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HOMOMORPHISMS OF PARTIAL UNARY ALGEBRAS

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1. PROBLEM

1.0. Notation. If A is a set we denote by $|A|$ the cardinal number of A. We denote by Ord the class of all ordinals. If $\alpha \in \text{Ord}$ then we put $W_\alpha = \{\beta \in \text{Ord};\ \beta < \alpha\}$. We denote by N the set of all finite ordinals.

Let φ be a partial map from the set A into the set B. We put $\text{dom}\ \varphi = \{x \in A;$ there exists $y \in B$ such that $(x, y) \in \varphi\}$. If $\text{dom}\ \varphi = A$ then we write $\varphi : A \to B$ and speak about a map φ. If $C \subseteq A$, $D \subseteq B$ then we put $\varphi(C) = \{\varphi(x);\ x \in C\}$; further, we define $\varphi^{-1}(D) = \{x \in A;\ \varphi(x) \in D\}$; finally, we denote by $\varphi \mid C$ the restriction $\varphi \cap (C \times B)$ of φ.

1.1. Definition. Let A be a non empty set, f a partial map from the set A into A. Then the ordered pair (A, f) is called a *unary algebra*.

1.2. Definition. Let (A, f) be a unary algebra. Then we put $D(A, f) = A - \text{dom} f$. If $D(A, f) = \emptyset$ then (A, f) is called a *complete unary algebra*.

1.3. Definition. Let (A, f), (B, g) be unary algebras and $F : A \to B$ a map. Then F is called a *homomorphism* of (A, f) into (B, g) if $x \in \text{dom} f$ implies $F(x) \in \text{dom}\ g$ and $F(f(x)) = g(F(x))$ for each $x \in A$. We write $F : (A, f) \to (B, g)$.

1.4. Problem. *Let (A, f), (B, g) be unary algebras. Find all homomorphisms $F : (A, f) \to (B, g)$.*

1.5. Definition. Let (A, f) be a unary algebra. We put $f^0 = \text{id}_A$. Suppose that we have defined a partial map f^{n-1} from A into A for $n \in N - \{0\}$. We denote by f^n the following partial map from A into A: if $x \in \text{dom} f^{n-1}$ and $f^{n-1}(x) \in \text{dom} f$ then we put $f^n(x) = f(f^{n-1}(x))$.

1.6. Lemma. *Let (A, f) be a unary algebra. Then the following assertions hold:*

(a) *(A, f) is complete iff* $\operatorname{dom} f^n = A$ *for all* $n \in N$.

(b) *If* $n \in N - \{0\}$, $x \in \operatorname{dom} f^n$ *then* $x \in \operatorname{dom} f^m$ *for each* $m \in \{0, 1, \ldots, n\}$ *and* $f^m(x) \in \operatorname{dom} f$ *for each* $m \in \{0, 1, \ldots, n-1\}$.

(c) *If* $n \in N - \{0\}$, $x_0, x_1, \ldots, x_n \in A$, $\{x_0, x_1, \ldots, x_{n-1}\} \subseteq \operatorname{dom} f$ *and* $f(x_i) = x_{i+1}$ *for each* $i \in \{0, 1, \ldots, n-1\}$ *then* $x_0 \in \operatorname{dom} f^n$ *and* $f^n(x_0) = x_n$.

(d) *Let* $n \in N$, $x \in A$ *be arbitrary. Then* $x \in \operatorname{dom} f^n$ *iff* $f^p(x) \in \operatorname{dom} f^{q-p}$ *for each* $p, q \in N$, $0 \leqq p \leqq q \leqq n$.

(e) *If* $m, n \in N$, $x \in \operatorname{dom} f^m$, $f^m(x) \in \operatorname{dom} f^n$ *then* $x \in \operatorname{dom} f^{m+n}$ *and* $f^{m+n}(x) = f^n(f^m(x))$.

(f) *If* $m, n \in N$, $x \in \operatorname{dom} f^m$, $f^m(x) \in \operatorname{dom} f^n$ *then* $x \in \operatorname{dom} f^n$, $f^n(x) \in \operatorname{dom} f^m$ *and* $f^m(f^n(x)) = f^n(f^m(x))$.

Proof of (a) is evident.

Proof of (b). The assertion follows directly from 1.5.

Proof of (c). Denoting by $V(n)$ the assertion (c) for $n \in N - \{0\}$ we see that $V(1)$ holds.

