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PAIRWISE SPLITTING LATTICE-ORDERED GROUPS

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Our terminology and notation in the theory of l-groups is mostly standard (as in [1]). There are some items which ought to be reviewed here at the outset. If M is a convex l-subgroup of the l-group G, and M is maximal with respect to not containing some element $0 \neq g \in G$ – all our groups being written additively unless otherwise indicated – then M is called a *value* in G, and also a *value* of g. The values in G are precisely the meet irreducibles of the lattice of convex l-subroups of G, ([1], theorem 1.6), and thus each value M has a cover M^*; in precise terms, M^* is the intersection of all the convex l-subgroups properly containing M.

G is *finite valued* if each non-zero element of G has at most finitely many values. G is *normal valued* if each value M in G is a normal subgroup of its cover M^*. It is well known that all finite valued l-groups are normal valued. There are now several known characterizations of normal valued l-groups. Shortly we shall need one which looks at the notion in the language of permutation groups.

Now to the central definition of this note: suppose G is an l-group; G is said to be *pairwise splitting* provided one has for all $0 \leqq x, y \in G$ a decomposition $x = x_1 + x_2$, where $x_1 \wedge x_2 = 0$, $x_1 \in G(y)$, the convex l-subgroup generated by y, and $x_2 \gg x_2 \wedge y$. (Recall that $a \ll b$ signifies that, for positive elements a and b, $na < b$ for each positive integer n.) With regard to $G(y)$ recall also that its positive cone consists of all $0 \leqq g \in G$ such that $g \leqq my$, for some natural number m. We should also like to make a local definition: for $0 < x, y \in G$, say that x *splits by* y if the above decomposition occurs. Thus G is pairwise splitting if and only if each $x > 0$ splits by each $y > 0$.

Recall that $0 < s \in G$ is *special* if it has only one value.

Lemma 1. *For an l-group G, each special element splits by any $y > 0$, and conversely, any $y > 0$ splits by any special element.*

Proof. Suppose $0 < y \in G$, and $0 < s \in G$ is special; let M be the lone value of s. If M is contained in a value of y then $s \in G(y)$; if M is incomparable to all the values of y, $s \wedge y = 0$, and certainly $s \gg s \wedge y$. Lastly, suppose M is strictly above values

$\{N_\lambda \mid \lambda \in \Lambda\}$ of y; then the values of $s \wedge y$ are precisely these N_λ, and it is clear that $s \gg s \wedge y$. This shows that s splits by y.

Next let us split y by s; if M is properly contained in some value of y we have $s \ll y$, i.e. $s \wedge y \ll y$. If M compares to no value of y, $s \wedge y = 0$, and once again we have what we need. And finally, suppose M contains values of y. There are two cases: if M is a value of y, then for a sufficiently large natural number n, $ns + M > y + M$. (Note: as s is special, M is normal in its cover M^*, [1], theorem 2.14, and we view the previous inequality in M^*/M, which is a subgroup of $\mathbb{R}$, the additive, naturally ordered reals.) Consider $y_1 = y \wedge ns$; clearly $y_1 \in G(s)$, and moreover M is its only value. It is easy to verify that $y_2 = y - y_1$ is disjoint from y_1, and finally, that $y_2 \gg y_2 \wedge s$, since in fact $y_2 \wedge s = 0$.

The other case occurs if M properly exceeds the set $\{N_\lambda \mid \lambda \in \Lambda\}$ of values of y. The reader can easily check that by taking $y_1 = y \wedge s$, y_1 has precisely the values N_λ $(\lambda \in \Lambda)$ and that if $y_2 = y - y_1$, $y_2 \wedge s = 0$, which is sufficient for our purposes.

It is conceptually easy to verify from lemma 1, albeit somewhat messy, that all finite valued l-groups are pairwise splitting. We present a more elegant argument a bit later. From the definitions it is evident that any hyper-archimedean l-group is pairwise splitting. (For the basic fact about hyper-archimedean l-groups, do consult [3].) In fact, if G is archimedean, it is pairwise splitting exactly when it is hyper-archimedean.

