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N-TH ORDER ORDINARY DIFFERENTIAL SYSTEMS UNDER STIELTJES BOUNDARY CONDITIONS*)

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1. Introduction. Several recent papers by KRALL [14], [15], [16], [17], and BROWN [2], [3], [5] have considered the problem of determining adjoints to differential-boundary systems with boundary conditions represented by Stieltjes measures. In a number of these papers a theory of linear relations in dually paired spaces has proved useful in allowing the construction of adjoint relations for operators defined on nondense domains.

Many of the results of this paper (with the exception of Theorem 4.10) have been proven in restricted cases or by the use of methods which can be considerably sharpened in one or more of the papers mentioned above.

In this paper the necessary improvements are made to generalize the previous results and make them more precise. In particular we show that the theory rests on an appropriate Green's relation and the so-called "*linear dependence principle.*"

A major advantage of the new technique is that it does not assume the existence of fundamental solutions or Green's functions. Thus our results hold both for conventionally "regular" operators and singular operators with quite irregular coefficients.

The results of this paper may also be applicable to the theory of selfadjoint extensions of symmetric differential operators which have been investgated by CODDINGTON [6], [7], [8], [9]. In another direction there are interesting applications to the theory of splines (already briefly explored in [4]). Furthermore generalized boundary value problems of the type considered here have been encountered in attempts to obtain "feedback type" formulas for controllers in the control theory of hyperbolic partial differential equations (see RUSSELL [20]). In subsequent papers we hope to explore these connections in more detail.

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2. Preliminaries. Let X denote the Banach space $\mathscr{L}_m^p[0, 1]$, $1 \leqq p < \infty$, consisting of m-dimensional vectors $y = (y_1, \ldots, y_m)^t$ under the norm

$$\|y\| = \left[\int_0^1 \left[\sum_{i=1}^m |y_i|^2\right]^{p/2} \mathrm{d}t\right]^{1/p}.$$

$X^* = \mathscr{L}_m^q[0, 1]$, $1/p + 1/q = 1$, denotes the dual space. In X we wish to consider the n-th order m-dimensional system consisting of the formal differential expression

$$ly = \sum_{k=0}^n a_k y^{(n-k)},$$

where the a_k, $k = 0, \ldots, n$, are $m \times m$ matrix coefficients with essentially bounded measurable (hence integrable) components, together with the boundary conditions

$$U_i y = \sum_{j=1}^n A_{ij} y^{(n-j)}(0) + \sum_{j=1}^n B_{ij} y^{(n-j)}(1) + \sum_{j=1}^n \int_0^1 \mathrm{d}K_{ij} y^{(n-j)} = 0,$$

$i = 1, \ldots, r$, $0 \leqq r \leqq mn$, (A_{ij}, B_{ij} are m-dimensional row vectors),

$$V_i y = \sum_{j=1}^n \int_0^1 \mathrm{d}L_{ij} y^{(n-j)} = 0,$$

$i = 1, \ldots, s$, $0 \leqq s < \infty$, where the $1 \times m$ matrix valued (m.v.) measures K_{ij} and L_{ij} are of bounded variation and nonatomic at the endpoints 0, 1 (i.e., $\mathrm{d}K_{ij}(0) = \mathrm{d}K_{ij}(1) = \mathrm{d}L_{ij}(0) = \mathrm{d}L_{ij}(1) = 0$). Also K_{ij} and L_{ij}, $j = 1, \ldots, n - 1$, are singular with respect to Lebesgue measure. The last assumption is for convenience, since, for example, if K_{ij} has an absolutely continuous part K_{ij}, it satisfies

$$\int_0^1 \mathrm{d}\tilde{K}_{ij} y^{(n-j)} = \tilde{K}'_{ij} y^{(n-j-1)}\Big|_0^1 - \int_0^1 \mathrm{d}\tilde{K}'_{ij} y^{(n-j-1)}.$$

By assuming that $\tilde{K}'_{ij}$ is also of bounded variation, etc., repeated integration by parts results in singular measures. The terms evaluated at 0 and 1 can be absorbed in the coefficients A_{ij} and B_{ij}.

