f) \tag{16}$$

for all n and $i = 1, 2, \ldots, n$, since $r(t) \to \infty$ for $t \to \infty$. The estimates (15) and (16) imply the required result (see (4)).

Owing to Lemma 3 and Lemma 4, we conclude from (10) that

$$\begin{gathered} Au_i \in H \subset (V \cap H)' \quad \text{for all} \quad i = 1, 2, \ldots, n \quad \text{and} \\ \|Au_i\|_{(V \cap H)'} \leqq \|Au_i\| \leqq C(u_0, f) \end{gathered} \tag{17}$$

for all $i = 1, 2, \ldots, n$.

Let us define a step function $\bar{u}^n(t)$ by

$$\bar{u}^n(t) = u_j \quad \text{for} \quad t_{j-1} < t \leqq t_j , \quad j = 1, 2, \ldots, n \quad \text{and} \quad \bar{u}^n(0) = u_0 .$$

If $u^n(t)$ is Rothe's function, i.e.,

$$u^n(t) = u_{j-1} + (t - t_{j-1})\, h^{-1}(u_j - u_{j-1}) \quad \text{for} \quad t_{j-1} \leqq t \leqq t_j ,$$

$j = 1, 2, \ldots, n$, then owing to Lemma 3 we have

$$\|u^n(t) - \bar{u}^n(t)\| \leqq C(u_0, f)\, n^{-1} \tag{18}$$

for all n and $t \in I$. From Lemma 3 we easily obtain the estimates

$$\|u^n\|_{L_\infty(I,V\cap H)} + \|\bar{u}^n\|_{L_\infty(I,V\cap H)} \leqq C(u_0, f) \tag{19}$$

for all n and $t \in I$.

Easily we find that $u^n \in W_2^1(I, H)$ and

$$\frac{\mathrm{d}u^n(t)}{\mathrm{d}t} = \frac{u_i - u_{i-1}}{h} \quad \text{for} \quad t_{j-1} < t < t_j\,, \quad j = 1, 2, \ldots, n\,.$$

Lemma 3 and Lemma 4 imply

$$\|u^n\|_{L_\infty(I,H)} \leqq C(u_0, f) \quad \text{for all} \quad n\,, \tag{20}$$

$$\left\|\frac{\mathrm{d}u^n(t)}{\mathrm{d}t}\right\|_{L_\infty(I,H)} \leqq C(u_0, f) \quad \text{for all} \quad n\,. \tag{21}$$

Thus, we have the estimate

$$\|u^n\|_{W_2{}^1(I,H)} \leqq C \quad \text{for all} \quad n\,. \tag{22}$$

Remark 2. We denote

$$\bar{f}^n(t) = f(t_{j-1}) + (t - t_{j-1})\, h^{-1}(f(t_j) - f(t_{j-1}))$$

for $t_{j-1} \leqq t \leqq t_j$, $j = 1, 2, \ldots, n$. The estimate (21) can be expressed also in the form (see the proof of Lemma 3)

$$\left\|\frac{\mathrm{d}u^n(t)}{\mathrm{d}t}\right\|_{L_\infty(I,H)} \leqq \|f(t_1)\| + \|Au_0\| + \int_0^T \left\|\frac{\mathrm{d}\bar{f}^n(\tau)}{\mathrm{d}\tau}\right\| \mathrm{d}\tau\,. \tag{21a}$$

Lemma 5. *There exists $u \in W_2^1(I, H)$ with u, $\mathrm{d}u/\mathrm{d}t \in L_\infty(I, H)$ and a subsequence $\{u^{n_k}(t)\}$ of $\{u^n(t)\}$ such that $u^{n_k} \to u$, $\mathrm{d}u^{n_k}/\mathrm{d}t \to \mathrm{d}u/\mathrm{d}t$ in $L_2(I, H)$ (weak convergence).*

Proof. $W_2^1(I, H)$ is a reflexive space and, thus, the assertion follows from (20), (21) and (22).