Let $n \in N - \{0, 1\}$ be arbitrary and let $V(n-1)$ hold. Further, let $\{x_0, x_1, \ldots, x_{n-1}\} \subseteq \operatorname{dom} f$ for $x_0, x_1, \ldots, x_n \in A$ and let $f(x_i) = x_{i+1}$ for each $i \in \{0, 1, \ldots, n-1\}$. Then the conditions of $V(n-1)$ are satisfied; thus, $x_0 \in \operatorname{dom} f^{n-1}$, $f^{n-1}(x_0) = x_{n-1}$. Further, $f^{n-1}(x_0) = x_{n-1} \in \operatorname{dom} f$ and we obtain, by 1.5, $x_0 \in \operatorname{dom} f^n$ and $f^n(x_0) = f(f^{n-1}(x_0)) = f(x_{n-1}) = x_n$. Thus, we have $V(n)$.

Proof of (d). The condition is sufficient for $p = 0$, $q = n$. The condition is necessary: By (b), $x \in \operatorname{dom} f^m$ for each $m \in \{0, 1, \ldots, n\}$ and $f^m(x) \in \operatorname{dom} f$ for each $m \in \{0, 1, \ldots, n-1\}$. Let $p, q \in N$, $0 \leqq p \leqq q \leqq n$ be arbitrary. Clearly, for $p = q$ the condition holds. Suppose that $p < q$. We put $x_i = f^{p+i}(x)$ for each $i \in \{0, 1, \ldots, q-p\}$. Then $\{x_0, x_1, \ldots, x_{q-p-1}\} \subseteq \operatorname{dom} f$ and, for each $i \in \{0, 1, \ldots, q-p-1\}$, $f(x_i) = f(f^{p+i}(x)) = f^{p+i+1}(x) = x_{i+1}$ by 1.5. Thus, $f^p(x) = x_0 \in \operatorname{dom} f^{q-p}$ by (c).

Proof of (e). We denote, for $n \in N$, the assertion (e) by $V(n)$. Clearly, $V(0)$ holds.

Let $n \in N - \{0\}$ be arbitrary and let $V(n-1)$ hold; further, if $x \in \operatorname{dom} f^m$, $f^m(x) \in \operatorname{dom} f^n$ then the conditions of $V(n-1)$ are satisfied which implies $x \in \operatorname{dom} f^{m+n-1}$, $f^{m+n-1}(x) = f^{n-1}(f^m(x))$. By (d), $f^m(x) \in \operatorname{dom} f^n$ implies $f^{m+n-1}(x) = f^{n-1}(f^m(x)) \in \operatorname{dom} f$; by 1.5, we obtain $x \in \operatorname{dom} f^{m+n}$ and $f^{m+n}(x) = f(f^{m+n-1}(x)) = f(f^{n-1}(f^m(x))) = f^n(f^m(x))$. Thus, $V(n)$ holds.

Proof of (f). By (e), we have $x \in \operatorname{dom} f^{m+n}$ and $f^{m+n}(x) = f^n(f^m(x))$. Hence $x \in \operatorname{dom} f^n$, $f^n(x) \in \operatorname{dom} f^m$ by (d) which implies $f^{n+m}(x) = f^m(f^n(x))$ by (e).

1.7. Definition. Let (A, f) be a unary algebra and let $x \in A$ be arbitrary. Then we define $[x]_{(A,f)} = \{f^n(x);\ x \in \operatorname{dom} f^n\}$.

1.8. Lemma. *Let (A, f), (B, g) be unary algebras, $F : (A, f) \to (B, g)$ a homomorphism. Then, for each $x \in A$, $n \in N$, $x \in \operatorname{dom} f^n$ implies $F(x) \in \operatorname{dom} g^n$ and $F(f^n(x)) = g^n(F(x))$.*

Proof. Let $x \in A$, $n \in N$ be arbitrary. We denote by $V(n)$ the assertion: if $x \in \operatorname{dom} f^n$ then $F(x) \in \operatorname{dom} g^n$ and $F(f^n(x)) = g^n(F(x))$.

$V(0)$ holds because $F(x) \in B = \operatorname{dom} g^0$ and $F(f^0(x)) = F(x) = g^0(F(x))$.

Let $n \in N - \{0\}$ be arbitrary and let $V(n-1)$ hold. Further, let $x \in \operatorname{dom} f^n$; then $x \in \operatorname{dom} f^{n-1}$ by 1.6 (d) and, by $V(n-1)$, $F(x) \in \operatorname{dom} g^{n-1}$, $F(f^{n-1}(x)) = g^{n-1}(F(x))$. Further, $x \in \operatorname{dom} f^n$ implies $f^{n-1}(x) \in \operatorname{dom} f$ by 1.6 (d). Thus, $F(f^{n-1}(x)) \in \operatorname{dom} g$ and $F(f(f^{n-1}(x))) = g(F(f^{n-1}(x)))$. We obtain $F(f^n(x)) = F(f(f^{n-1}(x))) = g(F(f^{n-1}(x))) = g(g^{n-1}(F(x))) = g^n(F(x))$ by 1.6 (e). Thus, $V(n)$ holds.