These boundary forms may be expressed in vector form as

$$Uy = \sum_{j=1}^n A_j y^{(n-j)}(0) + \sum_{j=1}^n B_j y^{(n-j)}(1) + \sum_{j=1}^n \int_0^1 \mathrm{d}K_j y^{(n-j)}, \quad Vy = \sum_{j=1}^n \int_0^1 \mathrm{d}L_j y^{(n-j)},$$

where

$$Uy = \begin{pmatrix} U_1 y \\ \vdots \\ U_r y \end{pmatrix}, \quad A_j = \begin{pmatrix} A_{1j} \\ \vdots \\ A_{rj} \end{pmatrix}, \quad B_j = \begin{pmatrix} B_{1j} \\ \vdots \\ B_{rj} \end{pmatrix}, \quad K_j = \begin{pmatrix} K_{1j} \\ \vdots \\ K_{rj} \end{pmatrix},$$

$$Vy = \begin{pmatrix} V_1 y \\ \vdots \\ V_s y \end{pmatrix}, \quad L_j = \begin{pmatrix} L_{1j} \\ \vdots \\ L_{sj} \end{pmatrix}.$$

Thus A_j, B_j are $r \times m$ matrices, K_j are $r \times m$ m.v. measures and L_j are $s \times m$ m.v. measures.

Our main result is the construction of the dual system in X^* (Theorem 4.10). In so doing, we will also exhibit Green's formula. (Theorem 4.6).

3. The Operator L. Let D' denote those elements $y \in X$ which satisfy

1. $y, y', \ldots, y^{(n-1)}$ exist.
2. $y^{(n-1)}$ is absolutely continuous on $[0, 1]$.
3. $ly = \sum_{k=0}^{n} a_k y^{(n-k)}$ exists a.e. and is in X.

D_0' is the set of all $y \in D'$ such that $y^{(j)}(0) = y^{(j)}(1) = 0$ $(j = 0, 1, \ldots, n-1)$. Let D denote those elements $y \in X$ which satisfy

1. $y \in D'$.
2. $Uy = 0$.
3. $Vy = 0$.

Additionally let $D_0 = D \cap D_0'$.

The operator L is defined by setting

$$Ly = ly$$

for all $y \in D$.

The operator L subsumes as special cases the operators considered by a number of authors (see [18] for a list) involving end point boundary conditions, interior point boundary conditions, integral boundary conditions or combinations thereof, as well as general Stieltjes integral conditions, which have appeared only recently (see [2] and [3] for the n-th order case).

4. The Adjoint Relation L^*. In order to calculate the adjoint it is convenient to rewrite the boundary conditions in a more appropriate form. Note that integration by parts (see McShane [19], pp. 332–335) yields for any $y \in D'$

$$\int_0^1 \mathrm{d}K_j y^{(n-j)} = K_j y^{(n-j)}\Big|_0^1 - \left[\int_0^1 K_j \,\mathrm{d}\xi_{j-1}\right] y^{(n-j+1)}(1) + \tag{4.1}$$
$$\cdots$$
$$+ (-1)^{j-1}\left[\int_0^1 \cdots \int_0^{\xi_{j-2}} K_j \,\mathrm{d}\xi_{j-1} \cdots \mathrm{d}\xi_1\right] y^{(n-1)}(1) +$$
$$+ (-1)^j \int_0^1 \left[\int_0^t \cdots \int_0^{\xi_{j-2}} K_j \,\mathrm{d}\xi_j \cdots \mathrm{d}\xi_1\right] y^{(n)} \,\mathrm{d}t,$$

and L_j satisfies a similar expression. Hence if $y \in D_0'$ the boundary conditions take the form (with j replaced by $j+1$)

$$\hat{U}y^{(n)} = \int_0^1 \left(\sum_{j=0}^{n-1} (-1)^{j+1} \left[\int_0^t \dots \int_0^{\xi_{j-1}} K_{j+1}\, d\xi_j \dots d\xi_1 \right] \right) y^{(n)}\, dt, \tag{4.2}$$

$$\hat{V}y^{(n)} = \int_0^1 \left(\sum_{j=0}^{n-1} (-1)^{j+1} \left[\int_0^t \dots \int_0^{\xi_{j-1}} L_{j+1}\, d\xi_j \dots d\xi_1 \right] \right) y^{(n)}\, dt.$$

4.1. Lemma. *Suppose $r \in X$. Then*

$$\hat{U}r = 0, \quad \hat{V}r = 0, \quad \int_0^1 r dt = 0, \quad \int_0^1 tr\, dt = 0, \dots, \int_0^1 t^{n-1} r\, dt = 0 \tag{4.3}$$

if and only if $r = y^{(n)}$ for some y in D_0.

