Czechoslovak Mathematical Journal

Jozef Kačur

A generalized maximum principle and estimates of max vrai u for nonlinear parabolic boundary value problems

Czechoslovak Mathematical Journal, Vol. 26 (1976), No. 4, 507–526

Persistent URL: `http://dml.cz/dmlcz/101425`

CZECHOSLOVAK MATHEMATICAL JOURNAL
Mathematical Institute of Czechoslovak Academy of Sciences
V. 26 (101), PRAHA 20. 12. 1976, No 4

A GENERALIZED MAXIMUM PRINCIPLE AND ESTIMATES OF max vrai u FOR NONLINEAR PARABOLIC BOUNDARY VALUE PROBLEMS

JOZEF KAČUR, Bratislava

(Received August 9, 1973)

In this paper we investigate estimates of max vrai $u(x, t)$ and min vrai $u(x, t)$ for the weak solution of nonlinear parabolic equations of the form

$$(1) \qquad \frac{\partial u}{\partial t} - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i\left(t, x, \frac{\partial u}{\partial x}\right) + a_0(t, x, u) = f(x, t)$$

in the domain $Q \equiv \Omega \times (0, T)$, where $\Omega \subset E^N$ (N-dimensional Euclidean space) is a bounded domain with a Lipschitzian boundary $\partial\Omega$. The initial-boundary conditions are given by a sufficiently smooth function $u_0(x, t)$ in Q:

$$(2) \qquad u(x, 0) = u_0(x, 0)\,, \quad u(x, t)\big|_{\partial\Omega \times (0,T)} = u_0(x, t)\big|_{\partial\Omega \times (0,T)}\,.$$

The growth of $a_i(t, x, \xi)$ in $\xi \in E^N$, $i = 1, \ldots, N$ and $a_0(t, x, s)$ in s is asufficiently general and is given below by th the relation (3) or (3'). The weak solution of the problem (1), (2) is an element of a space of the Orlicz-Sobolev type – see [1]. The estimate for max vrai $u(x, t)$ and min vrai $u(x, t)$ imply, in a special case ($f(x, t) \equiv 0$, $\operatorname{sgn} a_0(t, x, s) = \operatorname{sgn} s$), the maximum principle for the weak solution $u(x, t)$ even in a stronger form (Theorem 4). The estimates obtained allow us to investigate the order of convergence $u(x, t) \to 0$ for $t \to \infty$ in dependence on the order of convergence $u_0(x, t) \to 0$ and $f(x, t) \to 0$ for $t \to \infty$. In some special cases (see Theorem 8) we prove that the initial state (given by $u_0(x, t) \equiv u_0(x)$ in (2) damps in a finite time, i.e., there exists a t_0 ($t_0 \equiv t_0(u_0)$) such that $u(x, t) \equiv 0$ for $t \geqq t_0$.

In § 1 a generalized maximum principle is proved (Theorem 1). In § 2 the stabilization $u(x, t) \to 0$ for $t \to \infty$ is studied. In § 3 we prove some estimates for max vrai . . $u(x, t)$, min vrai $u(x, t)$ and the maximum principle also for more general equations of the form (1) with members $a_i(t, x, u, \partial u/\partial x)$ for $i = 0, 1, \ldots, N$.

The estimates obtained for max vrai $u(x, t)$ and min vrai $u(x, t)$ substantially depend on the behaviour of the member $a_0(t, x, s)$. If $a_0(t, x, s) \equiv 0$, then the estimates (in the case $f(x, t) \equiv 0$) are reduced to the maximum principle.

The results have been obtained by means of the direct variational methods for parabolic boundary value problems. When solving parabolic boundary value problems, the direct variational methods have been used by many authors, e.g., by O. A. Ladyženskaja [5, 10], K. Rektorys [6], P. P. Mosolov [4], author [1, 2] and others. In order that we may use the direct variational methods we must suppose the elliptic operator in (1) to be potential which is a restrictive assumption, but it lies in the substance of the method.

Notation and definitions. We suppose that $a_i(t, x, \xi)$, $i = 0, 1, \ldots, N$, $\xi \in E^N$ (for $i = 0$, $\xi \in E^1$) are continuous functions in all variables $x \in \Omega$, $t \in \langle 0, T \rangle$ and $|\xi| < \infty$. The growth of $a_i(t, x, \xi)$ $(i = 0, 1, \ldots, N)$ in ξ will be described by means of the functions of a certain class $\mathfrak{M}_3$ which is substantially larger then the class of polynomials $|u|^p$ – see [11].

$\mathfrak{M}_3$ is the set of all real functions $g(s)$ satisfying

I $s\, g(s)$ is even and convex for $s \geqq s_1$ and $\lim_{s \to \infty} (s\, g(s))' = \infty$.

II There exists a constant c such that

$$g(2s) \leqq c\, g(s) \quad \text{for each} \quad s \geqq s_1 .$$

III There exists $l > 1$ such that

$$g(s) < \tfrac{1}{2}\, g(ls) \quad \text{for each} \quad s \geqq s_1 ,$$

where s_1 is a sufficiently big positive number.

We assume that with respect to the equation (1) it is possible to find $g_i(s) \in \mathfrak{M}_3$ $(i = 0, 1, \ldots, N)$ satisfying $g_i(s) \geqq g_j(s)$ (or $g_i(s) \leqq g_j(s)$) for all $i, j = 1, \ldots, N$, $s \geqq s_1$ such that

$$(3) \qquad |a_i(t, x, \xi)| \leqq C(1 + \sum_{j=1}^{N} \min(|g_i(\xi_j)|, |g_j(\xi_j)|))$$

for $i = 1, \ldots, N$. If $i = 0$, then the right hand side in (3) consists of a single member $g_0(\xi_0)$ instead of the sum.

Other more general conditions (without the assumption $g_i \geqq g_j$ or $g_i \leqq g_j$) are of the form

$$(3') \qquad |a_i(t, x, \xi)| \leqq C(1 + \sum_{j=1}^{N} g_i(G_i^{-1}\, G_j(|\xi_j|)))$$

for $i = 1, \ldots, N$ where $G_i(s) = s\, g_i(s)$ and G_i^{-1} is its inverse function for $s \geqq 0$. (More exactly see in [6]). If $i = 0$, then the right hand side in (3′) consists of a single member $g_0(G_0^{-1}\, G_0(|\xi_0|))$ instead of the sum.

To ensure the ellipticity (monotonicity) we shall suppose

$$(4) \qquad \sum_{i=0}^{N} (\xi_i - \eta_i) [a_i(t, x, \xi) - a_i(t, x, \eta)] \geqq 0 \quad \text{for all} \quad \xi, \eta \in E^N .$$

The coerciveness of the elliptic operator in (1) will be guaranteed by

$$\sum_{i=0}^{N} \xi_i\, a_i(t, x, \xi) \geqq C_1 \sum_{i=0}^{N} \xi_i\, g_i(\xi_i) + C_2 \xi_0\, g_0(\xi_0) - C_3\,, \tag{5}$$

where $C_1 > 0$, $C_3 \geqq 0$ and $C_2 > 0$. If $a_0(t, x, s) \equiv 0$, then $C_2 = 0$ and we put $g_0(s) = s$.

The potentiality of the elliptic operator in (1) will be guaranteed by the symmetry

$$\frac{\partial a_i(t, x, \xi)}{\partial \xi_j} = \frac{\partial a_j(t, x, \xi)}{\partial \xi_i} \quad \text{for} \quad i, j = 1, \ldots, N\,. \tag{6}$$

The existence of the weak solution (for the definition see [1]) of the problem (1), (2) is proved in [1]. The solution is an element of the space $W(Q)$:

$$W(Q) \equiv \left\{ v \in L_{G_0}(Q) \cap L_2(Q);\ \frac{\partial v}{\partial x_i} \in L_{G_i}(Q),\ i = 1, \ldots, N \right\},$$

where $\partial v / \partial x_i$ are derivatives in the sense of distributions and $L_{G_i}(Q)$ is the Orlicz's space generated by the function $G_i(s) = s\, g_i(s)$.

The basic means which allows us to use the variational methods is Rothe's method (method of lines) which we apply in the following way.