We now present a class of (abelian) l-groups which are pairwise splitting, yet are neither hyper-archimedean nor finite valued. First, let us review a definition: a p.o. set Λ is a *root system* if no two incomparable elements have a common lower bound. For each root system Λ construct the *v-group* $V = V(\Lambda, R_\lambda)$, where for each $\lambda \in \Lambda$ $R_\lambda = \mathbb{R}$, consisting of all real valued functions whose supports satisfy the ascending chain condition. It is well known that this is an l-group, and that any abelian l-group can be embedded in one of these. ([1], theorem 4.3)

Let $P = P(\Lambda, R_\lambda) = \{v \in V \mid \{v_\lambda \mid \lambda \in \Lambda\}$ is a bounded set of real numbers$\}$. It is easy to verify that P is an l-subgroup of $V(\Lambda', R_\lambda)$, and that P is pairwise splitting.

Our first theorem makes some later arguments easier.

Theorem 2. *A pairwise splitting l-group is normal valued.*

Before giving the proof, a few comments are in order. We will need a result of John Read (see [5]) which gives a characterization of normal valuedness in terms of permutation groups. So we should review briefly the basic aspects of that theory. If T is a totally ordered set, we let $\mathscr{A}(T)$ denote the l-group of order preserving permutations of T under composition. Holland's well known theorem ([1], theorem 1.10) states that each l-group may be embedded in some $\mathscr{A}(T)$.

Read's theorem can then be stated thusly: G is a normal valued l-group if and only if in G, thought of as an l-subgroup of $\mathscr{A}(T)$ for a suitable chain T, the following *doesn't* occur: there are elements g, $h > 1$, and an $s \in T$ so that $sh^{-1} < s < sg$, $sgh = sg$ and $sh^{-1}g = sh^{-1}$.

Now for the proof of theorem 2: suppose G is pairwise splitting but not normal valued. Then, with the same notation of the previous paragraph, there are elements $1 < g, h \in G$, so that $sh^{-1} < s < sg$, $sgh = sg$ and $sh^{-1}g = sh^{-1}$, for some $s \in T$.

Split g by h: $g = ab$, with $a \wedge b = 1$, $a \in G(h)$ and $b \gg b \wedge h$. Since a and b are disjoint they commute and so $tg = ta$ or tb, for all $t \in T$. Suppose $sg = sa$; by iterating h, $sah^n = sa$, for each natural number n. Since $a \in G(h)$ this implies that $sa^2 = sa$, ie. $sa = s$, which is a contradiction. We conclude therefore that $sg = sb$.

As $b \gg b \wedge h$, we have that if $tb > t$, then $tb > th$. For suppose to the contrary, that $tb \leqq th$; then $tb^2 \leqq thb$, $tbh \leqq th^2$, and $\min\{tb^2, tbh\} \leqq tb$. Since $tb > t$, the inequality $tb^2 \leqq tb$ is impossible, whence $tbh \leqq tb$, and in fact $tbh = tb$. As $tb \leqq th$ we get that $tbh \leqq th$, and from that $tb \leqq t$, which is a contradiction. We conclude then that $tb > th$ as claimed.

In summation then: if $tb > t$ it follows that $tb > th$. But we know that $sb = sg > s$, and deduce immediately that $sb > sh$, ie. $sg > sh$. By symmetry, splitting h by g, we obtain $sh > sg$. This absurdity implies that G must be normal valued.

One immediate consequence of theorem 2 is the uniqueness of the splitting in a pairwise splitting l-group.

Corollary. *Suppose G is pairwise splitting; the splitting decomposition of $x > 0$ by $y > 0$ is unique.*

Proof. Suppose $x = x_1 + x_2 = a_1 + a_2$, where $x_1, a_1 \in G(y)$, $x_1 \wedge x_2 = a_1 \wedge a_2 = 0$, and $x_2 \gg x_2 \wedge y$, $a_2 \gg a_2 \wedge y$. $a_2 = (a_2 \wedge x_1) + (a_2 \wedge x_2)$, and if $a_2 \wedge x_1 > 0$, let Q be one of its values. Q is then a value of x_1 or a_2.