If we denote by $f^n(t)$ the step function $f^n(t) = f(t_j)$ for $t_{j-1} < t \leqq t_j$, $j = 1, 2, \ldots, n$ and $f^n(0) = f(0)$ then the identity (11) can be rewritten into the form

$$\left(\frac{\mathrm{d}u^n(t)}{\mathrm{d}t}, v\right) + [A\,\bar{u}^n(t), v] = (f^n(t), v) \tag{23}$$

for all $v \in V \cap H$.

Lemma 6. *$u^n \to u$ in the norm of the space $C(I, H)$ and the estimate*

$$\|u^n(t) - u(t)\|^2 \leqq C(u, f)\, n^{-1}$$

is valid for all n and $t \in I$.

Proof. Let us subtract (23) for $n = r$ and $n = s$ where $v = \bar{u}^r(t) - \bar{u}^s(t)$. We obtain

$$\left(\frac{\mathrm{d}u^r(t)}{\mathrm{d}t} - \frac{\mathrm{d}u^s(t)}{\mathrm{d}t}, \quad \bar{u}^r(t) - \bar{u}^s(t)\right) + [A\,\bar{u}^r(t) - A\,\bar{u}^s(t),\ \bar{u}^r(t) - \bar{u}^s(t)] =$$

$$= (f^r(t) - f^s(t),\ \bar{u}^r(t) - \bar{u}^s(t))$$

from which we deduce by virtue of (3)

(24)

$$\left(\frac{\mathrm{d}(u^r(t) - u^s(t))}{\mathrm{d}t}, u^r(t) - u^s(t)\right) \leqq \left(\frac{\mathrm{d}(u^r(t) - u^s(t))}{\mathrm{d}t}, u^r(t) - \bar{u}^r(t) + u^s(t) - \bar{u}^s(t)\right) +$$

$$+ (f^r(t) - f^s(t),\ \bar{u}^r(t) - \bar{u}^s(t))\,.$$

Since

$$\int_0^t \left(\frac{\mathrm{d}(u^r(t) - u^s(t))}{\mathrm{d}t}, u^r(t) - u^s(t)\right) \mathrm{d}t = \tfrac{1}{2}\|u^r(t) - u^s(t)\|^2$$

and

$$\|f^r(t) - f^s(t)\| \leqq L\left(\frac{1}{r} + \frac{1}{s}\right),$$

integrating (24) in $(0, t)$ and using the estimate (18) and Lemma 3 we conclude

$$\|u^r(t) - u^s(t)\|^2 \leqq C\left(\frac{1}{r} + \frac{1}{s}\right). \tag{25}$$

Thus, there exists $v \in C(I, H)$ such that $u^n \to v$ in $C(I, H)$. But $u^n \to v$ also in the space $L_2(I, H)$ and thus, $v = u$ because of Lemma 5. By the limiting proces; for $s \to \infty$ in (25) we obtain the required estimate and the proof is complete.

As a consequence of Lemma 6 we have $u(0) = u_0$. From Lemma 6 and (18) we also deduce $\bar{u}^n \to u$ in the norm of the space $L_\infty(I, H)$.

We shall use the following assertion:

Lemma 7. *If $v \in L_\infty(I, V \cap H)$ then $Av \in L_\infty(I, (V \cap H)')$ and*

$$\int_I [A(v + \lambda w), z]\,\mathrm{d}t \to \int_I [Av, z]\,\mathrm{d}t \quad \textit{for} \quad \lambda \to 0$$

(λ is a real number), where $v, w, z \in L_\infty(I, V \cap H)$.

Proof. From the boundedness of A we deduce

$$\|A\,v(t)\|_{(V \cap H)'} \leqq C \quad \text{for a.e.} \quad t \in I\,.$$

We prove that $A\,v(t)$ is a measurable abstract function (see [15]). To this aim it suffices to prove that $[A\,v(t), w]$ is a measurable function of t for all $w \in V \cap H$ since $V \cap H$ is a separable reflexive space (see [15]). There exists a sequence $\{v^n(t)\}$ of simple functions such that $v^n(t) \to v(t)$ in $V \cap H$ for a.e. $t \in I$. Thus, $[A\,v^n(t), w]$ is a measurable function and from (2) we obtain $[A\,v^n(t), w] \to [A\,v(t), w]$ for a.e. $t \in I$ and, hence, $A\,v(t)$ is a measurable abstract function. Owing to (2),