Let $\{t_i\}_{i=0}^{m}$ be a partition of the interval $\langle 0, T \rangle$, $\Delta t_i = t_i - t_{i-1}$ and $u_0 \equiv u_0(x, 0)$. Successively for $j = 1, \ldots, m$ we solve the noninear elliptic equations

$$\frac{u_j - u_{j-1}}{\Delta t_j} - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i \left(t_j, x, \frac{\partial u_j}{\partial x} \right) + a_0(t_j, x, u_j) = f(t_j, x) \tag{1'}$$

with Dirichlet boundary conditions

$$u_j(s)|_{\partial\Omega} = u_0(t_j, s)|_{\partial\Omega} \tag{2'}$$

u_j, $j = 1, \ldots, m$ being elements of the space $W(\Omega)$:

$$W(\Omega) \equiv \left\{ v \in L_{G_0}(\Omega) \cap L_2(\Omega);\ \frac{\partial v}{\partial x_i} \in L_{G_i}(\Omega),\ i = 1, \ldots, N \right\}$$

with the norm

$$\|v\|_W = \sum_{i=1}^{N} \left\| \frac{\partial v}{\partial x_i} \right\|_{G_i} + \|v\|_{L_2} + \|v\|_{G_0}\,,$$

where $\|\cdot\|_{G_i}$ is the corresponding Orlicz's norm in the space L_{G_i}.

We denote $\mathring{W}(\Omega) = \overline{C_0^\infty(\Omega)}$, where the closure is in the norm of $W(\Omega)$. ($C_0^\infty(\Omega)$ is the set of infinitely differentiable functions with support in Ω.)

By means of the functions u_j, $j = 1, \ldots, m$ we construct Rothe's function $u^{(m)}(x, t)$ in Q:

$$u^{(m)}(x, t) = u_{j-1} + \frac{t - t_{j-1}}{\Delta t_j}(u_j - u_{j-1}), \quad \text{for} \quad t_{j-1} \leqq t \leqq t_j, \quad j = 1, \ldots, m .$$

We suppose

$$u_0(x, t) \in C^2(\bar{Q}), \tag{7}$$

$$f(x, t) \in L_2(Q) \quad \text{and} \quad \left\| \frac{\partial f}{\partial t} \right\|_{L_2(\Omega)} \leqq C \quad \text{for a.e.} \quad t \in (0, T). \tag{8}$$

The results of the present paper are based on the following result proved in [1].

Theorem A. *Let the conditions* (4)–(8) *be fulfilled. Suppose that* $a_i(t, x, \xi)$ *and the derivatives* $\partial a_i/\partial t$, $\partial^2 a_i/\partial t^2$, $\partial^2 a_i/\partial t\, \partial \xi_\alpha$, $\partial^2 a_i/\partial \xi_\alpha\, \partial \xi_\beta$ $(i = 0, \ldots, N,\ \alpha, \beta = 1, \ldots, N)$ *are continuous functions in all variables and satisfy* (3) (or (3')). *Then there exists a unique weak solution* $u(x, t)$ *of* (1), (2) *and the estimates*

$$\|u^{(m)} - u\|_{C(\langle 0,T\rangle,\, L_2(\Omega))} \leqq C \,.\, \max_{j=1,\ldots,m} \sqrt{(\Delta t_j)}$$

hold. Moreover, we have $u \in L_\infty(\langle 0, T\rangle, W(\Omega))$ *and* $\partial u/\partial t \in L_\infty(\langle 0, T\rangle, L_2(\Omega))$.

If a stationary parabolic boundary value problem (1), (2) is considered, i.e., if $a_i(t, x, \xi) \equiv a_i(x, \xi)$ and $u_0(x, t) \equiv u_0(x)$, then Theorem A holds without the assumption that the derivatives $\partial a_i/\partial t$, $\partial^2 a_i/\partial t^2$, $\partial^2 a_i/\partial t\, \partial \xi_\alpha$ and $\partial^2 a_i/\partial \xi_\alpha\, \partial \xi_\beta$ satisfy (3) (or (3')).

1.

In this section we prove a generalized maximum principle for the weak solution $u(x, t)$ of (1), (2).

We shall suppose

$$\sum_{i=1}^{N} \xi_i\, a_i(t, x, \xi) \geqq 0 , \tag{9}$$

$$f(x, t) \in L_\infty(Q) \quad \text{and} \quad \max_{x\in\Omega} \operatorname{vrai} f(x, t), \quad \min_{x\in\Omega} \operatorname{vrai} f(x, t) \tag{10}$$

are integrable over $\langle 0, T\rangle$ in the sense of Riemann.

Let us denote by $S_\tau(S_\tau^*)$ the set of all continuous functions $\gamma(t)$ $(\gamma^*(t))$ in $\langle \tau, T\rangle$ satisfying

$$\gamma(t) \geqq \max_{x\in\partial\Omega} u_0(x, t)\ \left(\gamma^*(t) \leqq \min_{x\in\partial\Omega} u_0(x, t)\right) \quad \text{for} \quad t \in \langle \tau, T\rangle .$$

Lemma 1. *Let the assumptions of Theorem* A *and the conditions* (9), (10) *be satisfied. Let* $\{u_j\}_{j=1}^m$ *be weak solutions of the problem* (1′), (2′) *with* $u_0 = u_0(x, 0)$. *Then, for* $i = 1, \ldots, m$,

$$\max_{x\in\Omega} \operatorname{vrai} u_i \leqq \max \{\gamma(t_i); \max_{x\in\Omega} \operatorname{vrai} u_{i-1} + \tag{11}$$

$$+ \max_{x\in\Omega} \operatorname{vrai} f(x, t_i)\, \Delta_i - \min_{\substack{x\in\bar{\Omega}\\ s\geqq\gamma(t_i)}} a_{\mathrm{C}}(t_i, x, s)\, \Delta t_i\}$$

and

$$\min_{x\in\Omega} \operatorname{vrai} u_i \geqq \min \{\gamma^*(t_i); \min_{x\in\Omega} \operatorname{vrai} u_{i-1} + \tag{12}$$

$$+ \min_{x\in\Omega} \operatorname{vrai} f(x, t_i)\, \Delta t_i - \max_{\substack{x\in\bar{\Omega}\\ s\leqq\gamma^*(t_i)}} a_0(t_i, x, s)\, \Delta t_i\}\,,$$

where $\gamma \in S_0$ *and* $\gamma^* \in S_0^*$ *are arbitrary.*

Proof. Let us construct the functional

$$\Phi(t, v) = \int_0^1 \mathrm{d}\tau \int_\Omega \sum_{i=1}^N \frac{\partial v}{\partial x_i}\, a_i\left(t, x, \tau \frac{\partial v}{\partial x}\right) \mathrm{d}x$$

for $v \in W(\Omega)$ and the functionals

$$\Psi(t_i, v, u_{i-1}) = \Phi(t_i, v) + \frac{1}{2\, \Delta t_i} \left\|v - u_{i-1}\right\|_{L_2}^2 +$$

$$+ \int_0^1 \mathrm{d}\tau \int_\Omega v\, a_0(t_i, x, v)\, \mathrm{d}x - \int_\Omega v\, f(x, t_i)\, \mathrm{d}x$$

for $v \in W(\Omega)$, $i = 1, \ldots, m$ ($f(x, t_i)$ is the trace of $f(x, t)$ for $t = t_i$, which exists due to the condition (8) – see [8]). From the assumptions of Theorem A it follows that u_i is the unique point of the minimum of $\Psi(t_i, v, u_{i-1})$ on the set $u_0(x, t_i) + \mathring{W}(\Omega)$, $i = 1, \ldots, m$. First we prove the estimate (11). We denote $M_i = \max\limits_{x\in\Omega} \operatorname{vrai} u_i$ and

$$u|^r = \begin{cases} u(x)\,, & \text{if } u(x) \leqq r \\ r\,, & \text{if } u(x) > r\,. \end{cases}$$

$u_i|^r \in W(\Omega)$ for each real r (for proof see [8] or [11] (Lemma 1)). Let us assume that $M_{i-1} < \infty$. We shall consider the functions $u_i|^r$, where $r \geqq \gamma(t_i) \geqq \max\limits_{x\in\partial\Omega} u_0(x, t_i)$. It is evident that $u_i|^r\big|_{\partial\Omega} = u_i\big|_{\partial\Omega}$ holds in the sense of the traces. We prove that

$$\Psi(t_i, u_i|^r, u_{i-1}) \leqq \Psi(t_i, u_i, u_{i-1}) \tag{13}$$

holds for $r > M_{i-1} + \max\limits_{x\in\Omega} \text{vrai} f(x, t_i)\, \Delta t_i - \min\limits_{\substack{x\in\bar\Omega \\ s\geqq \gamma(t_i)}} a_0(t_i, x, s)\, \Delta t_i$ and hence with respect to the uniquenes of the minimum we have $u_i|^r \equiv u_i$ which implies the estimate (11). Let us investigate $r \geqq \gamma(t_i)$ such that (13) is true. From (9) we deduce $\Phi(t_i, u_i|^r) \leqq \leqq \Phi(t_i, u_i)$ for $i = 1, \ldots, m$. Using the mean value theorem for integrals we obtain

$$\int_0^1 \int_\Omega u_i\, a_0(t_i, x, \tau u_i)\, \mathrm{d}x\, \mathrm{d}\tau - \int_0^1 \int_\Omega u_i|^r\, a_0(t_i, x, \tau u_i|^r)\, \mathrm{d}x\, \mathrm{d}\tau =$$

$$= \int_\Omega (u_i - u_i|^r)\, a_0(t_i, x, u_i|^r + \vartheta_i(x)\,(u_i - u_i|^r))\, \mathrm{d}x\,.$$