If Q is a value of x_1 then $y \notin Q$, and Q is a value of x with $x + Q = x_1 + Q$. Since $y \wedge a_2 \ll a_2$ and G is normal valued, Q is properly contained in a value of a_2, which is a contradiction since a value of a_2 is also a value of x. If on the other hand Q is a value of a_2, and $x_1 \notin Q^*$, then a value N of x_1 – and hence x – exceeds Q properly. Since N and Q are values of x this is absurd.

So $a_2 = a_2 \wedge x_2$, and by symmetry $x_2 = a_2 \wedge x_2$. Thus, $x_2 = a_2$ and hence $x_1 = a_1$.

The next lemma is important in a technical sense.

Lemma 3. *Suppose G is normal valued and $0 < x, y \in G$; the following are equivalent.*

a) *x splits by y;*

b) *there is a positive integer n so that for each value M of y, not properly contained in a value of x, we have $ny + M \geqq x + M$;*

c) *there is a plenary subset $\mathscr{P}$ of values, and a positive integer n, so that if M is a value of y in $\mathscr{P}$, not properly contained in a value of x (in $\mathscr{P}$), then $ny + M \geqq x + M$.*

(Recall: a subset $\mathscr{P}$ of values is *plenary* if 1) each $0 \neq g \in G$ has a value in $\mathscr{P}$, and 2) if $x \notin M \in \mathscr{P}$ there is a value $N \in \mathscr{P}$ of x containing M.)

Proof. b) → c) is obvious. To see that a) implies b), split x by y: $x = x_1 + x_2$ with $x_1 \wedge x_2 = 0$, $x_1 \in G(y)$ and $x_2 \gg x_2 \wedge y$. There is a natural number n such that $ny \geqq x_1$. If M is a value of y, not properly contained in a value of x, and $x \in M$ it is clear that $ny + M > x + M$. If $x \notin M$ then M is also a value of x; in fact, M must be a value of x_1 since otherwise x_2 would have a value in common with $x_2 \wedge y$. As $x_2 \gg x_2 \wedge y$ and G is normal valued, this would be a contradiction. Since $ny \geqq x_1$, $ny + M \geqq x_1 + M = x + M$.

c) → a) Suppose $\mathscr{P}$ is the plenary subset of values with the stipulated condition; suppose further that m is the smallest natural number satisfying condition c) with a strict inequality. Put $z_n = (n + 1) y \wedge x - ny \wedge x$ $(n \geqq m)$. If M is a value of z_n in $\mathscr{P}$, then either $(n + 1) y \wedge x$ or $ny \wedge x$ are not in M; eitherway $x, y \notin M$. M is contained in a value N (resp. Q) of x (resp. y).

If $N \subset Q$ then $ny \wedge x + N = x + N$, and $z_n \in N$, ie. $M \subset N$. Likewise if $N = Q$, $M \subset N$. But $z_n \in M^*$ and so $ny \wedge x + M^* = (n + 1) y \wedge x + M^*$. The cosets modulo M^* form a chain, and the canonical map from G/M^* to G/Q is order preserving; consequently, by the choice of $n \geqq m$, $ny + M^* > x + M^*$, and $ny + M > x + M$ as well. Then $z_n + M = [(n + 1) y \wedge x + M] - [ny \wedge x + M] = M$, a contradiction.

It follows that $Q \subset N$ and $ny \wedge x + Q = ny + Q$, which implies that $z_n + Q = y + Q$ and hence that $M = Q$. To summarize: every value of z_n in $\mathscr{P}$ is properly contained in a value of x, ie. $z_n \ll x$. But we've also shown that every value of z_n in $\mathscr{P}$ coincides with one of y and $y + M = z_n + M$ for each such value. This, and additional details, as in the previous paragraph show that $y \geqq z_n$, and $z_n \wedge (y - z_n) = 0$.

