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SOME REMARKS ABOUT THE THREE SPACES PROBLEM IN BANACH ALGEBRAS

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Let X be a Banach algebra, not necessarily commutative, with a unit element, and let I be a closed two-sided ideal in X. Thus X/I is also a Banach algebra with unit. Suppose (α) is some property of Banach algebras; we shall say that (α) is a regular property if the fact that I and X/I have property (α) implies that X has property (α). In the present paper we shall show the following properties to be regular:

(α_1) X is a B^*-equivalent algebra;

(α_2) X is a function algebra.

1. Let X be a Banach algebra with involution and a unit element. X is a B^*-equivalent algebra if it admits a norm (defining the correct topology) which is submultiplicative and for which

$$|x^*x| = |x|^2 \quad \text{for all} \quad x \in X .$$

It is well-known [1] that a Banach *-algebra X is B^*-equivalent if and only if there exists a constant $C > 0$ such that

$$\|x^*x\| \geqq C\|x\|^2$$

for all $x \in X$, in the original norm. Thus, in a B^*-equivalent algebra, $x_n^* x_n \to 0$ implies $x_n \to 0$.

Theorem 1.1. *Let X be a Banach algebra with a unit element and involution, let I be a closed *-ideal in X. If I and X/I are both B^*-equivalent algebras (being in a natural way Banach algebras with involution), then X is a B^*-equivalent algebra.*

Proof. Suppose on the contrary that X is not B^*-equivalent. Then there is a sequence (y_n) in X such that $\|y_n\| = 1$ and $y_n^* y_n \to 0$.

Let $\varkappa : X \to X/I$ be the quotient map. One has

$$\varkappa(y_n^* y_n) = \varkappa(y_n^*)\, \varkappa(y_n) = \varkappa(y_n)^*\, \varkappa(y_n) \to 0\,.$$

Hence, since X/I is a B^*-equivalent algebra, $\varkappa(y_n) \to 0$. Therefore there is a sequence (i_n) in I such that $y_n + i_n^* \to 0$ in X. Certainly the sequence (i_n^*) must be bounded and, therefore, (i_n) is bounded since the involution is continuous on I. Hence

$$(y_n i_n)^*\, (y_n i_n) = i_n^* y_n^* y_n i_n \to 0\,.$$

Since I is a B^*-equivalent algebra, $y_n i_n \to 0$. But now

$$i_n^* i_n = (y_n + i_n^*)\, i_n - y_n i_n \to 0$$

(for (i_n) is bounded and $y_n + i_n^* \to 0$). Therefore, $i_n \to 0$, since I is a B^*-equivalent algebra.

However, it is impossible that $i_n^* \to 0$ because $\|y_n\| = 1$ and $y_n + i_n^* \to 0$. So it must be the case that X is a B^*-equivalent algebra, which proves Theorem 1.1.

2. Now let X be a commutative Banach algebra with a unit element. X is called a *function algebra* if the norm in X is equivalent to the spectral norm $\|\cdot\|_{sp}$, defined by

$$\|x\|_{sp} = \lim_{n\to\infty} \|x^n\|^{1/n} = \sup_{f\in\mathfrak{M}(X)} |f(x)|$$

(where $\mathfrak{M}(X)$ is the structure space of X, i.e. the space of all nonzero multiplicative linear functionals).

Lemma 2.1. *The following conditions are equivalent:*

(i) *X is a function algebra,*

(ii) *if $x_n \in X$, $n = 1, 2, \ldots$, and $x_n^2 \to 0$, then $x_n \to 0$,*

(iii) *if $x_n \in X$, $n = 1, 2, \ldots$, and $x_n^2 \to 0$, then (x_n) is bounded,*

(iv) *there exists $C > 0$ such that $\|x\|^2 \leqq C\|x^2\|$ for every $x \in X$.*

Proof. The implications (i) ⇒ (ii) and (ii) ⇒ (iii) are trivial. Suppose that the condition (iii) holds. If (iv) is not true, then there exists a sequence (x_n) in X such that $\|x_n\| = 1$ and $\|x_n\|^2 > n^3\|x_n^2\|$, $n = 1, 2, \ldots$. Then $\|x_n^2 \,.\, n^2/\|x_n\|^2\| < 1/n$ and $(x_n \,.\, n/\|x_n\|)^2 \to 0$, but the sequence $(x_n \,.\, n/\|x_n\|)$ is not bounded. It remains to show (iv) ⇒ (i), but this is obvious.

Theorem 2.2. *Let X be a commutative*) Banach algebra with a unit element and I a closed ideal in X. If I and X/I are function algebras, so is X. (In other words, the property of being a function algebra is regular.)*

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*) Reviewer's remark: this assumption is superfluous.

Proof. We show that the condition (iii) of Lemma 2.1. holds. Suppose instead that $x_n^2 \to 0$ and the sequence (x_n) is not bounded. We may assume that $\|x_n\| \to \infty$, and let $y_n = x_n/\|x_n\|$, $(n = 1, 2, \ldots)$. Then certainly $y_n^2 \to 0$. If $\varkappa : X \to X/I$ is the quotient homomorphism, then

$$\varkappa(y_n^2) = (\varkappa(y_n))^2 \to 0 .$$

Therefore, since X/I is a function algebra, $\varkappa(y_n) \to 0$. Consequently, there exists a sequence (i_n) in I such that $y_n + i_n \to 0$, and so necessarily (i_n) is bounded. Thus $(i_n y_n)^2 = i_n^2 y_n^2 \to 0$, and since I is a function algebra, $i_n y_n \to 0$. Hence

$$i_n^2 = i_n(y_n + i_n) - i_n y_n \to 0 \quad \text{and} \quad i_n \to 0 .$$

But this is absurd, as $y_n + i_n \to 0$ and $\|y_n\| = 1$ for all n.

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Reference

[1] *V. Pták:* Banach algebras with involution, Manuscripta math. *6* (1972), p. 245—290.

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