$$[A(v(t) + \lambda\, w(t)), z(t)] \to [A\,v(t), z(t)] \tag{29}$$

for $\lambda \to 0$ and a.e. $t \in I$. We suppose that $0 < \lambda < 1$. From the estimate

$$\|v + \lambda w\|_{L_\infty(I,V\cap H)} \leqq C(v, w)$$

(the constant $C(v, w)$ is independent of λ) we deduce

$$\|A(v + \lambda w)\|_{L_\infty(I,(V\cap H)')} \leqq C(v, w)$$

and hence

$$|[A\,(v(t) + \lambda\, w(t)), z(t)]| \leqq C(v, w)\, \|z\|_{L_\infty(I,V\cap H)}\,.$$

From this estimate, (29) and the Lebesgue Theorem we obtain the required result.

In the sequel we prove that $u(t)$ from Lemma 5 is a solution of (1). From the definition of $L_p(I, X)$ and from the definition of the Bochner integral (see 15) it follows that $L_\infty(I, X)$ is a dense set in $L_p(I, X)$ ($p \geqq 1$).

Let $v \in L_\infty(I, V \cap H)$. Integrating (23) over the interval I, where $v = v(t)$, we obtain

$$\int_I \left(\frac{\mathrm{d}u^n(t)}{\mathrm{d}t},\ v(t)\right) \mathrm{d}t + \int_I [A\,\bar{u}^n(t), v(t)]\, \mathrm{d}t = \int_I (f^n(t), v(t))\, \mathrm{d}t\,. \tag{26}$$

The estimate (17) implies

$$\|A\bar{u}^n\|_{L_\infty(I,(V\cap H)')} \leqq \|A\bar{u}^n\|_{L_\infty(I,H)} \leqq C(u_0, f) \tag{27}$$

for all n.

Theorem 1. *There exists a solution $u(t)$ of* (1), (2) (*in the sense of Definition* 3) *and the estimate*

$$\|u^n(t) - u(t)\|^2 \leqq C(u_0, f)\, n^{-1}$$

holds.

Proof. Since $L_\infty(I, (V \cap H)')$ is the dual space to the separable space $L_1(I, V \cap H)$, there exists $\chi \in L_\infty(I, (V \cap H)')$ (moreover, $\chi \in L_\infty(I, H)$) such that

$$\int_I [A\,\bar{u}^{n_k}(t). v(t)]\, \mathrm{d}t \to \int_I [\chi(t), v(t)]\, \mathrm{d}t \tag{28}$$

for all $v \in L_1(I, V \cap H)$, where $\{\bar{u}^{n_k}\}$ is a suitable subsequence of $\{\bar{u}^n\}$, i.e., $Au^{n_k} \underset{w*}{\to} \chi$

in $L_\infty(I, (V \cap H)')$ (weak* convergence in $L_\infty(I, (V \cap H)')$). Thus, by the limiting process in (26), where $v \in L_\infty(I, V \cap H)$, we obtain

$$(29)\qquad \int_I \left(\frac{\mathrm{d}u(t)}{\mathrm{d}t}, v(t)\right) \mathrm{d}t + \int_I [\chi(t), v(t)] \,\mathrm{d}t = \int_I (f(t), v(t)) \,\mathrm{d}t$$

because of (28) and Lemma 5. By the same argument as in (28), we obtain from (19) that there exists $g \in L_\infty(I, V \cap H)$ and a subsequence $\{\bar{u}^{n_k}\}$ such that $\bar{u}^{n_k} \underset{w^*}{\rightarrow} g$ in $L_\infty(I, V \cap H)$ (weak* convergence). But $\bar{u}^n \to u$ in $L_2(I, H) \supset L_\infty(I, V \cap H)$ and thus $u = g$.

Now, we prove $A\, u(t) = \chi(t)$ using some technique of monotone operators.