Next, if M is a value in $\mathscr{P}$ for z_n then by the above $ky \wedge x + M = ky + M$, for all $k \geqq m$, so that $z_m + M = z_n + M$. It is also clear by now that all the z_n $(n \geqq m)$ have exactly the same values, and in fact that $z_n = z_m$, for all $n \geqq m$.

So we have $c = z_m \leqq y$ such that $c \wedge (y - c) = 0$ and $c \ll x$. This is equivalent to the splitting of x by y, and we prove it in the following lemma. As soon as that has been done the proof of lemma 3 will be complete.

Lemma 4. *Take G to be normal valued, and $0 < x, y \in G$. x splits by y if and only if there is an element $c \geqq 0$ such that $c \leqq y$, $c \wedge (y - c) = 0$, $c \ll x$ and $c = (n + 1) y \wedge x - ny \wedge x$, for all n exceeding a suitably chosen natural number m.*

Proof. *Necessity.* Let us suppose that x splits by y: let $x = x_1 + x_2$ be the splitting decomposition, with $x_1 \in G(y)$. Select n sufficiently large, so that $ny \geqq x_1$. Then $ny \wedge x = x_1 \vee (ny \wedge x_2)$ and $n(x_2 \wedge y) = x_2 \wedge n(x_2 \wedge y) = x_2 \wedge ny$, so that $ny \wedge x = x_1 \vee n(y \wedge x_2) = x_1 + n(y \wedge x_2)$. Thus, $(n + 1) y \wedge x - ny \wedge$

$\wedge\, x = y \wedge x_2$ which is independent of n. Set $c = y \wedge x_2$; clearly $y \geqq c$ and $c \ll x_2 \leqq x$. If $d = y - c$ then d is disjoint from $x_2 - c$, and since $x_2 \gg c$, and G is normal valued, x_2 and $x_2 - c$ have the same values, proving that $d \wedge x_2 = 0$, ie. $d \wedge c = 0$.

Sufficiency. Write $y = c + d$, with $c \wedge d = 0$, $c \ll x$ and so that $c = (n + 1)\, y \wedge x - ny \wedge x$, for all positive integers exceeding some m. If M is a value of d, $c \in M$ and $d + M = y + M$, with $y \in M^*$. So $(n + 1)\, y \wedge x$ and $ny \wedge x$ are both in M^*, $(n + 1)\, y \wedge x + M = ny \wedge x + M$, whence $ny + M \geqq x + M$ and $x \in M^*$.

For each positive integer k, $x - (x \wedge kd)$ and $kd - (x \wedge kd)$ are disjoint, and if M is a value of kd then $x \in M^*$ by the above paragraph. For large enough k, $kd - (x \wedge kd) + M = ky + M - (x \wedge ky) + M = ky - x + M > M$, ie. M is a value of $kd - (x \wedge kd)$. Conversely, it is easily seen that if k is large enough, any value of $kd - (x \wedge kd)$ is a value of kd. Thus, for a sufficiently large positive integer k, kd and $kd - (x \wedge kd)$ have precisely the same values. Consequently, kd and $x - (x \wedge kd)$ are disjoint, and so are $x \wedge kd$ and $x - (x \wedge kd)$. For such a k pick $x_1 = x \wedge kd$ and $x_2 = x - x_1$; then $x_1 \wedge x_2 = 0$, $x_1 \leqq kd \leqq ky$ and $x_2 \wedge y = x_2 \wedge c + x_2 \wedge d = x_2 \wedge c + (x - (x \wedge kd)) \wedge d = x_2 \wedge c$. Also $c = x_1 \wedge c + x_2 \wedge c = (x \wedge kd \wedge c) + x_2 \wedge c = x_2 \wedge c$; hence $c \ll x_2$ and therefore $x_2 \wedge y \ll x_2$.