The function $\vartheta_i(x)$ satisfying $0 \leqq \vartheta_i(x) \leqq 1$ can be chosen to be measurable — see [12] (footnote to Lemma 5,1). The inequality (13) is fulfilled provided

$$(14) \quad \int_0^1 \int_\Omega u_i|^r\, a_i(t_i, x, u_i|^r)\, \mathrm{d}x + \frac{1}{2\,\Delta t_i} \|u_i|^r - u_{i-1}\|^2_{L_2} - \int_\Omega u_i|^r f(x, t_i)\, \mathrm{d}x \leqq$$

$$\leqq \int_0^1 \int_\Omega u_i\, a_0(t_i, x, u_i)\, \mathrm{d}x + \frac{1}{2\,\Delta t_i} \|u_i - u_{i-1}\|^2_{L_2} - \int_\Omega u_i f(x, t_i)\, \mathrm{d}x\,.$$

With respect to the above we can write (14) in the form

$$(15) \quad \int_{A_i{}^r} (u_i - u_i|^r)\,(u_i + u_i|^r - 2\,u_{i-1} - 2\,\Delta t_i f(x, t_i) +$$

$$+ 2\,\Delta t_i\, a_0(t_i, x, u_i|^r + \vartheta_i(x)\,(u_i - u_i|^r)))\, \mathrm{d}x \geqq 0\,,$$

where $A_i^r \equiv \{x \in \Omega;\ u_i(x) > r\}$.

If $\operatorname{mes} A_i^r = 0$ for $r = \gamma(t_i)$, then the estimate (11) is proved. If $\operatorname{mes} A_i^r > 0$ for $r \geqq \gamma(t_i)$ then we have $u_i - u_i|^r \geqq 0$ and $u_i + u_i|^r \geqq 2r$ on the set A_i^r. Thus, the inequality (15) holds for

$$(16) \qquad r \geqq \max_{x\in A_i{}^r} \text{vrai}\, u_{i-1} + \Delta t_i \max_{x\in A_i{}^r} \text{vrai} f(x, t_i) - \min_{\substack{x\in\bar\Omega \\ s\geqq\gamma(t_i)}} a_0(t_i, x, s)\, \Delta t_i$$

and hence the estimate (11) is proved under the assumption $M_{i-1} < \infty$. In (16) the evident inequality

$$\inf_{x\in A_i{}^r} a_0(t_i, x, u_i|^r + \vartheta_i(x)\,(u_i - u_i|^r)) \geqq \min_{\substack{x\in\bar\Omega \\ s\geqq\gamma(t_i)}} a_0(t, x, s)$$

has been used, since $r \geqq \gamma(t_i)$. From $\max\limits_{x\in\Omega} u_0(x, 0) = M_0 < \infty$ we obtain that the estimate (11) holds for $i = 1, \ldots, m$.

Analogously we prove the estimate (12). For this purpose we consider the functions $u_i|_r$, where

$$u_i|_r = \begin{cases} u_i(x), & \text{if } u_i(x) \geqq r \\ r, & \text{if } u_i(x) < r \end{cases}$$

and $r \leqq \gamma^*(t_i)$.

For simplicity let us denote

$$\varphi(\tau) = \max_{x \in \Omega} \operatorname{vrai} f(x, \tau), \quad \varphi^*(\tau) = \min_{x \in \Omega} \operatorname{vrai} f(x, \tau),$$

$$a_\gamma(\tau) = \min_{\substack{x \in \bar\Omega \\ s \geqq \gamma(\tau)}} a_0(\tau, x, s) \quad \text{and} \quad a^*_{\gamma^*}(\tau) = \max_{\substack{x \in \bar\Omega \\ s \leqq \gamma^*(\tau)}} a_0(\tau, x, s),$$

where $0 \leqq \tau \leqq T$.

The main result is the following generalized maximum principle.

Theorem 1. *If the assumptions of Theorem A, (9) and (10) are satisfied, then the estimates*

$$(17) \qquad u(x, t) \leqq \inf_{\gamma \in S} \max \left\{ \max_{0 \leqq \xi \leqq t} \left(\gamma(\xi) + \int_\xi^t \varphi(\tau)\, d\tau - \int_\xi^t a_\gamma(\tau)\, d\tau \right); \right.$$
$$\left. \max_{x \in \bar\Omega} u_0(x, 0) + \int_0^t \varphi(\tau)\, d\tau - \int_0^t a_\gamma(\tau)\, d\tau \right\},$$

$$(18) \qquad u(x, t) \geqq \sup_{\gamma^* \in S^*} \min \left\{ \min_{0 \leqq \xi \leqq t} \left(\gamma(\xi) + \int_\xi^t \varphi^*(\tau)\, d\tau - \int_\xi^t a^*_{\gamma^*}(\tau)\, d\tau \right); \right.$$
$$\left. \min_{x \in \bar\Omega} u_0(x, 0) + \int_0^t \varphi^*(\tau)\, d\tau - \int_0^t a^*_{\gamma^*}(\tau)\, d\tau \right\}$$

hold for a.e. $x \in \Omega$, where $u(x, t)$ is the weak solution of the problem (1), (2).

Proof. Let us consider $\gamma(t) \in S_0$ and $\gamma^*(t) \in S_0^*$. The estimates (11) and (12) in Lemma 1 are recurrent. Using Lemma 1 let us estimate $\max \operatorname{vrai} u_{i-1}$ in (11). Then we have

$$\max \operatorname{vrai} u_i \leqq \max \{\gamma(t_i);\ \gamma(t_{i-1}) + \varphi(t_i)\, \Delta t_i - a_\gamma(t_i)\, \Delta t_i;$$

$$\max \operatorname{vrai} u_{i-2} + \sum_{l=i-1}^{i} \varphi(t_l)\, \Delta t_l - \sum_{l=i-1}^{i} a_\gamma(t_l)\, \Delta t_l\}$$

and successively we obtain

$$\max \operatorname{vrai} u_i \leqq \max \{ \max_{j=1,\dots,i} (\gamma(t_j) + \sum_{l=j+1}^{i} \varphi(t_l) \Delta t_l - \sum_{l=j+1}^{i} a_\gamma(t_l) \Delta t_l) ; \quad \max_{x \in \bar{\Omega}} u_0(x, 0) + \sum_{l=1}^{i} \varphi(t_l) \Delta t_l - \sum_{l=1}^{i} a_\gamma(t_l) \Delta t_l \} \tag{19}$$

where $i = 1, \dots, m$.

Similarly, using Lemma 1 in the estimate (12) successively we have

$$\min \operatorname{vrai} u_i \geqq \min \{ \min_{j=1,\dots,i} (\gamma^*(t_j) + \sum_{l=j+1}^{i} \varphi^*(t_l) \Delta t_l - \sum_{l=j+1}^{i} a^*_{\gamma*}(t_l) \Delta t_l) ; \quad \min_{x \in \bar{\Omega}} u_0(x, 0) + \sum_{l=1}^{i} \varphi^*(t_l) \Delta t_l - \sum_{l=1}^{i} a^*_{\gamma*}(t_l) \Delta t_l \} . \tag{20}$$

In the estimates (19) and (20) a fixed partition D of the interval $\langle 0, T \rangle$ is considered. Now, let us consider a sequence $\{D_n\}$ of partitions D_n of the interval $\langle 0, T \rangle$ such that $\nu(D_n) \to 0$ with $n \to \infty$, where $\nu(D_n)$ is the norm of D_n. We suppose that a fixed point $t_0 \in \langle 0, T \rangle$ is in D_n for all n. Let $\{t_j^n\}_{j=1}^{m_n}$ be the points of the partition D_n and let $\{u_j^n\}_{j=1}^{m_n}$ be the weak solutions of (1'), (2') corresponding to the partition D_n. Let us consider the sequence $\{u_{r_n}^n\}_{n=1}^{\infty}$ of the weak solutions $u_{r_n}^n$ of (1'), (2'), where $t_{r_n}^n = t_0$ for each n ($u_{r_n}^n$ is the weak solution of (1'), (2') on the section $t = t_0$ with respect to D_n).'