By substituting $v(t) = \bar{u}^n(t)$ into (26) we obtain

$$\int_I [A\, \bar{u}^n(t), \bar{u}^n(t)] \,\mathrm{d}t \to - \int_I \left(\frac{\mathrm{d}u(t)}{\mathrm{d}t}, u(t)\right) \mathrm{d}t + \int_I (f(t), u(t)) \,\mathrm{d}t$$

because of Lemma 5 and $\bar{u}^n \to u$ in $L_2(I, H)$. From this fact and (29) we deduce

$$(30)\qquad \int_I [A\, \bar{u}^n(t), \bar{u}^n(t)] \,\mathrm{d}t \to \int_I [\chi(t), u(t)] \,\mathrm{d}t$$

for $n \to \infty$. Owing to (3) we have

$$\int_I [A\, \bar{u}^n(t) - A\, v(t), \bar{u}^n(t) - v(t)] \,\mathrm{d}t \geqq 0$$

which together with (30) yields

$$(31)\qquad \int_I [\chi(t) - A\, v(t), u(t) - v(t)] \,\mathrm{d}t \geqq 0$$

for all $v \in L_\infty(I, V \cap H)$. Putting $v(t) = u(t) + \lambda\, w(t)$ into (31), where $\lambda > 0$ and $w \in L_\infty(I, V \cap H)$, and by the limiting process $\lambda \to 0$ we have

$$\int_I [\chi(t) - A\, u(t), w(t)] \,\mathrm{d}t \geqq 0 \quad \text{for all} \quad w \in L_\infty(I, V \cap H)$$

because of Lemma 7. Thus,

$$\int_I [\chi(t) - A\, u(t), w(t)] \,\mathrm{d}t = 0 \quad \text{for all} \quad v \in L_1(I, V \cap H)$$

must hold ($L_\infty(I, V \cap H)$ is a dense set in $L_1(I, V \cap H)$) and hence $u(t)$ is a solution of (1), (2). The rest of the proof follows from Lemma 6.

Theorem 2. *There exists a unique solution of the problem* (1) (*in the sense of Definition* 3).

Proof. If u_1, u_2 are two solutions of (1), then $u = u_1 - u_2 \in L_\infty(I, V \cap H)$ satisfies

$$\int_I \left(\frac{du(t)}{dt}, w(t)\right) dt + \int_I [A\,u_1(t) - A\,u_2(t), w(t)]\, dt = 0 \tag{32}$$

for all $w \in L_1(I, V \cap H)$. For an arbitrary $t_0 \in I$ let us put

$$w_{t_0}(t) = \begin{cases} u(t) & \text{for} \quad 0 \leqq t \leqq t_0 \leqq T \\ 0 & \text{for} \quad t_0 < t \leqq T \end{cases}$$

into (32) and we obtain

$$\int_0^{t_0} \left(\frac{du(t)}{dt}, u(t)\right) dt + \int_0^{t_0} [A\,u_1(t) - A\,u_2(t),\; u_1(t) - u_2(t)]\, dt = 0\,.$$

Thus, we deduce from (3) that

$$\int_0^{t_0} \left(\frac{du(t)}{dt}, u(t)\right) dt = \tfrac{1}{2}\|u(t_0)\|^2 - \tfrac{1}{2}\|u(0)\|^2 \leqq 0$$

and hence $u(t_0) = 0$ since $u(0) = 0$.

Remark 3. If $u(t)$ is the solution of (1) and $f(t) \in C^1(I, H)$ then, due to (21a), the estimate

$$\left\|\frac{du(t)}{dt}\right\|_{L_\infty(I,H)} \leqq \|Au_0\| + \|f(0)\| + \int_0^T \left\|\frac{df(\tau)}{d\tau}\right\| d\tau$$

holds.

2. APPLICATIONS

In this section we shall apply the abstract results from § 1 to a sufficiently wide class of nonlinear parabolic equations from the introduction.

Let $a_i(x, \xi)$, $\xi \in E^d$ for $|i| \leqq k$ be real functions defined for $x \in \Omega$ and $|\xi| < \infty$, continuous in ξ for a.e. $x \in \Omega$ and measurable in x for fixed ξ (the Carathéodory condition).