This lemma says that in a pairwise splitting l-group G there is a "second splitting" besides the usual one. If $0 < x$, $y \in G$ we can split x by y: $x = x_1 + x_2$ with $x_1 \in G(y)$, $x_1 \wedge x_2 = 0$ and $x_2 \gg x_2 \wedge y$. According to lemma 4 we can also write $x = a + b$, with $a \ll y$, $a \wedge b = 0$ and so that no value of b is properly contained in one of x. It is clear that $a \leqq x_1$, so that $x_1 = a + (x_1 \wedge b)$, and $x = a + (x_1 \wedge b) + x_2$. We therefore get a splitting of x by y into a trio of pairwise disjoint elements, so that every value of the first is a value of x properly contained in one of y, every value of the second is a value of x coinciding with a value of y, while every value of the third is a value of x which contains a value of y properly, or else is incomparable to all of them.

As a consequence of lemma 3 we obtain an embedding theorem for abelian pairwise splitting l-groups.

Theorem 5. *If G is an abelian pairwise splitting l-group then there is an l-embedding σ into $V(\Lambda, R_\lambda)$, for a suitable root system Λ, so that for each pair $0 < x$, $y \in G$ there is a positive integer n, such that for each maximal component λ of $y\sigma$ not strictly below a maximal component of $x\sigma$, we have $ny\sigma_\lambda \geqq x\sigma_\lambda$.*

Sketch of proof and remarks. The l-embeddings of abelian l-groups into v-groups arise out of plenary subsets of values of the l-group, and each plenary subset is a root system. (See [1], chapter 4 for details.) The maps themselves are *v-embeddings*; ie. for each $g > 0$ there is a one to one correspondence between its values in the plenary subset and the maximal components of its image $g\sigma$, so that if $\lambda \in \Lambda$ and the value M correspond, then $g\sigma_\lambda = g + M$. (Note: R_λ is in fact

M^*/M.) With these remarks it is evident how lemma 3 gives the desired result immediately. We should only observe that in view of our comments every such embedding has the property of the theorem.

Our next result concerns special elements. Call $g > 0$ in the l-group G *indecomposable* if $g = a + b$ with $a \wedge b = 0$ implies that $a = 0$ or $b = 0$. Each special element is indecomposable, and if values of special elements form a plenary subset then ([2]) indecomposable elements are special.

Proposition 6. *If G is a pairwise splitting l-group and $0 < g \in G$ is indecomposable, then it is special.*

Proof. Suppose g has two values M and N; it easy to pick strictly positive elements a and b so that $a \wedge b = 0$, $a \in M^* \setminus M$, $b \in M$ and $b \in N^* \setminus N$, $a \in N$. Consider $z = (g \wedge a) + (g \wedge b)$; notice that neither $g \wedge a$ nor $g \wedge b$ is zero. We split g by z; but by the indecomposability of g either $g \in G(z)$ or $g \gg g \wedge z$, and the latter is impossible. So there is a positive integer m such that $g \leqq mz = m(g \wedge a) + m(g \wedge b)$. The Riesz interpolation property then gives $g = g_1 + g_2$ with $g_1 \leqq m(g \wedge a)$ and $g_2 \leqq m(g \wedge b)$, and as g is indecomposable and $g_1 \wedge g_2 = 0$, we conclude that $g_1 = 0$ or $g_2 = 0$. Thus, $g \leqq m(g \wedge a)$ or else $g \leqq m(g \wedge b)$; but $g \wedge a \in N$ while $g \notin N$, and $g \wedge b \in M$ while $g \notin M$. This is a contradiction, proving that g is special.

Our final results have to do with the theory of torsion classes (see [4]); the author first looked at pairwise splitting l-groups with torsion classes in mind. It turns out that the class $\mathscr{P}\mathscr{s}$ of pairwise splitting l-groups do form a torsion class, but the proof is non-trivial.