Owing to Theorem A we have $u_{r_n}^n \to u(x, t_0)$ in the norm of the space $L_2(\Omega)$, where $u(x, t_0)$ is the trace of $u(x, t)$ for $t = t_0$ (see [1, 8]). Regarding this fact we deduce

$$\max_{x \in \Omega} \operatorname{vrai} u(x, t_0) \leqq \limsup_{n \to \infty} \max_{x \in \Omega} \operatorname{vrai} u_{r_n}^n . \tag{21}$$

In the following we prove (17), (18) by the limiting proces in (19) and (20).

Let us denote by $t_{j_M}^n$ the point in D_n (next to 0) satisfying

$$\max_{j=1,\dots,r_n} (\gamma(t_j^n) + \sum_{l=j+1}^{r_n} \varphi(t_l^n) \Delta t_l^n - \sum_{l=j+1}^{r_n} a_\gamma(t_l^n) \Delta t_l^n) = \gamma(t_{j_M}^n) + \sum_{l=j_M+1}^{r_n} \varphi(t_l^n) \Delta t_l^n - \sum_{l=j_M+1}^{r_n} a_\gamma(t_l^n) \Delta t_l^n . \tag{22}$$

From $t_{j_M}^n$ it is possible to choose a subsequence (we denote it again by $\{t_{j_M}^n\}_{n=1}^{\infty}$) such that $t_{j_M}^n \to \xi_0 \in \langle 0, T \rangle$ with $n \to \infty$. Now, we estimate

$$R_n = \left| \gamma(\xi_0) + \int_{\xi_0}^{t} \varphi(\tau) \, d\tau - \int_{\xi_0}^{t} a_\gamma(\tau) \, d\tau - (\gamma(t_{j_M}^n) + \sum_{l=j_M+1}^{r_n} \varphi(t_l^n) \Delta t_l^n - \sum_{l=j_M+1}^{r_n} a_\gamma(t_l^n) t_l^n) \right| .$$

The function $\varphi(\tau)$ is bounded and measurable in $\langle 0, T\rangle$. $\gamma(t)$ is uniformly continuous in $\langle 0, T\rangle$. From (5) and with respect to the definition of the class $\mathfrak{M}_3$ (the condition I), we find easily that $a_0(t, x, s) \to \infty$ for $s \to \infty$ and $a_0(t, x, s) \to -\infty$ for $s \to -\infty$ and hence the function $a_\gamma(\tau)$ is bounded and continuous in $\langle 0, T\rangle$. Hence we conclude $R_n \to 0$ with $n \to \infty$ which implies

$$\text{(23)} \qquad \limsup_{n\to\infty} \max \{ \max_{j=1,\dots,r_n} (\gamma(t_j^n) + \sum_{l=j+1}^{r_n} \varphi(t_l^n)\, \Delta t_l^n - \sum_{l=j+1}^{r_n} a_\gamma(t_l^n)\, \Delta t_l^n) ;$$

$$\max_{x\in\bar\Omega} u_0(x, 0) + \sum_{l=1}^{r_n} \varphi(t_l^n)\, \Delta t_l^n - \sum_{l=1}^{r_n} a_\gamma(t_l^n)\, \Delta t_l^n\} \leqq$$

$$\leqq \max \left\{ \max_{0\leqq\xi\leqq t} (\gamma(\xi) + \int_\xi^t \varphi(\tau)\, d\tau - \int_\xi^t a_\gamma(\tau)\, d\tau ;\right.$$

$$\left. \max_{x\in\bar\Omega} u_0(x, 0) + \int_0^t \varphi(\tau)\, d\tau - \int_0^t a_\gamma(\tau)\, d\tau \right\}.$$

Thus, (21) and (23) imply (17). Analogously, using (20) we prove (18) and the proof is complete.

In the following theorem we prove that the assumption (9) is not substantial in Theorem 1.

Theorem 2. *Let the assumptions of Theorem A, (8) and (10) be fulfilled. Then the estimates (17), (18) hold with*

$$\varphi(\tau) = \max_{x\in\Omega} \operatorname{vrai} \left(f(x, \tau) - \sum_{i=1}^N \frac{\partial a_i(\tau, x, 0)}{\partial x_i} \right),$$

$$a_\gamma(\tau) = \min_{\substack{x\in\bar\Omega \\ s\geqq\gamma(\tau)}} (a_0(\tau, x, s) - a_0(\tau, x, 0)),$$

$$\varphi^*(\tau) = \min_{x\in\Omega} \operatorname{vrai} \left(f(x, \tau) - \sum_{i=1}^N \frac{\partial a_i(\tau, x, 0)}{\partial x_i} \right),$$

$$a_{\gamma^*}^*(\tau) = \max_{\substack{x\in\bar\Omega \\ s\leqq\gamma^*(\tau)}} (a_0(\tau, x, s) - a_0(\tau, x, 0)\,.$$

Proof. The assumptions of Theorem A imply $\partial a_i(t, x, 0)/\partial x_i \in L_\infty(\Omega)$ for $i = 1, \dots, N$. Let us consider the problem (1), (2) with $a_i^*(t, x, \xi) = a_i(t, x, \xi) - a_i(t, x, 0)$ for $i = 1, \dots, N$ and $f^*(x, t) = f(x, t) - \sum_{i=1}^N (\partial a_i(t, x, 0)/\partial x_i)$.

$a_i^*(t, x, \xi)$, $i = 0, 1, \dots, N$ and $f^*(x, t)$ satisfy all the conditions of Theorem 1. Indeed, (4) for $a_i(t, x, \xi)$ yields (9) for $a_i^*(t, x, \xi)$. Further, from (5) we deduce

$$\text{(24)} \qquad \sum_{i=0}^N a_i^*(t, x, \xi)\, \xi_i \geqq C_1 \sum_{i=0}^N \xi_i\, g_i(\xi_i) - C_2(\varepsilon) \sum_{i=0}^N \xi_i\, g_i(\xi_i) - C_3(\varepsilon)$$

where $C_2(\varepsilon) \to 0$ with $\varepsilon \to 0$. Indeed, using Young's inequality (see [7], proof of Theorem 7) we obtain

$$\sum_{i=0}^{N} \xi_i\, a_i(t, x, 0) = \sum_{i=0}^{N} \varepsilon\xi_i \frac{a_i(t, x, 0)}{\varepsilon} \leqq \sum_{i=0}^{N} \left(G_i(\varepsilon\xi_i) + P_i\left(\frac{a_i(t, x, 0)}{\varepsilon}\right)\right),$$

where $G_i(s) = s\, g_i(s)$ (for sufficiently big s) and $G_i(s) \leqq s\, g_i(s) + C$ for all $s \in E^1$ with C independent of s. $P_i(s)$ is the conjugate function to $G_i(s)$ ($P_i(s) = \max_r |rs - G(r)|$). Convexity of $G_i(s)$ implies $G_i(\varepsilon s) \leqq \varepsilon\, G_i(s)$ for $0 \leqq \varepsilon \leqq 1$ and hence from (24) and (3) (or (3')) we obtain the condition (5) for $a_i^*(t, x, \xi)$, $i = 0, \ldots, N$. The other conditions of Theorem 1 are evidently fulfilled. By virtue of Green's theorem we find easily that the weak solution $u(x, t)$ of (1), (2) with $a_i^*(t, x, \xi)$ and $f^*(x, t)$ is at the same time the weak solution of (1), (2) with $a_i(t, x, \xi)$ and $f(x, t)$ and hence Theorem 2 is a consequence of Theorem 1.

Remark 1. Theorem 1 can be formulated in a more general form.

Theorem 1'. *Let the assumptions of Theorem* A, (9) *and* (10) *be satisfied. Then, for* $t \geqq t_1$ *we hawe*

$$(17')\qquad u(x, t) \leqq \inf_{\gamma \in S_{t_1}} \max \left\{ \max_{t_1 \leqq \xi \leqq t} \left(\gamma(\xi) + \int_\xi^t \varphi(\tau)\, d\tau - \int_\xi^t a_\gamma(\tau)\, d\tau \right);\right.$$
$$\left. \max_{x \in \Omega} \operatorname{vrai} u(x, t_1) + \int_{t_1}^t \varphi(\tau)\, d\tau - \int_{t_1}^t a_\gamma(\tau)\, d\tau \right\}$$

and

$$(18')\qquad u(x, t) \geqq \sup_{\gamma^* \in S_{t_1}^*} \min \left\{ \min_{t_1 \leqq \xi \leqq t} \left(\gamma(\xi) + \int_\xi^t \varphi^*(\tau)\, d\tau - \int_\xi^t a_{\gamma^*}^*(\tau)\, d\tau \right);\right.$$
$$\left. \min_{x \in \Omega} \operatorname{vrai} u(x, t_1) + \int_{t_1}^t \varphi^*(\tau)\, d\tau - \int_{t_1}^t a_{\gamma^*}^*(\tau)\, d\tau \right\}.$$

2.