The growth of $a_i(x, \xi)$ in ξ is described by functions of a certain class $\mathcal{M}_3$, which is essentially larger than the class of polynomials $|u|^p$ – see [2].

Definition 4. $\mathcal{M}_3$ is the set of all real, continuous functions $g(u)$, for which there exists $u_1 > 0$ such that:

i) $u\,g(u)$ is convex and even for $u \geqq u_1$ and $\lim_{u\to\infty} (u\,g(u))' = \infty$;

ii) there exists a constant C such that

$$g(2u) \leqq C\,g(u) \quad \text{for} \quad u \geqq u_1\,,$$

iii) there exists $l > 1$ such that

$$g(u) \leqq \tfrac{1}{2}\,g(lu) \quad \text{for} \quad u \geqq u_1\,.$$

Let $g_i(u) \in \mathcal{M}_3$ for $|i| \leqq k$ be such that $g_i(u) \leqq g_j(u)$ (or $g_i(u) \geqq g_j(u)$) for $u \geqq u_1$ and for each pair i, j with $|i|, |j| \leqq k$. Then the growth conditions are of the form

$$|a_i(x, \xi)| \leqq C(1 + \sum_{|j|\leqq k} \min(|g_i(\xi_j)|, |g_j(\xi_j)|)) \tag{33}$$

for all $|i| \leqq k$ and $\xi \in E^d$.

In the papers [1] and [2] even more general growth conditions are considered. Monotonicity (ellipticity) of our operator will be guaranteed by

$$\sum_{|i|\leqq k} (\xi_i - \eta_i)\,[a_i(x, \xi) - a_i(x, \eta)] \geqq 0 \tag{34}$$

for all $\xi, \eta \in E^d$. We assume coerciveness in the form

$$\sum_{i\leqq k} \xi_i\, a_i(x, \xi) \geqq C_1 \sum_{|i|\leqq k} \xi_i\, g_i(\xi_i) - C_2\,. \tag{35}$$

By means of $G_i(u) = u\,g_i(u)$, $|i| \leqq k$ we construct the Orlicz space $L_{G_i}(\Omega)$ – see [14] and [1]. Now we define the space $W^k_G(\Omega)$:

$$W = W^k_G = \{u \in L_2(\Omega) : D^i u \in L_{G_i}(\Omega) \quad \text{for all} \quad |i| \leqq k\}$$

($D^i u$ are distribution derivatives) with the norm

$$\|\cdot\|_W = \|u\|_{L_2} + \sum_{|i|\leqq k} \|D^i u\|_{G_i}$$

where $\|\cdot\|_{G_i}$ is the Orlicz norm in the space $L_{G_i}(\Omega)$. Let $C^\infty_0(\Omega)$ be the set of all functions defined on Ω having derivatives of all orders and with support in Ω.

Let us denote

$$\mathring{W} = \mathring{W}^k_G = \overline{C^\infty_0(\Omega)}\,,$$

where the closure is taken in the norm $\|\cdot\|_W$. Now, by the form

$$(Au, v) = \sum_{|i|\leqq k} D^i v\, a_i(x, Du)\,\mathrm{d}x \quad \text{where} \quad u, v \in W$$

we define an operator $A : W \to W'$ (W' is the dual space to W). Indeed, (33) implies that $a_i(x, Du)$ is a continuous and bounded operator from W into $L_{P_i}(\Omega)$, where

$L_{P_i}(\Omega)$ is the dual space to $L_{G_i}(\Omega)$. (For the proof see [1] Lemma 3, § 1 and Lemma 3 § 2). Thus

a) $A : W \to W'$ is a continuous and bounded operator.

(34) implies that

b) A is a monotone operator.