Proof. If $r = y^{(n)}$ for some y in D_0 it is clear from (4.1), (4.2) that the first two conditions of (4.3) are satisfied. Repeated integration by parts also shows that the last n orthogonality conditions of the lemma are satisfied. Conversely if $r \in X$ satisfies (4.3), define the n-fold integral

$$y = \int_0^t \dots \int_0^{\xi_{j-1}} r\, dt.$$

It is easily checked that ly exists a.e. in X and that $y(0), y'(0), \dots, y^{(n-1)}(0) = 0$. The last n orthogonality conditions of (4.3) show (again by repeated integration by parts) that $y(1), y'(1), \dots, y^{(n-1)}(1) = 0$. Next substituting r into the first two conditions of (4.3) and using (4.1) it follows that

$$\sum_{j=1}^{n} \int_0^1 dK_j y^{(n-j)} = 0,$$

$$\sum_{j=1}^{n} \int_0^1 dL_j y^{(n-j)} = 0.$$

This shows that r is the n^{th} derivative of a function in D_0, and completes the proof of the Lemma.

4.2. Remark. $\hat{U}y$, $\hat{V}y$ and the integrals $\int_0^1 r\, dt \dots, \int_0^1 t^{(n-1)} r\, dt$ constitute a system of $r + s + mn$ functionals $\psi_i : X \to \mathbb{C}$. Lemma 4.1 says that *the intersection of the nullspaces of ψ_i ($\cap \operatorname{Ker}' \psi_i$) is precisely $D_0^{(n)} \cap X$* where $D_0^{(n)}$ denotes the set of n^{th} derivatives of D_0.

We now introduce a sequence of "partial adjoints". Let ϕ and ψ be fixed parametric vectors in $\mathbb{C}^r$, $\mathbb{C}^s$ respectively, and define

$$
\begin{aligned}
l_0^+ z &= \quad a_0^* z\,,\\
l_1^+ z &= -[l_0^+ z + K_1^* \phi + L_1^* \psi]' + a_1^* z\,,\\
l_2^+ z &= -[l_1^+ z + K_2^* \phi + L_2^* \psi]' + a_2^* z\,,\\
&\dots\\
l_n^+ z &= -[l_{n-1}^+ z + K_n^* \phi + L_n^* \psi]' + a_n^* z\,.
\end{aligned}
$$

It is assumed that z, ϕ and ψ are appropriately chosen so the indicated differentiations can be performed, and that $l_j^+ z + K_{j+1}^* \phi + L_{j+1}^* \psi$, $j = 0, \dots, n-1$ are absolutely continuous functions. Note that since $K_1 \dots, K_{n-1}$ and $L_1, \dots, L_{n-1}$ are singular and of bounded variation, hence differentiable almost everywhere, that

$$
\begin{aligned}
l_0^+ z &= \quad a_0^* z\,,\\
l_1^+ z &= -[l_0^+ z]' + a_1^* z\,, \quad \text{a.e.}\,,\\
l_2^+ z &= -[l_1^+ z]' + a_2^* z\,, \quad \text{a.e.}\,,\\
&\dots\\
l_{n-1}^+ z &= -[l_{n-2}^+ z]' + a_{n-1}^* z\,, \quad \text{a.e.}\,,\\
l_n^+ z &= -[l_{n-1}^+ z]' + a_n^* a + K_n^{*\prime} \phi + L_n^{*\prime} \psi\,, \quad \text{a.e.}\,,
\end{aligned}
$$

where $K_n^{*\prime}$ and $L_n^{*\prime}$ denote the derivatives of the absolutely continuous parts of K_n^* and L_n^*. Note further that $-l_{n-1}^+ z' + a_n^* z$ denotes the traditional formal adjoint of ly and that

$$
l_j^+ z = \sum_{i=0}^{j} (-1)^{j-i} (a_i^* z)^{(j-i)} \quad \text{a.e.}\,, \qquad j = 0, \dots, n-1\,.
$$

The proof of the following purely algebraic result may be found in KELLEY and NAMIOKA [12] p. 7.

4.3. Lemma. (*Linear dependence principle.*) *Suppose λ, $\psi_1, \dots, \psi_n$ are a finite collection of linear functionals (possibly unbounded) defined on a (linear) space. If* $\ker \lambda \supset \cap \ker \psi_i$ *then λ is a linear combination of the functionals ψ_i.*

In what follows let D^* denote the domain of the adjoint relation $L^* \subset X^* \times X^*$ and $L^* z$ an element in $R(L^*)$ such that $\langle z, L^* z\rangle$ is in the graph of L^* [1]).