In this section we present some consequences of Theorem 1 (or Theorem 1'). Let us denote $Q_t = \Omega \times \langle 0, t \rangle$ and $\Gamma_t = \partial\Omega \times \langle 0, t \rangle \cup \Omega \times \{0\}$.

For the weak solution of (1), (2), the following maximum principle holds.

Theorem 3. *Let the assumptions of Theorem A be satisfied and suppose* $f(x, t) \equiv 0$, $\operatorname{sgn} s = \operatorname{sgn} a_0(t, x, s)$. *If* $0 \leqq \max_{Q_T} \operatorname{vrai} u(x, t)$ ($\min_{Q_T} \operatorname{vrai} u(x, t) \leqq 0$), *then* $\max_{Q_T} \operatorname{vrai} u(x, t) \leqq \max_{\Gamma_T} \operatorname{vrai} u(x, t)$ ($\min_{Q_T} \operatorname{vrai} u(x, t) \geqq \min_{\Gamma_T} \operatorname{vrai} u(x, t)$).

Proof. Consider the functions $\gamma(s) = \max \{0, \max_{x \in \partial\Omega} u_0(x, s)\}$ and $\gamma^*(s) = \min \{0, \min_{x \in \partial\Omega} u_0(x, s)\}$ for $s \in \langle 0, T \rangle$. It is clear that $\gamma \in S_0$ and $\gamma^* \in S_0^*$. Owing to the assumptions we have $a_\gamma(\tau) \geqq 0$ and $a^*_{\gamma*}(\tau) \leqq 0$. Hence due to the estimates (17) and (18) of Theorem 1 we conclude

$$\min\{0; \min_{\Gamma_t} u_0(x, t)\} \leqq u(x, t) \leqq \max\{0; \max_{\Gamma_t} u_0(x, t)\} \tag{25}$$

a.e. in Q_t. With regard to the fact that $u(x, t) = u_0(x, t)$ on Γ_t (in the sense of traces — see [8]) the proof of Theorem 3 is complete.

Formulation of Theorem 3 is analogous to the maximum principle for the smooth solution of linear equations (see [9]): The nonnegative maximum (nonpositive minimum) of $u(x, t)$ is attained on the boundary Γ_T. A stronger formulation is: The solution $u(x, t)$ cannot attain its nonnegative maximum (nonpositive minimum) in the interior of the domain Q_T. Now, we present a strengthened form of Theorem 3.

Theorem 4. *Let the assumptions of Theorem* 3 *be fulfilled. If* $0 < \max_{x\in\partial\Omega} u_0(x, t_0) < \max_{\Gamma_{t_0}} u_0(x, t)$ $(0 > \min_{x\in\partial\Omega} u_0(x, t_0) > \min_{\Gamma_{t_0}} u_0(x, t))$, *then* $\max_{x\in\Omega} \operatorname{vrai} u(x, t_0) < \max_{\Gamma_{t_0}} \operatorname{vrai} u(x, t)$ $(\min_{x\in\Omega} \operatorname{vrai} u(x, t_0) > \min_{\Gamma_{t_0}} \operatorname{vrai} u(x, t))$, *where* $u(x, t_0)$ *is the trace of* $u(x, t)$ *for* $t = t_0 \in \langle 0, T\rangle$.

Proof. Let us consider $\gamma \in S_0$ and $\gamma^* \in S_0^*$ defined in Theorem 3. It follows from (4) and $\operatorname{sgn} a_0(t, x, s) = \operatorname{sgn} s$ that $a_0(t, x, s) > 0$ for $s > 0$ and also $a_\gamma(\tau) > 0$ for $\gamma(\tau) > 0$. Hence the continuity of $a_\gamma(\tau)$ in τ implies

$$\int_\xi^{t_0} a_\gamma(\tau)\, d\tau > 0 \quad \text{for all } \xi\,, \quad 0 \leqq \xi < t_0\,.$$

If $\gamma(t_0) = \max_{0\leqq\xi\leqq t_0} \gamma(\xi)$, then

$$\max_{0\leqq\xi\leqq t_0}\left(\gamma(\xi) - \int_\xi^{t_0} a_\gamma(\tau)\, d\tau\right) = \gamma(t_0)$$

and due to the assumption there exists $\varepsilon > 0$ such that

$$\max\left\{\gamma(t_0);\ \max u_0(x, 0) - \int_0^{t_0} a_\gamma(\tau)\, d\tau < \max_{\Gamma_{t_0}} u_0(x, t) - \varepsilon\right.$$

which implies the assertion.

If $\gamma(t_0) < \max_{0\leqq\xi\leqq t_0} \gamma(\xi)$, then there exists $\delta > 0$ such that

$$\gamma(\xi) < \max_{0\leqq\xi\leqq t_0} \gamma(\xi) - \varepsilon \quad \text{for} \quad t_0 - \delta < \xi \leqq t_0\,,$$

where $0 < \varepsilon < \frac{1}{2}(\max_{0\leqq\xi\leqq t_0} \gamma(\xi) - \gamma(t_0))$.

Let us denote $\int_{t_0-\delta}^{t_0} a_\gamma(\tau)\,d\tau = \delta_1 > 0$. Then we have

$$\max_{0\leqq\xi\leqq t_0}\left(\gamma(\xi) - \int_\xi^{t_0} a_\gamma(\tau)\,d\tau\right) \leqq \max_{0\leqq\xi\leqq t_0}\gamma(\xi) - \min(\varepsilon, \delta_1) \leqq \max_{\Gamma_{t_0}} u_0(x, t) - \min(\varepsilon, \delta_1)\,.$$

Further, we obtain

$$\max u_0(x, 0) - \int_0^{t_0} a_\gamma(\tau)\,d\tau \leqq \max_{\Gamma_{t_0}} u_0(x, t) - \min(\varepsilon, \delta_1)\,.$$

Hence with regard to the previous inequality, Theorem 1 implies Theorem 4.

In the following, using Theorem 1 (or Theorem 1′) we shall estimate $\max\limits_{x\in\Omega}\operatorname{vrai} u(x, t)$ and $\min\limits_{x\in\Omega}\operatorname{vrai} u(x, t)$ for $t > 0$. If we are interested in the estimate of $u(x, t)$ in a neighbourhood of the point $t = \infty$, then we assume that $u_0(x, t)$, $f(x, t)$ and $a_i(t, x, \xi)$ $i = 0, 1, \ldots, N$ are defined for $t \in \langle 0, \infty)$ and the assumptions of Theorem 1 are satisfied on every interval $\langle 0, T\rangle$ $(T < \infty)$.

For simplicity we suppose that there exist continuous functions $a_i(t, y)$ $i = 1, 2$ (in the variables t, y) defined for $t \in \langle 0, T\rangle$ and $|y| < \infty$ such that

$$a_0(t, x, s) \geqq a_1(t, s) \quad \text{for all} \quad x \in \Omega \quad \text{and} \quad t \geqq 0\,, \tag{26}$$

$$a_0(t, x, s) \leqq a_2(t, s) \quad \text{for all} \quad x \in \Omega\,, \quad t \geqq 0\,. \tag{26'}$$

Further we assume

$$\max_{x\in\Omega}\operatorname{vrai} f(x, t) \leqq f_1(t)\,, \tag{27}$$

$$\min_{x\in\Omega}\operatorname{vrai} f(x, t) \geqq f_2(t) \tag{27'}$$

where f_1, f_2 are continuous functions for $t \in \langle 0, T)$.

Remark 2. From 4 we deduce that $a_0(t, x, s)$ is nondecreasing in s and hence

$$a_\gamma(\tau) = \min_{\substack{x\in\bar\Omega\\ s\geqq\gamma(t)}} a_0(t, x, s) = \min_{x\in\bar\Omega} a_0(t, x, \gamma(t)) \geqq a_1(t, \gamma(t))\,.$$

Similarly we obtain $a^*_{\gamma*}(t) \leqq a_2(t, \gamma^*(t))$.