Let us denote

$$[u] = \sum_{|i| \leqq k} \|D^i u\|_{G_i} .$$

In [1] and [2] we have proved that

$$\lim_{[u]\to\infty} ([u])^{-1} \int_\Omega \sum_{|i|\leqq k} D^i u \, a_i(x, D(u + u_0)) \, dx = \infty \tag{36}$$

for an arbitrary $u_0(x) \in W$. Thus, defining a function

$$r(t) = \inf_{[u]=t} \sum_{|i|\leqq k} \int_\Omega D^i u \, a_i(x, Du) \, dx$$

(see also [5]) we obtain

c) $[Au, u] \geqq [u] \, r([u])$ with $r(t) \to \infty$ for $t \to \infty$. Let $u_0(x) \in W$ and

$$D^i a_i(x, Du_0) \in L_2(\Omega) \quad \text{for all} \quad |i| \leqq k . \tag{37}$$

Now we apply the result of § 1 to the solution of the following problem:

$$\frac{\partial u}{\partial t} + \sum_{|i|\leqq k} (-1)^{|i|} D^i a_i(x, Du) = f(x, t) , \tag{1'}$$

$$u(x\ 0) = u_0(x) \in \mathring{W} , \tag{2'}$$

$$D_\nu^l u(x, t)|_{\partial\Omega\times(0,T)} = 0 \quad \text{for} \quad l = 0, 1, \ldots, k - 1 .$$

We shall assume

$$f(x, t) \in L_2(Q) , \quad \left\| \frac{\partial f(x, t)}{\partial t} \right\|_{L_2(\Omega)} \leqq C \quad \text{for a.e.} \quad t \in I \tag{38}$$

($\partial f(x, t)/\partial t$ is in the sense of distributions).

Theorem 3. *If the assumptions* (33), (34), (35), (37) *and* (38) *are satisfied then there exists a unique solution $u(x, t)$ (in the sense of Definition* 3) *of the problem* (1'), (2') *and the estimate*

$$\|u^n(x, t) - u(x, t)\|^2_{L_2(\Omega)} \leqq C(u_0, f) \, n^{-1}$$

holds, where $u^n(x, t)$ is Rothe's function.

Proof. First, we verify that (38) implies that $F(t) = f(x, t)$ is a Lipschitz continuous abstract function from $\langle 0, T\rangle \to L_2(\Omega)$. From (38) we easily deduce that

$$\|f(x, t) - f(x, t')\|^2_{L_2(\Omega)} \leqq |t - t'| \cdot \int_t^{t'} \left\| \frac{\partial f(x, s)}{\partial s} \right\|^2_{L_2(\Omega)} \mathrm{d}s \leqq C|t - t'|^2 .$$

If we put $H \equiv L_2(\Omega)$ and $V = \mathring{W}$ then our operator A satisfies the assumptions (2)–(4) from § 1, where $V \cap H = V$. From the estimate

$$\sup_{\substack{v \in W \\ \|v\|_{L_2} \leqq 1}} |[Au_0, v]| = \sup_{\substack{v \in W \\ \|v\|_{L_2} \leqq 1}} \left| \int_\Omega \sum_{|i| \leqq k} D^i v \, a_i(x, Du_0) \, \mathrm{d}x \right| \leqq$$

$$= \sum_{|i| \leqq k} \sup_{\substack{v \in W \\ \|v\|_{L_2} \leqq 1}} \left| \int_\Omega v D^i \, a_i(x, Du_0) \, \mathrm{d}x \right| \leqq C$$

we deduce that $Au_0 \in L_2(\Omega) \equiv H$ and hence (5) is satisfied, Thus, Theorem 1 and Theorem 2 imply Theorem 3.

Remark 4. If we replace (38) by a weaker assumption

$$(38') \qquad f(x, t) \in L_2(Q) \, , \quad \left\| \frac{\partial f(x, t)}{\partial t} \right\| \in L_2(Q) \, ,$$