A class $\mathscr{T}$ closed with respect to isomorphic copies is a *torsion class* provided it is closed under taking a) convex l-subgroups, b) l-homomorphic images and c) the join of convex l-subgroups contained in $\mathscr{T}$.

Theorem 7. *$\mathscr{P}\mathscr{s}$ is a torsion class.*

Proof. The closure relative to convex l-subgroups and l-homomorphic images is obvious. As for the third property, consider an l-group which is the join of convex l-subgroups C_i $(i \in I)$ belonging to $\mathscr{P}\mathscr{s}$. In view of theorem 2, and since the class of normal valued l-groups is a torsion class, we conclude that G is normal valued. The rest of the proof depends upon the lemma:

Lemma 8. *Suppose G is normal valued, and $0 < x_1, x_2, y \in G$. If x_1 and x_2 split by y then so does $x_1 + x_2$.*

Proof. We use lemma 3: select a value M of y not properly contained in any value of $x_1 + x_2$, then M is not properly contained in any value of $x_1 + x_2$, then M is not properly contained in any values of x_1 or x_2. There exist positive integers m_1

and m_2 (which are independent of M) such that $m_1 y + M \geqq x_1 + M$ and $m_2 y + M \geqq x_2 + M$. So if $m = m_1 + m_2$, $my + M \geqq x_1 + x_2 + M$. By lemma 3, $x_1 + x_2$ splits by y.

Now we continue with the proof of theorem 7: let $P = \{0 \leqq x \in G \mid x$ splits by each $0 < y \in G$, and each $0 \leqq z \leqq x$ has the same property$\}$. By lemma 8 $x \in P$ if and only if $G(x) \in \mathscr{P}_{\mathscr{S}}$; this is admittedly not obvious, but we shall leave its verification to the reader anyway.

So we have $C = \bigvee_{i \in I} C_i$, with each $C_i \in \mathscr{P}_{\mathscr{S}}$. If $0 < g \in G$ then $g = g_1 + g_2 + \ldots + g_n$, with $g_k \in C_{i_k}$. $G(g_k) \in \mathscr{P}_{\mathscr{S}}$ and so $g_k \in P$, for each $k = 1, 2, \ldots, n$. Again by lemma 8, $g \in P$ and therefore splits by each positive element of G. Conclusion: G is pairwise splitting.

Not every torsion class is closed under taking l-subgroups, but $\mathscr{P}_{\mathscr{S}}$ is.

Proposition 9. *If $G \in \mathscr{P}_{\mathscr{S}}$ and H is an l-subgroup of G, then $H \in \mathscr{P}_{\mathscr{S}}$.*

Proof. Suppose $0 < x, y \in H$; split x by y in G: $x = x_1 + x_2$, with $x_1 \wedge x_2 = 0$, $x_1 \in G(y)$ and $x_2 \gg x_2 \wedge y$. Since $x_2 \wedge y = (n + 1) y \wedge x - ny \wedge x$, for a sufficiently large positive integer n, $x_2 \wedge y \in H$. Also $y - x_2 \wedge y \in H$, and by the proof of lemma 4, $x_1 = x \wedge k[y - x_2 \wedge y]$, for a suitable natural number k, so $x_1 \in H$ and hence $x_2 \in H$.

By way of closing comments we should mention that $\mathscr{P}_{\mathscr{S}}$ is not closed under products or extensions, since the torsion class of hyper-archimedean l-groups is not. On the other hand it is easy to show that $\mathscr{P}_{\mathscr{S}}$ is closed with respect to lexicographic extensions, and hence restricted wreath products.

The author also suspects in view of proposition 6, that the assumption of complete distributivity on a pairwise splitting l-group should give some further restrictions on the l-group. For example, it can be shown that if G is pairwise splitting, completely distributive and has no infinite descending chains $a_1 \gg a_2 \gg \ldots \gg a_n \gg \ldots$ of strictly positive elements, then each $0 < g \in G$ exceeds a special element.

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