Theorem 5. *Let $u(x, t)$ be a solution of* (1), (2) *and let the assumptions of Theorem A*, (9) *and* (10) *be fulfilled.*

i) *If* (26), (27) *hold and if $\gamma \in S_{t_1}$ is an arbitrary absolutely continuous function in $\langle t_1, T)$ $(t_1 < T \leqq \infty)$ satisfying the differential inequality*

$$\gamma'(t) \geqq f_1(t) - a_1(t, \gamma(t)) \quad \textit{for} \quad t_1 \leqq t < T \quad \textit{with} \quad \gamma(t_1) \geqq \max_{x\in\Omega}\operatorname{vrai} u(x, t_1) \tag{28}$$

then $u(x, t) \leqq \gamma(t)$ for a.e. $x \in \Omega$ and $t_1 \leqq t < T$.

ii) *If* (26′), (27′) *hold and if* $\gamma^* \in S^*_{t_1}$ *is an arbitrary absolutely continuous function in* $\langle t_1, T)$ $(t_1 < T \leqq \infty)$ *satisfying the differential inequality*

$$\gamma^{*\prime}(t) \leqq f_2(t) - a_2(t, \gamma^*(t)) \quad \textit{for} \quad t_1 \leqq t < T \quad \textit{with} \tag{29}$$

$$\gamma^*(t_1) \leqq \min_{x \in \Omega} \operatorname{vrai} u(x, t_1),$$

then $\gamma^*(t) \leqq u(x, t)$ *for a.e.* $x \in \Omega$, $t_1 \leqq t < T$.

Proof. i) From (26), (27) we obtain

$$\gamma(\xi) + \int_\xi^t \varphi(\tau)\,d\tau - \int_\xi^t a_\gamma(\tau)\,d\tau \leqq \gamma(\xi) + \int_\xi^t f_1(\tau)\,d\tau - \int_\xi^t a_1(\tau, \gamma(\tau))\,d\tau$$

where $t_1 \leqq \xi \leqq t < T$. The inequality (28) implies

$$\max_{t_1 \leqq \xi \leqq t} \left(\gamma(\xi) + \int_\xi^t f_1(\tau)\,d\tau - \int_\xi^t a_1(\tau, \gamma(\tau))\,d\tau \right) = \gamma(t)$$

and

$$\max_{x \in \Omega} \operatorname{vrai} u(x, t_1) + \int_{t_1}^t f_1(\tau)\,d\tau - \int_{t_1}^t a_\gamma(\tau)\,d\tau \leqq$$

$$\leqq \max_{x \in \Omega} \operatorname{vrai} u(x, t_1) + \int_{t_1}^t \gamma'(\tau)\,d\tau \leqq \gamma(t).$$

Hence due to Theorem 1′ we obtain $u(x, t) \leqq \gamma(t)$ for $t \in \langle t_1, T)$ and a. e. $x \in \Omega$.

Assertion ii) can be proved analogously as Assertion i).

Theorem 6. *Let the assumptions of Theorem A,* (9) *and* (10) *be fulfilled.*

i) *If* (26′), (27′) *are satisfied and if* $\gamma \in S_{t_1}$ *is an arbitrary absolutely continuous function in* $\langle t_1, T)$ $(t_1 < T \leqq \infty)$ *satisfying the differential inequality*

$$\gamma(t) \leqq f_2(t) - a_2(t, \gamma(t)) \quad \textit{for} \quad t_1 \leqq t < T \quad \textit{with} \tag{30}$$

$$\gamma(t_1) \geqq \max_{x \in \Omega} \operatorname{vrai} u(x, t_1)$$

then

$$u(x, t) \leqq \gamma(t_1) - \int_{t_1}^t a_\gamma(s)\,ds + \int_{t_1}^t \varphi(s)\,ds$$

for a.e. $x \in \Omega$ *and* $t_1 \leqq t < T$.

ii) *If* (26), (27) *are satisfied and if* $\gamma^* \in S^*_{t_1}$ *is an arbitrary absolutely continuous function in* $\langle t_1, T)$ $(t_1 < T \leqq \infty)$ *satisfying the differentail inequality*

$$\gamma^{*\prime}(t) \geqq f_1(t) - a_1(t, \gamma^*(t)) \quad \textit{for} \quad t_1 \leqq t < T \quad \textit{with} \tag{31}$$
$$\gamma^*(t_1) \leqq \min_{x \in \Omega} \operatorname{vrai} u(x, t_1)$$

then

$$u(x, t) \geqq \gamma^*(t_1) - \int_{t_1}^{t} a^*_{\gamma^*}(s) \, ds + \int_{t_1}^{t} \varphi^*(s) \, ds$$

for a.e. $x \in \Omega$ *and* $t_1 \leqq t < T$.

Proof. i) From (30) we deduce

$$\max_{t_1 \leqq \xi \leqq t} \left(\gamma(\xi) + \int_{\xi}^{t} f_2(\tau) \, d\tau - \int_{\xi}^{t} a_2(\tau, \gamma(\tau)) \, d\tau \right) =$$

$$= \gamma(t_1) - \int_{t_1}^{t} a_2(\tau, \gamma(\tau)) \, d\tau + \int_{t_1}^{t} f_2(\tau) \, d\tau \leqq \gamma(t_1) - \int_{t_1}^{t} a_\gamma(\tau) \, d\tau + \int_{t_1}^{t} \varphi(\tau) \, d\tau \, .$$

Hence Assertion i) is a consequence of Theorem 1′. Assertion ii) can be proved analogously as Assertion i).

Remark 3. In general we combine Theorem 5 and Theorem 6 applying then to certain parts of the half line $t \geqq 0$. The possibility of such a combination follows from Theorem 1′.

Remark 4. Let the assumptions of Theorem 1 be fulfilled. If $u_0(x, t) \equiv u_0(t)$, $f(x, t) \equiv f(t)$ and $a_0(t, x, s) = a_0(t, s)$ and if $u_0(t)$ is a solution of the differential equation

$$y'(t) + a_0(t, y(t)) = f(t) \, , \quad y(0) = u_0(0)$$

then the estimates (17) and (18) imply $u(x, t) \equiv u_0(t)$, where $u(x, t)$ is a solution of (1), (2).

Theorem 7. *Let the assumptions of Theorem A*, (9), (10) *and* $\operatorname{sgn} a_0(t, x, s) = \operatorname{sgn} s$ *be fulfilled. If*

$$|u_0(x, t)| + |f(x, t)| \leqq \Phi(t) \quad \textit{for} \quad t \geqq t_1 \geqq 0 \, ,$$

where $\Phi(t)$ *is a continuous non-increasing function satisfying* $\Phi(t) \to 0$ *with* $t \to \infty$, *then*

$$\max_{x \in \Omega} \operatorname{vrai} |u(x, t)| \to 0 \quad \textit{with} \quad t \to \infty \, ,$$

where $u(x, t)$ *is the solution of* (1), (2).

Proof. From the definition of the class $\mathfrak{M}_3$ (the condition I) and from 5 we deduce

$$s\,a_0(t, x, s) \geqq \frac{C_1}{2}\,s\,g_0(s) \quad \text{for} \quad s \geqq s_1 > 0\,.$$

Let us define the function

$$g_0^*(s) = \inf_{x\in\bar{\Omega}, t\geqq 0} a_0(t, x, s) \quad \text{for} \quad 0 \leqq s \leqq s_1\,.$$

From (4) it follows (see Remark 4) that $g_0^*(s)$ is an increasing function. Further, the assumption on $a_0(t, x, s)$ implies that $g_0^*(s)$ is continuous for $0 \leqq s \leqq s_1$. Now, we continuously extend the function $g_0^*(s)$ for $s \geqq s_1$ such that

$$g_0^*(s) \leqq \frac{C_1}{2}\,g_0(s) \quad \text{for} \quad s_1 \leqq s \leqq s_2\,,$$

$$g_0^*(s) = \frac{C_1}{2}\,g_0(s) \quad \text{for} \quad s \geqq s_2 \quad \text{and} \quad g_0^*(s)$$

is increasing for $s \geqq 0$. Let us put

$$g_0^*(s) = -g_0^*(-s) \quad \text{for} \quad s < 0\,.$$

It is obvious that

$$s\,a_0(t, x, s) \geqq s\,g_0^*(s) \quad \text{for} \quad s \in E^1\,.$$

Now, let us consider the differential inequality

$$\gamma'(\xi) + g_0^*(\gamma(\xi)) \geqq \Phi(\xi) \quad \text{for} \quad \xi \geqq t_1\,. \tag{32}$$

Owing to the properties of $\Phi(\xi)$ and $g_0^*(s)$ we conclude that for each $K > 0$ there exists a non-increasing function $0 < \gamma_K(\xi) \in S_{t_1}$ (absolutely continuous, e.g. broken half line) such that $\gamma_K(t_1) \geqq K$, $\gamma_K(\xi) \to 0$ with $\xi \to \infty$ and (32) is satisfied. Now, let us put $a_1(t, y) = g_0^*(y)$ for $y \geqq 0$, $t > 0$. (In this case it suffices to consider $a_1(t, y)$ for $y \geqq 0$.) In virtue of Theorem 1 we deduce that $\max\text{vrai}\,|u(x, t_1)| < \infty$. Thus, Theorem 5 yields

$$\max\text{vrai}\,u(x, t) \leqq \gamma_{K_0}(t) \quad \text{for} \quad t \geqq t_1 \quad \text{and a.e.} \quad x \in \Omega\,, \tag{33}$$

where $K_0 \geqq \max\text{vrai}\,u(x, t_1)$.