4.4. Theorem. *Let $z \in D^*$ then for any element $L^* z$ there exist parameters ϕ, ψ such that $l_j^+ z$, $j = 0, \dots, n$ exists, and*

$$
L^* z = l_n^+ z\,.
$$

Proof. Let $y \in D_0$. If $z \in D^*$, then

$$
[y, L^* z] = [Ly, z]\,,
$$

[1]) For the basic facts concerning linear relations on Hilbert space consult ARENS [1].

or

$$\int_0^1 (L^*z)^* \, y \, \mathrm{d}t = \int_0^1 z^*\Big(\sum_{k=0}^{n} a_k y^{(n-k)}\Big) \, \mathrm{d}t \, .$$

Hence

$$(4.4) \qquad \int_0^1 \Big[z^* a_0 + \sum_{j=1}^{n} (-1)^j \int_0^t \ldots \int_0^{\xi_{j-1}} z^* a_j \, \mathrm{d}\xi_j \ldots \mathrm{d}\xi_1 + \\ + (-1)^{n+1} \int_0^t \ldots \int_0^{\xi_{n-1}} (L^*z) \, \mathrm{d}\xi_n \ldots \mathrm{d}\xi_1 \Big] \, y^{(n)} \, \mathrm{d}t = 0 \, .$$

The expression in brackets on the left side of (4.4) determines a functional $\lambda :$ $: \mathscr{L}^p_m[0, 1] \to C$ whose null space contains $D_0^{(n)} \cap X$. Now $D_0^{(n)} \cap X = \cap \ker \psi_i$. (cf. Lemma 4.1 and Remark 4.2). Therefore by the linear dependence principle (Lemma 4.3) λ is a linear combination of the functionals ψ_i. That is, there exist constants $C_0, \ldots, C_{n-1}$ and vectors $\phi \in C^r$, $\psi \in C^s$ such that

$$(4.5) \qquad a_0^* z + \sum_{j=1}^{n} (-1)^j \int_0^t \ldots \int_0^{\xi_{j-1}} a_j^* z \, \mathrm{d}\xi_j \ldots \mathrm{d}\xi_1 + \\ + (-1)^{n+1} \int_0^t \ldots \int_0^{\xi_{n-1}} (L^*z) \, \mathrm{d}\xi_n \ldots \mathrm{d}\xi_1 = \\ = \sum_{j=0}^{n-1} (-1)^{j+1} \Big[\int_0^t \ldots \int_0^{\xi_{j-1}} K_{j+1}^* \, \mathrm{d}\xi_j \ldots \mathrm{d}\xi_1 \Big] \phi + \\ + \sum_{j=0}^{n-1} (-1)^{j+1} \Big[\int_0^t \ldots \int_0^{\xi_{j-1}} L_{j+1}^* \, \mathrm{d}\xi_j \ldots \mathrm{d}\xi_1 \Big] \psi + \sum_{j=0}^{n-1} t^j C_j \, .$$[2]

If this is unraveled, successive differentiations are possible, and the result follows.

4.5. Remark. For a fixed z in D^* the parameters ϕ and ψ shown to exist in the previous theorem are not necessarily unique. The right side of (4.5) is unchanged if we substitute for ϕ, ψ a pair of vectors $\langle \phi', \psi' \rangle$ satisfying the condition

$$(4.6) \qquad K_{j+1}^*(\phi' - \phi) + L_{j+1}^*(\psi - \psi') = 0 \, , \quad j = 0, \ldots, n-1 \, .$$

Consequently the expressions $l_{j+1}z$, $j = 0, \ldots, n-1$, defined with reference to $\langle \phi', \psi' \rangle$ also are unchanged, and satisfy the equivalent structural properties

$$l_j^+ z + K_{j+1}^* \phi + L_{j+1}^* \psi \in AC_m[0, 1] \, ,$$

[2]) Note that if no interior boundary conditions are present (i.e., K_j, L_j are zero), $D_0 = D_1'$ and the transition from (4.4) to (4.5) rests on a well known generalization of the Fundamental Lemma of the Calculus of Variations (cf. HESTENES [12], p. 105).

$j = 0, \ldots, n - 1$. To summarize $\langle \phi', \psi' \rangle$ will determine the same expressions $l_0^+ z, \ldots, l_n^+ z$ if (4.6) is satisfied. For this reason we call a pair of vectors $\langle \phi', \psi' \rangle$ satisfying (4.6) a pair of *admissible vectors* for z. In particular we call the vectors ϕ in $\mathbb{C}^r$ corresponding to the K_j *admissible boundary vectors for* z.