Theorem 3 remains true. Indeed, from (38′) it follows that $f(x, t_i)$ is well defined in the sense of traces (see [4]) and following the estimate before the relation (12) we find out that the estimate

$$\sum_{i=1}^{j} \|f(x, t_i) - f(x, t_{i-1})\|_{L_2} = \sum_{i=1}^{j} \left\| \frac{f(x, t_i) - f(x, t_{i-1})}{h} \right\|_{L_2} h \leqq C \left\| \frac{\partial f}{\partial t} \right\|_{L_2(Q)}$$

holds and hence (compare with (12), (13))

$$\left\| \frac{u_i - u_{i-1}}{h} \right\|_{L_2(\Omega)} = \|Au_0\|_{L_2(\Omega)} + \|f(x, t_1)\|_{L_2(\Omega)} + \left\| \frac{\partial f}{\partial t} \right\|_{L_2(Q)} \leqq$$

$$\leqq Au_0 + C \left(\|f\|_{L_2(Q)} + \left\| \frac{\partial f}{\partial t} \right\|_{L_2(Q)} \right).$$

If we solve (1′), (2′) with nonhomogeneous boundary conditions in (2′) then we suppose that (2′) is given by means of a function $u_0(x) \in W$, i.e.,

$$(2'') \qquad u(x, 0) = u_0(x) \quad \text{and} \quad D^l_\nu \, u(x, t)_{|\partial\Omega \times (0,T)} = D^l_\nu \, u_0(x)_{|\partial\Omega}$$

for $l = 1, 2, \ldots, k - 1$ and $t \in (0, T)$.

Theorem 3'. *If the assumptions* (33), (34), (35), (37) *and* (38)' *are satisfied, then there exists a unique solution* $u(x, t)$ *of* (1'), (2'') (*in the sense of Definition* 3) *and the estimate*

$$\|u^n(x, t) - u(x, t)\|^2_{L_2(\Omega)} \leqq C(u_0, f) \,.\, n^{-1}$$

holds, where $u^n(x, t)$ is Rothe's function defined in the introduction.

In this case we shall consider the problem

$$\frac{\partial z}{\partial t} + \sum_{|i| \leqq k} (-1)^{|i|} D^i a_i(x, D(u_0 + z)) = f(x, t) \,,$$

$$z(x, 0) = 0 \,, \quad D^l_\nu z(x, t)|_{\partial\Omega \times (0,T)} = 0$$

for $l = 0, 1, \ldots, k - 1$, $t \in (0, T)$. Now we define the operator A by means of the duality form

$$(Az, v) = \int_\Omega \sum_{|i| \leqq k} D^i v \, a_i(x, D(u_0 + z)) \, dx \,.$$

Easily we find that A satisfies all the assumptions of § 1 and hence Theorem 3' is a consequence of Theorem 1 and Theorem 2, where $u(x, t) = z(x, t) + u_0(x)$.

Remark 5. The assumption that $\{t_j\}_{j=1}^n$ is a uniform partition of the interval $\langle 0, T \rangle$ is not essential in this paper. All theorems, lemmas and estimates are valid if we consider an arbitrary partition $\{t_j\}_{j=1}^n$ of $\langle 0, T \rangle$, whose norm converges to zero with $m \to \infty$. In this case in Lemma 6, Theorem 1, Theorem 3 and 3' the estimates

$$\|u^n(t) - u(t)\|^2 \leqq C \max_{j=1,2,\ldots,n} |t_j - t_{j-1}|$$

hold.

The results from § 1 and § 2 can be applied to the following examples:

1. $$\frac{\partial u}{\partial t} + \sum_{i \in M} (-1)^{|i|} D^i[l_i(x) \, g_i(D^i u)] = f(x, t)$$

where $M \supset \{i, |i| = k\}$ is a subset of multiindices $\{i, |i| \leqq k\}$, $0 < c_1 = l_i(x) \in L_\infty(\Omega)$, $g_i(u) \in \mathcal{M}_3$ for $i \in M$ and $g_i'(s) \geqq 0$ for $|s| < \infty$.

The assumption (5) can be guaranted by the following properties:

i) $(d^i/ds^i) \, g_i(s)$ for $i \in M$ are continuous for $|s| < \infty$,

ii) $u_0(x) \in W^{2k}_\infty(\Omega)$ (Sobolev's space),

iii) $l_i(x) \in C^{|i|}(\bar{\Omega})$.

2.

$$\frac{\partial u}{\partial t} - \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\right) = f(x, t) \tag{39}$$

where $p \geqq 2$. For the condition (5) it suffices to assume $u_0(x) \in C^2(\bar{\Omega})$.