Now, let us consider the differential inequality

$$\gamma^{*\prime}(\xi) + g_0^*(\gamma^*(\xi)) \leqq -\Phi(\xi) \quad \text{for} \quad \xi \geqq t_1 \tag{34}$$

with $\gamma^*(t_1) \leqq -K$, $K > 0$.

Let us put $a_2(t, y) = g_0^*(y)$ for $y \leqq 0$. Similarly as in the preceding, we conclude from the properties of $g_0^*(s)$ and $\Phi(s)$ that for each $K > 0$ there exists $0 > \gamma_K^*(\xi) \in S^*$ such that (33) is satisfied ($\gamma_K^*(\xi) = -\gamma_K(\xi)$, where $\gamma_K(\xi)$ is from (32). Let us put $K_1 = \max \{0, \max_{x \in \Omega} \operatorname{vrai} - u(x, t_1)\}$. Due to Theorem 5 we obtain

$$\min_{x \in \Omega} \operatorname{vrai} u(x, t) \geqq -\gamma_{K_1}(t) . \tag{35}$$

The proof now follows from (33) and (35).

In order to estimate the rate of the stabilization $u(x, t) \to 0$ for $t \to \infty$, Theorem 5 will be applied.

Theorem 8. *Let $u(x, t)$ be a weak solution of (1), (2) and let the assumptions of Theorem A, (9) and (10) be satisfied. Suppose* $\operatorname{sgn} a_0(t, x, s) = \operatorname{sgn} s$ *and*

$$s\, a_0(t, x, s) \geqq C|s|^\alpha \ (\alpha > 1) \quad \text{for all} \quad t \geqq 0\,, \quad |s| < \infty \quad \text{and} \quad x \in \Omega\,. \tag{36}$$

i) *Let $\alpha < 2$.* a) *If $f(x, t) \equiv u_0(x, t) \equiv 0$ for $t \geqq t_1 > 0$, then there exists $t_2 \geqq t_1$ such that $u(x, t) \equiv 0$ for $t \geqq t_2$, where t_2 depends on C, α, t_1, u_0 and f.*

b) *If* $\max_{x \in \Omega} \operatorname{vrai} (|u_0(x, t)| + |f(x, t)|) = O(t^{-\beta})$ $(\beta > 1)$ *then* $\max_{x \in \Omega} \operatorname{vrai} |u(x, t)| = O(t^{-\beta+1})$.

ii) *Let $\alpha = 2$.* a) *If* $\max_{x \in \Omega} \operatorname{vrai} (|u_0(x, t)| + |f(x, t)|) = O(t^{-\beta})$ $(\beta > 1)$ *then* $\max_{x \in \Omega} \operatorname{vrai} |u(x, t)| = O(t^{-\beta+1})$.

b) *If* $\max_{x \in \Omega} \operatorname{vrai} (|u_0(x, t)| + |f(x, t)|) = O(e^{-\lambda t})$ $(\lambda > 0)$ *then* $\max_{x \in \Omega} \operatorname{vrai} |u(x, t)| = O(e^{-\delta t})$ *where $0 < \delta < \min(\lambda, C)$ and C is from (36).*

iii) *Let $\alpha > 2$. If* $\max_{x \in \Omega} \operatorname{vrai} (|u_0(x, t)| + |f(x, t)|) = O(t^{-\beta})$ $(\beta > 1)$ *then* $\max_{x \in \Omega} \operatorname{vrai} |u(x, t)| = O(t^{-\delta})$ *where* $0 < \delta < \min(1/(\alpha - 2), \beta/(\alpha - 1))$.

Proof. i) From (36) we deduce $a_1(t, s) = Cs^{\alpha-1}$ for $t \geqq 0$, $s \geqq 0$ and $a_2(t, s) = -C|s|^{\alpha-1}$ for $t \geqq 0$, $s < 0$, where $a_1(t, s)$, $a_2(t, s)$ are from (26), Let $\gamma(t)$ be a solution of the equation

$$\gamma'(t) = -C\, \gamma^{\alpha-1}(t) \quad \text{for} \quad t \geqq t_1\,, \quad \gamma(t_1) \geqq \max_{x \in \Omega} \operatorname{vrai} |u(x, t_1)|\,. \tag{37}$$

Since $\gamma(t) \in S_{t_1}$ and $-\gamma(t) \in S_{t_1}^*$ (for $t \geqq t_1$), we obtain from Theorem 5 that $\max_{x \in \Omega} \operatorname{vrai} |u(x, t)| \leqq \gamma(t)$. But $\gamma(t) = 0$ for $t \geqq t_2 = \gamma(t_1)^{2-\alpha}/C(2 - \alpha)$ (C is from (37)) and hence Assertion i), a) is proved.

b) Let us denote by $\varphi(t)$ a continuous function satisfying

$$\varphi(t) \geqq \max_{x \in \Omega} \operatorname{vrai} (|u_0(x, t)| + |f(x, t)|)\,.$$

By virtue of the differential inequality

$$y'(t) \geqq -C\, y^{\alpha-1}(t) + \varphi(t) \quad \text{for} \quad t \geqq t_1\,, \tag{38}$$

where $y(t_1) = B_1 = \max\limits_{x\in\Omega} \operatorname{vrai} u(x, t_1)$, we can deduce by an elementary computation that the function

$$\gamma(t) = B_1 e^{-\varepsilon(t-t_2)} \,.\, e^{-\varepsilon(t_2-t_1)} + A t^{-(\beta-1)}$$

satisfies $\gamma(t_1) = B_1$, $\gamma(t) \in S_{t_1}$, $-\gamma(t) \in S^*_{t_1}$ and the differential inequality (38) for sufficiently large $t_2 \geqq t_1$ and sufficiently small $\varepsilon > 0$. This fact and Theorem 5 imply Assertion b).

ii) Assertion a) can be proved analogously as Assertion i), b). If $\varphi(t) = Ae^{-\lambda t}$ (for $t \geqq t_1$) then in the case b) we deduce easily that the function

$$\gamma(t) = Be^{-\delta(t-t_1)}\,, \quad \text{where} \quad 0 < \delta < \min\,(C, \lambda)\,,$$

satisfies (38) for sufficiently big $B \geqq B_1$. Thus, Theorem 5 implies b).

iii) If $\varphi(t) = At^{-\beta}$ $(\beta > 1)$ then the function

$$\gamma(t) = Be^{-\varepsilon(t-t_2)} \,.\, e^{-\varepsilon(t_2-t_1)} + At^{-\delta}$$

where $0 < \delta < \min\,(1/(\alpha-2), \beta/(\alpha-1))$ is an element of S_{t_1} and satisfies (38) for sufficiently big $t_2 \geqq t_1$, $B \geqq B_1$ and sufficiently small $\varepsilon < 0$. Hence, Assertion iii) is a consequence of Theorem 5.

3.

The maximum principle can be obtained for the weak solution of a more general class of the equations (1). Let us consider the nonlinear members $a_i(t, x, u, \partial u/\partial x)$ for $i = 0, 1, \ldots, N$ in the equation (1). We assume that the conditions (3) ((3')), (4), (5), (6) and (9) are fulfilled for $a_i(t, x, \xi)$ where $\xi \in E^{N+1}$, $i = 0, \ldots, N$ and the sums are supposed to range through $i = 0, 1, \ldots, N$. (In the condition (6) $i, j = 0, 1, \ldots, N$.) Moreover, we shall suppose that $a_0(t, x, u, \partial u/\partial x)$ satisfies

i) $s\, a_0(t, x, s, \eta) \geqq 0$ for $\eta \in E^N$, $s \in E^1$,
ii) $|a_0(t, x, s, 0)| \leqq |a_0(t, x, s, \eta)|$ for $\eta \in E^N$.