4.6. Theorem (*Green's Formula*). *Let* $y, \ldots, y^{(n)}$, $l_0^+ z, \ldots, l_n^+ z$ [3]) *exist and let the products* $(l_j^+ z)^* y^{(n-j)}$, $j = 1, \ldots, n$, *be integrable. Then*

$$\int_0^1 [z^*(ly) - (l_n^+ z)^* y] \, \mathrm{d}t =$$

$$= \sum_{j=1}^n (l_{j-1}^+ z)^* y^{(n-j)}\Big|_0^1 + \phi^* \sum_{j=1}^n \int_0^1 \mathrm{d}K_j y^{(n-j)} + \psi^* \sum_{j=1}^n \int_0^1 \mathrm{d}L_j y^{(n-j)} .$$

Proof. Consider

$$\int_0^1 (l_j^+ z)^* y^{(n-j)} \, \mathrm{d}t = \int_0^1 \{-[l_{j-1}^+ z + K_j^* \phi + L_j^* \psi]' + a_j^* z\}^* y^{(n-j)} \, \mathrm{d}t =$$

$$= -[l_{j-1}^+ z] + K_j^* \phi + L_j^* \psi]^* y^{(n-j)}\Big|_0^1 +$$

$$+ \int_0^1 [l_{j-1}^+ z + K_j^* \phi + L_j^* \psi]^* y^{(n-j+1)} \, \mathrm{d}t + \int_0^1 (a_j^* z)^* y^{(n-j)} \, \mathrm{d}t .$$

If the terms involving K_j^* and L_j^* under the integral are integrated by parts,

$$\int_0^1 (l_j^+ z)^* y^{(n-j)} \, \mathrm{d}t = -(l_{j-1}^+ z)^* y^{(n-j)}\Big|_0^1 + \int_0^1 (l_{j-1}^+ z)^* y^{(n-j+1)} \, \mathrm{d}t -$$

$$- \phi^* \int_0^1 \mathrm{d}K_j y^{(n-j)} - \psi^* \int_0^1 \mathrm{d}L_j y^{(n-j)} + \int_0^1 z^* a_j y^{(n-j)} \, \mathrm{d}t ,$$

the terms to be evaluated at 0 and 1 cancelling. If this is summed from $j = 1$ to n, the result follows.

Note that in Green's formula the terms to be evaluated at 0 and 1 on the right side involve only $l_0^+ z, \ldots, l_{n-1}^+ z$, which depend upon ϕ and ψ only on a set of measure zero. Hence these terms may be considered to be independent of these parameters.

4.7. Lemma. *Suppose* ξ_1, ξ_2 *are vectors in* $\mathbb{C}^r$ *and* $\mathbb{C}^s$ *respectively. If*

$$\xi_1^* \sum_{j=1}^n \int_0^1 \mathrm{d}K_j y^{(n-j)} + \xi_2^* \sum_{j=1}^n \int_0^1 \mathrm{d}L_j y^{(n-j)} = 0 \tag{4.7}$$

for all y *in* D', *then*

$$K_{j+1}^* \xi_1 + L_{j+1}^* \xi_2 = 0 , \quad j = 0, \ldots, n - 1 ,$$

on $[0, 1]$.

---

[3]) I.e., there exist parameters ϕ, ψ so these expressions are defined.

Proof. D' contains the space $C^{(n-1)}[0, 1]$ of $n - 1$ fold continuously differentiable functions y under the norm

$$\|y\| = \sum_{i=0}^{n-1} \|y^{(i)}\|_\infty$$

Thus (4.7) defines a continuous functional on $C^{(n-1)}[0, 1]$ which vanishes everywhere and which is represented by the $1 \times m$ measures $\xi_1^* K_{j+1} + \xi_2^* L_{j+1}$. It follows by a form of the Riesz Representation theorem (see DUNFORD and SCHWARTZ [11] p. 344) that these measures must be zero measures. Taking transposes completes the proof of the lemma.

4.8. Theorem. *If $z \in D^*$, z satisfies the "parametric" endpoint conditions*

$$l_j^+ z(0^+) = -A_{j+1}^* \xi$$
$$l_j^+ z(1^-) = \quad B_{j+1} \xi\,, \quad j = 0, \ldots, n-1\,,$$

where ξ is an admissible boundary vector for z.