3.

$$\frac{\partial u}{\partial t} - \Delta u + a(x, u) = f(x, t),$$

$a(x, t)$ satisfy the Carathéodory conditions and

i) there exists $g(s) \in \mathscr{M}_3$ such that $|a(x, s)| \leqq C(1 + |g(s)|)$,

ii) $s\, a(x, s) \geqq C_1 s\, g(s) - C_2$,

iii) $(s_1 - s_2)\,[a(x, s_1) - a(x, s_2)] \geqq 0$ for a.e. $x \in \Omega$.

For the condition (5) it suffices to assume $u_0(x) \in W_2^2(\Omega)$ and $a(x, u_0) \in L_2(\Omega)$.

4. Let us consider the equation (39) with Neumann boundary conditions

$$u(x, 0) = u_0(x), \tag{39'}$$

$$\sum_{i=1}^{N} \left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i} \cos(\nu, x_i)\Big|_{\partial\Omega \times (0,T)} = 0$$

where $u_0(x) \in C^2(\bar{\Omega})$.

In this case we define A by the duality

$$(Au, v) = \int_\Omega \sum_{i=1}^{N} \frac{\partial v}{\partial x_i} \left|\frac{\partial(u_0 + u)}{\partial x_i}\right|^{p-2} \frac{\partial(u_0 + u)}{\partial x_i}\, dx.$$

We put $H = L_2(\Omega)$, $V = W_p^1(\Omega)$ and $u_0 = 0$ (in (1)). If we denote by $u(t) \equiv u(x, t)$ the solution of (1) (guaranteed by Theorem 1 and Theorem 2), then $u(x, t) + u_0(x)$ is the weak solution of (39), (39').

References

[1] *J. Kačur:* On existence of the weak solution for non-linear partial differential equations of elliptic type. I. Comment. Math. Univ. Carolinae, *11*, 1 (1970), 137–181.

[2] *J. Kačur:* On existence of the weak solution for non-linear partial differential equations of elliptic type. II. Comment. Math. Univ. Carolinae, *13*, 2 (1972), 211–225.

[3] *K. Rektorys:* On application of direct variational methods to the solution of parabolic boundary value problems of arbitrary order in the space variables. Czech. Math. Journal, *21 (96)*, (1971), 318–339.

[4] *J. Nečas:* Les méthodes directes en théorie des équations elliptiques. Prague, 1967.

[5] *J. Nečas:* Les équations elliptiques non linéaires. L'école d'été, Tchécoslovaquie, 1967. Czech. Math. Journal *2* (1969), 252—274.
[6] *E. Rothe:* Zweidimensionale parabolische Randweraufgaben. Math. ann. *102* (1930).
[7] *Т. Д. Вентцель:* Первая краевая задача для квазилинейново уравнения со многими пространственными переменными. Матем. сб. *41* (*83*), (1957), 499—520.
[8] *О. А. Ладыженская:* Решение в целом первой краевой задачи для квазилинейных параболических уравнений, ДАН СССР *107*, (1965), 636—639.
[9] *А. М. Ильин, А. С. Калашников, О. А. Олейник:* Линейные уравнения второго порядка параболического типа, УМН *17*, вып. 3, (1962), 3—146.
[10] *П. П. Мосолов:* Вариационные методы в нестационарных задачах. (Параболический случай.) Изв. АН СССР, *34* (1970), 425—457.
[11] *G. J. Minty:* On a "monotonicity" method for the solution of nonlinear equation in Banach spaces. Proc. N.A.S. USA *50* (1963), 1038—1041.
[12] *F. E. Browder:* Nonlinear elliptic boundary value problems. Bull. Amer. Math. Soc. 69, N. 6 (1963), 862—874.
[13] *F. E. Browder:* Strongly nonlinear parabolic boundary value problems. Amer. Journ. of Math. 86, 2 (1964).
[14] *М. А. Красносельский, Я. Б. Рутицкий:* Выпуклые функции и пространства Орлича Москва, 1958.
[15] *K. Yosida:* Functional analysis. Springer-Verlag, 1965.

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