Theorem 9. *Let the assumptions of Theorem A,* (9), (10), i) *and* ii) *be satisfied. Then the estimates*

$$u(x, t) \leqq \max \left\{ \max_{0\leqq\xi\leqq t} \left(\gamma(\xi) + \int_\xi^t \max_{x\in\Omega} \operatorname{vrai} f(x, \tau)\, d\tau\right); \right.$$

$$\left. \max_{x\in\bar\Omega} u_0(x, 0) + \int_0^t \max_{x\in\Omega} \operatorname{vrai} f(x, \tau)\, d\tau \right\}$$

and

$$u(x, t) \geqq \min \left\{ \min_{0 \leqq \xi \leqq t} \left(\gamma^*(\xi) + \int_\xi^t \min_{x \in \Omega} \operatorname{vrai} f(x, \tau) \, d\tau \right); \right.$$

$$\left. \min_{x \in \bar{\Omega}} u_0(x, 0) + \int_0^t \min_{x \in \Omega} \operatorname{vrai} f(x, \tau) \, d\tau \right\}$$

hold for the weak solution $u(x, t)$ *of* (1), (2), *where* $\gamma(\xi) = \max \{\max_{x \in \partial\Omega} u_0(x, \xi); 0\}$ *and* $\gamma^*(\xi) = \min \{\min_{x \in \partial\Omega} u_0(x, \xi); 0\}$.

Proof. The method of the proof is the same as for Theorem 1. We put

$$\Phi^*(t, v) = \sum_{i=0}^N \int_0^1 \int_\Omega D^i v \, a_i \left(t, x, \tau v, \tau \frac{\partial v}{\partial x} \right) d\tau \, dx$$

for $v \in W(\Omega)$. Due to the conditions (9), i) and ii) we obtain

$$\Phi^*(t, v|^r) \leqq \Phi^*(t, v) \quad \text{for} \quad r \geqq 0$$

and

$$\Phi^*(t, v|_r) \leqq \Phi^*(t, v) \quad \text{for} \quad r \leqq 0 .$$

Indeed, the condition (4) implies that $a_0(t, x, s, 0)$ is an increasing function in s and hence from i) and ii) we conclude

$$\int_0^1 \int_\Omega v|^r \, a_0(t, x, \tau v|^r, 0) \, d\tau \, dx \leqq \int_0^1 \int_\Omega v \, a_0(t, x, \tau v, 0) \, d\tau \, dx \leqq$$

$$\leqq \int_0^1 \int_\Omega v \, a_0 \left(t, x, \tau v, \tau \frac{\partial v}{\partial x} \right) d\tau \, dx \quad \text{for} \quad r \geqq 0 .$$

Similarly we prove the other inequality. Further, we put

$$\Psi^*(t_i, v, u_{i-1}) = \Phi^*(t_i, v) + \frac{1}{2\Delta t_i} \|v - u_{i-1}\|_{L_2}^2 + \int_\Omega v f(x, t_i) \, dx$$

for $v \in W(\Omega)$. Analogously as in Lemma 1 we prove

$$\Psi^*(t_i, u_i, u_{i-1}) \geqq \Psi^*(t_i, u_i|^r, u_{i-1})$$

for

$$r \geqq \max \{\gamma(t_i); \max_{x \in \Omega} \operatorname{vrai} u_{i-1} + \Delta t_i \max_{x \in \Omega} \operatorname{vrai} f(x, t_i)\}$$

and hence by the same argument as in Theorem 1 we prove Theorem 9.

Corollary (maximum principle). *Let the assumptions of Theorem* 9 *and* $f(x, t) = 0$ *be fulfilled. If* $\max\limits_{Q_T} \operatorname{vrai} u(x, t) \geqq 0$ $(\min\limits_{Q_T} \operatorname{vrai} u(x, t) \leqq 0)$, *then* (25) *takes place for the weak solution* $u(x, t)$ *of* (1), (2).

The results obtained can be applied for instance in the following cases.

Examples. 1.

$$\frac{\partial u}{\partial t} - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left[l_i(t, x)\, g_i\left(\frac{\partial u}{\partial x_i}\right) \right] + l_0(t, x)\, g_0(u) = f(x, t)$$

where $g_i(s) \in \mathfrak{M}_3$ for $i = 0, 1, \ldots, N$ are increasing functions and $g_i(s) \in C^2(-\infty, \infty)$. $l_i(t, x) \geqq C > 0$ and $\partial l_i(t, x)/\partial t$, $\partial l_i(t, x)/\partial t^2$ are in $C(\bar{Q})$ for $i = 0, \ldots, N$. (The conditions on smoothness of $l_i(t, x)$ and $g_i(s)$ are introduced in order that Theorem A might be applied.)

2.
$$\frac{\partial u}{\partial t} - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left[l_i(t, x) \left(\frac{\partial u}{\partial x_i}\right)^p \right] + a_0(t, x, u) = f(x, t)$$

where $p = 2k - 1$, with k positive integer.

3.
$$\frac{\partial u}{\partial t} - \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\, a_i\left(t, x, \frac{\partial u}{\partial x}\right) + a_0(t, x, u) = f(x, t)$$

where

$$a_i\left(t, x, \frac{\partial u}{\partial x}\right) \equiv l_i(t, x) \frac{\partial u}{\partial x_i} \left(1 + \sum_{i=1}^{N} \left(\frac{\partial u}{\partial x_i}\right)^2\right)^{m/2-1}$$

$m \geqq 2$, $i = 1, \ldots, N$.

4.
$$\frac{\partial u}{\partial t} - \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\, a_i\left(t, x, u, \frac{\partial u}{\partial x}\right) + a_0\left(t, x, u, \frac{\partial u}{\partial x}\right) = f(x, t)$$

where

$$a_i\left(t, x, u, \frac{\partial u}{\partial x}\right) = l_i(t, x) \frac{\partial u}{\partial x_i} \left(1 + u^2 + \sum_{i=1}^{N} \left(\frac{\partial u}{\partial x_i}\right)^2\right)^{m/2-1}$$

for $i = 0, \ldots, N$, $m \geqq 2$. (For $i = 0$ we put $\partial u/\partial x_0 \equiv u$.)

5.
$$\frac{\partial u}{\partial t} - \Delta u + c\, u|u|^{\alpha-2} = 0 \quad \text{in} \quad \Omega \subset E^N,$$

$$u(x, 0) = \varphi(x), \quad u|_{\partial\Omega} = 0 \quad \text{for} \quad t \geqq 0$$

where $c > 0$, $1 < \alpha < 2$ and $\varphi(x) \in C_0^2(\Omega)$. The identity $u(x, t) \equiv 0$ holds for $t \geqq t_1$ where $t_1 = (\max\limits_{x \in \Omega} |\varphi(x)|)^{2-\alpha}/c(2 - \alpha)$.

References

[1] *J. Kačur:* Method of Rothe and nonlinear parabolic boundary value problems of arbitrary order. (I, II).

[2] *J. Kačur:* Application of Rothe's method to nonlinear evolution equations, Mat. Časopis Sloven. Akad. Vied, *25*, (1975), N-1, 63—81.

[3] *П. П. Мосолов:* Вариационные методы в нестационарных задачах (Параболический случай) Изв. АН СССР, *34* (№ 1970), 425—457.

[4] *K. Rektorys:* On application of direct variational methods to the solution of parabolic boundary value problems of arbitrary order in the space variables. Czech. Math. Journal, *21* (*96*) 1971, 318—339.

[5] *О. А. Ладыженская:* Решение в целом первой краевой задачи для квазилинейных параболических уравнений, ДАН СССР *107*, 1956, 635—639.

[6] *J. Kačur:* On existence of the weak solution for non-linear partial differential equations of elliptic type. I. Comment. Math. Univ. Carolinae, *11*, 1 (1970), 137—181.

[7] *J. Kačur:* On existence of the weak solution for non-linear partial differential equations of elliptic type. II. Comment. Math. Univ. Carolinae, *13*, 2 (1972), 211—225.

[8] *J. Nečas:* Les méthodes directes en théorie équations elliptiques. Prague, 1967.

[9] *A. Friedman:* Partial differential equations of parabolic type, Prentice-Hall, INC, 1964.

[10] *О. А. Ладыженская, В. А. Солонников, Н. Н. Уральцева:* Линейные и квазилинейные уравнения параболического типа. „Наука", Москва 1967.

[11] *J. Kačur:* On boundedness of the weak solution for some class of quasilinear partial differential equations. Časopis Pěst. Mat., *98* (1973), 43—55.

[12] *М. А. Красносельский:* Топологические методы в теории нелинейных интегральных уравнений, Москва, 1956.

Author's address: 816 31 Bratislava, Mlynská dolina, ČSSR (Prírodovedecká fakulta UK).