Proof. This is another exercise involving the linear dependence principle. Fix $z \in D^*$. By Theorem 4.4 there exists $\phi \in \mathbb{C}^r$, $\psi \in \mathbb{C}^s$ such that $l_j^+ z$, $j = 0, \ldots, n$, exists. Moreover, since $l_j^+ z + K_{j+1}^* \phi + L_{j+1}^* \phi$, $j = 0, \ldots, n-1$, are absolutely continuous functions by definition, it is clear that $l_j^+ z$ are functions of bounded variation. Therefore the limits $l_j^+ z(0^+)$, $l_j^+ z(1^-)$ exist. Also it is clear that the products $l_j^+ z\, y^{(n-j)}$ are integrable for y in D'. From Green's formula (Theorem 4.6.)

$$\sum_{j=1}^{n} (l_{j-1}^+ z)^* \, y^{(n-j)}\Big|_0^1 + \phi^* \sum_{j=1}^{n} \int_0^1 \mathrm{d}K_j y^{(n-j)} + \psi^* \sum_{j=1}^{n} \int_0^1 \mathrm{d}L_j y^{(n-j)}$$

can be viewed as a functional $\lambda : D' \to \mathbb{C}$ whose kernel contains D. On the other hand, D is the intersection of the kernels of the $r + s$ functionals ψ_i determined by Uy, Vy. Therefore by Lemma 4.3 there exists a vector ξ in $\mathbb{C}^r$ and a vector δ in $\mathbb{C}^s$ such that

$$\lambda y = \xi^* U y + \delta^* V y\,.$$

Writing this out and simplifying we get

$$\sum_{j=1}^{n} [l_{j-1}^{+*} z(1^-) - \xi^* B_j]\, y^{(n-j)}(1^-) - [\sum_{j=1}^{n} l_{j-1}^{+*} z(0^+) + \xi^* A_j]\, y^{(n-j)}(0^+) +$$
$$+ (\phi^* - \xi^*) \sum_{j=1}^{n} \int_0^1 \mathrm{d}K_j y^{(n-j)} + (\psi^* - \delta^*) \sum_{j=1}^{n} \int_0^1 \mathrm{d}L_j y^{(n-j)} = 0\,.$$

or

$$(4.8) \quad \sum_{j=1}^{n} [l_{j-1}^{+*} z(1^-) - \xi^* B_j]\, y^{(n-j)}(1^-) - \sum_{j=1}^{n} [l_{j-1}^{+*} z(0^+) + \xi^* A_j]\, y^{(n-j)}(0^+) =$$
$$= (\xi^* - \phi^*) \sum_{j=1}^{n} \int_0^1 \mathrm{d}K_j y^{(n-j)} + (\delta^* - \psi^*) \sum_{i=1}^{n} \int_0^1 \mathrm{d}L_j y^{(n-j)}$$

for all y in D'. Observe that both sides of (4.8) determine functionals which have the same values on D' and thus on the space $C^{(n)}[0, 1] \subset D'$. Since the functional on the left is defined by a linear combination of point evaluators at the endpoints, and since the measures K_j, L_j have support in $(0, 1)$ it follows from the same representation theorem used in Lemma 4.7 ([11], p. 344) that both functionals vanish identically on $C^{(n)}[0, 1]$; i.e.,

$$\sum_{j=1}^{n} [l_j^{+*} z(1^-) - \xi^* B_j] \; y^{(n-j)}(1^-) - \sum_{j=1}^{n} [l_{j-1}^{+*} z(0^+) + \xi^* A_j] \, y^{(n-j)}(0^+) = 0 ,$$

$$(\xi^* - \phi^*) \sum_{j-1}^{n} \int_0^1 \mathrm{d}K_j y^{(n-j)} + (\delta^* - \psi^*) \sum_{j=1}^{n} \mathrm{d}L_j y^{(n-j)} = 0 .$$

In the first equation the variability of the y forces the desired endpoint conditions

$$l_{j-1}^{+} z(1^-) = B_j^* \xi , \quad l_{j-1}^{+} z(0^+) = -A_j^* \xi , \quad j = 0, \ldots, n-1$$

to hold. Moreover, it follows from the second equation and Lemma 4.7 that ξ is an admissible boundary vector for z.

4.9. Theorem. *If the matrix* $[A_1, \ldots, A_n, B_1, \ldots, B_n]$ *has full rank (i.e. rank r) there exist boundary forms* $M_1 y$, $M_2 y$, $N_1 z$, $N_2 z$, *where*

$$M_1 y = \sum_{j=1}^{n} A_j y^{(n-j)}(0) + \sum_{j=1}^{n} B_j^{(n-j)}(1)' ,$$

$$M_2 y = \sum_{j=1}^{n} C_j y^{(n-j)}(0) + \sum_{j=1}^{n} D_j y^{(n-j)}(1) ,$$

$$N_1 z = \sum_{j=1}^{n} \tilde{A}_j l_{n-j}^{+} z(0) + \sum_{j=1}^{n} \tilde{B}_j l_{n-j}^{+} z(1) ,$$

$$N_2 z = \sum_{j=1}^{n} \tilde{C}_j l_{n-j}^{+} z(0) + \sum_{j=1}^{n} \tilde{D}_j l_{n-j}^{+} z(1) ,$$

such that

$$\sum_{j=1}^{n} (l_{j-1}^{+} z)^* \, y^{(n-j)} \Big|_0^1 = -(N_2 z)^* (M_1 y) - (N_1 a)^* (M_2 y) ,$$

where the $2mn\text{-}r \times 2mn$ *matrices* $[C_1, \ldots, C_n, D_1, \ldots, D_n]$, $[\tilde{A}_1, \ldots, \tilde{A}_n, \tilde{B}_1, \ldots, \tilde{B}_n]$ *and the* $r \times 2mn$ *matrix* $[\tilde{C}_1, \ldots, \tilde{C}_n, \tilde{D}_1, \ldots, \tilde{D}_n]$ *have full rank.*

Proof. We refer the reader to CODDINGTON and LEVINSON's text [10; page 284] where a similar derivation is found.

4.10. Theorem. *Let* $z \in D^*$. *Then (provided the full rank condition of Theorem* 4.9 *holds)*

$$-\xi = N_2 z$$

where ξ is an admissible boundary vector for z [4]*). Further z satisfies the boundary condition*

$$N_1 z = 0 .$$

Proof. Employing Theorem 4.9 in Green's formula,

$$\int_0^1 [z^*(Ly) - (L^*z)^* y] \, dt = -(N_2 z)^* (M_1 y) - (N_1 z)^* (M_2 y) + \tag{4.9}$$
$$+ \phi^* \sum_{j=1}^{n} \int_0^1 dK_j y^{(n-j)} + \psi^* \sum_{j=1}^{n} \int_0^1 dL_j y^{(n-j)}$$

for all y in D'. As before we fix z in D^*. The right side of (4.9) is a functional whose kernel includes D. Proceeding exactly as in Theorem 4.8 (D is the intersection of the kernels of the $r + s$ functionals ψ_i determined by Uy, Vy), there exists (by Lemma 4.3) $\xi \in \mathbb{C}^r$ and $\delta \in \mathbb{C}^s$ such that

$$-[N_2 z + \xi]^* M_1 y - (N_1 z)^* M_2 y +$$
$$+ (\phi^* - \xi^*) \sum_{j=1}^{n} \int_0^1 dK_j y^{(n-j)} + (\psi^* - \delta^*) \sum_{j=1}^{n} \int_0^1 dL_j y^{(n-j)} = 0$$

for all y in D'. Also the first two terms (involving $M_1 y$, $M_2 y$) define a functional which agrees with the functional defined by $(\phi^* - \xi^*) \sum_{j=1}^{n} \int_0^1 dK_j y^{(n-j)} + (\psi^* - \delta^*) \cdot \sum_{j=1}^{n} \int_0^1 dL_j y^{(n-j)}$ on D'. By the same reasoning as before

$$[N_2 z + \xi]^* M_1 y + (N_1 z)^* M_2 y = 0 ,$$

and ξ is an admissible vector. Since y and its derivatives to be on arbitrary values at the endpoints and the complementary forms $M_1 y$, $M_2 y$ are represented by matrices of full rank, we conclude that

$$N_2 z = -\xi , \quad N_1 z = 0 .$$

In fact Theorems 4.4 and 4.8 or 4.10 completely characterize L^*. To summarize:

4.11. Theorem. *The domain of L^*, D^*, consists of those elements $z \in X^*$ which satisfy:*
There exist parameters $\xi \in \mathbb{C}^r$ and $\delta \in \mathbb{C}^s$ such that

1. $l_j^+ z + K_{j+1}^* \xi + L_{j+1}^* \delta$ *are absolutely continuous*, $j = 1, \ldots, n - 1$.
2. $l_n^+ z$ *exists a.e. and is in* X^*.
3. $l_j^+ z(0^+) = -A_{j+1}^* \xi$, $l_j^+ z(1^-) = B_{j+1}^* \xi$.

3′. $N_1 z = 0$, $\xi = N_2 z$.

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[4]) Here ξ is not necessarily the same boundary vector given in Theorem 4.8.

Moreover L^* is a closed linear relation with the graph

$$G(L^*) = \{(z, l_n^+ z) : z \in D^*\} .$$

Proof. The fact that L^* is closed follows from the standard theory of adjoints (e.g. [1]). Denote the totality of functions satisfying the stated conditions by D^+. An easy application of Green's formula shows that $D^+ \subset D^*$ and

$$\{\langle z, l_n^+ z\rangle : z \in D^+\} \subset G(L^*) . \tag{4.10}$$

Conversely consider z in D^*. There exists (Theorem 4.4) a vector pair $\langle \phi, \psi\rangle$ such that $l_j^+ z$, $j = 0, \ldots, n$ exists,

$$l_j^+ z + K_{j+1}^* \phi + L_{j+1}^* \psi , \quad j = 1, \ldots, n - 1 ,$$

are absolutely continuous functions, and $l_n^+ z \in R(L^*)$. By theorems 4.8 and 4.10 there exist admissible vector pairs $\langle \xi, \delta\rangle$, $\langle \xi', \delta'\rangle$ for z such that the endpoint conditions 3, 3′ are satisfied .In either case it follows at once from the definition of admissibility (cf. Remark 4.5) that the $l_j^+ z$ are unchanged and that the functions

$$l_j^+ z + K_{j+1}^* \xi + L_{j+1}^* \delta , \quad j = 0, \ldots, n - 1$$

or

$$l_j^+ z + K_{j+1}^* \xi' + L_{j+1}^* \delta'$$

are absolutely continuous. This shows $D^* \subset D^+$ and that

$$G(L^*) \subset \{(z, l_n^+ z) : z \in D^+\} . \tag{4.11}$$

Putting (4.10) and (4.11) together, the theorem is proved.

4.12. Remark. If the functions K_{ij} and L_{ij} do not have absolutely continuous parts with derivatives of bounded variation, then the partial adjoints, and, hence, the boundary functions $M_1 z$ and $M_2 z$ depend on ϕ and ψ. Since in this case these parameters cannot be isolated, the parametric form of endpoint conditions for D^* (Theorem 4.11−3) seems preferable. The partial adjoints $l_j^+ z$, $j = 1, \ldots, n - 1$ also depend upon ϕ and ψ in this case.

5. Conclusion. Theorem 4.6 describes D^* parametrically in terms of its continuity properties and end point conditions. For "Extended Hermite-Birkhoff" (multipoint) conditions it has been possible to eliminate ψ and obtain nonparametric boundary conditions (see [2; § 4]). If ly has solutions in the sense of Caratheodory one can explicitly write out the functions in D^* in terms of Green's functions (see [3; § 5]).

In order to amplify a remark in the introduction concerning splines, we point out that the functions in D^* are "generalized splines" which include polynomial, hyperbolic trigonometric, Lg, histosplines, etc., as special cases. How the optimality properties of spline interpolation follow from the construction of an appropriate adjoint is outlined in [4].

Also the results of this paper extend without difficulty to a wider class of generalized boundary value problems than explicitly considered here. We have nowhere used arguments that depend on the "regularity" (e.g., existence of fundamental solutions) of ly. Since Theorem 4.4 (from which everything else follows) depends only on integration by parts and the linear dependence principle, our characterization of L^* remains true for example even if the coefficients $\{a_j\}$ are non-square $k \times m$ matrices with components in $\mathscr{L}^p[0, 1]$. The reader may verify this claim by simply retracing the proofs under the new hypotheses. L now transforms elements in $\mathscr{L}^p_m[0, 1]$ into $\mathscr{L}^p_k[0, 1]$ and L^* maps from $\mathscr{L}^q_k[0, 1]$ into $\mathscr{L}^q_m[0, 1]$. We feel that this generalization is all the more remarkable because almost nothing seems known about such operators with such coefficients.

In a subsequent paper we shall develop an application of this type of generalization to the problem of minimizing sums of $\mathscr{L}^2$ norms of operators under Stieltjes constraints.

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