1.9. Definition. Let (A, f) be a unary algebra. For arbitrary $x, y \in A$, we put

$$(x, y) \in \varrho(A, f) \quad \text{iff there exist} \quad m, n \in N \quad \text{such that} \quad x \in \operatorname{dom} f^m,$$
$$y \in \operatorname{dom} f^n \quad \text{and} \quad f^m(x) = f^n(y).$$

If $\varrho(A, f) = A \times A$ then (A, f) is called a *connected* unary algebra and we refer to it briefly as to a c-algebra.

2. c-ALGEBRAS

First, we shall solve Problem 1.4 for c-algebras.

2.1. Lemma. *Let (A, f) be a c-algebra. Then $|D(A, f)| \leqq 1$.*

Proof. Suppose, on the contrary, $x, y \in D(A, f)$ and $x \neq y$. Then there are $m, n \in N$ such that $x \in \operatorname{dom} f^m$, $y \in \operatorname{dom} f^n$ and $f^m(x) = f^n(y)$. We see that $m = 0$, $n = 0$ cannot occur because, in this case, $x = f^0(x) = f^0(y) = y$. Let, for example, $m \neq 0$. Then we obtain $x \in \operatorname{dom} f$ by 1.6 (d) which is a contradiction. Similarly, we obtain a contradiction for $n \neq 0$.

2.2. Definition. Let (A, f) be a c-algebra such that $D(A, f) \neq \emptyset$. Then we put $\{d(A, f)\} = D(A, f)$.

2.3. Lemma. *Let (A, f) be a c-algebra such that $D(A, f) \neq \emptyset$. Then, for arbitrary $x \in A$, there is $m \in N$ such that $x \in \operatorname{dom} f^m$ and $f^m(x) = d(A, f)$.*

Proof. For $x \in A$, $d(A, f) \in A$, there exist $m, n \in N$ such that $x \in \operatorname{dom} f^m$, $d(A, f) \in \operatorname{dom} f^n$ and $f^m(x) = f^n(d(A, f))$. Hence $n = 0$ by 1.6 (d) and we have $f^m(x) = f^0(d(A, f)) = d(A, f)$.

2.4. Definition. Let (A, f) be a c-algebra and $x \in A$ arbitrary. Then we define $Z(x) = \{y \in A;$ there exists an infinite set $N(y) \subseteq N$ such that $x \in \operatorname{dom} f^n$ and $f^n(x) = y$ for each $n \in N(y)\}$.

2.5. Lemma. *Let (A, f) be a c-algebra such that $D(A, f) \neq \emptyset$. Then, for arbitrary $x \in A$, $Z(x) = \emptyset$.*

Proof. Suppose, on the contrary, $Z(x) \neq \emptyset$ and $y \in Z(x)$. Then there is an infinite set $N(y) \subseteq N$ such that, for each $n \in N(y)$, $x \in \operatorname{dom} f^n$, $f^n(x) = y$. Further, by 2.3, there is $n_0 \in N$ such that $x \in \operatorname{dom} f^{n_0}$ and $f^{n_0}(x) = d(A, f)$. Since $N(y)$ is infinite, there is $n_1 \in N(y)$ such that $n_1 > n_0$. Thus, by 1.6 (d), the conditions of 1.6 (e) are fulfilled and by 1.6 (e), $y = f^{n_1}(x) = f^{n_1 - n_0}(f^{n_0}(x)) = f^{n_1 - n_0}(d(A, f))$. In virtue of $n_1 - n_0 > 0$, we obtain, by 1.6 (d), $d(A, f) \in \operatorname{dom} f$ which is a contradiction.

2.6. Lemma. *Let (A, f) be a c-algebra. Then $Z(x) = Z(y)$ for any $x, y \in A$.*

Proof. For $D(A, f) = \emptyset$, (A, f) is a complete c-algebra and the assertion follows from [2], 1.2.

Let $D(A, f) \neq \emptyset$. Then $Z(x) = \emptyset = Z(y)$ by 2.5.

2.7. Definition. Let (A, f) be a c-algebra. Then we put $Z(A, f) = Z(x)$ where $x \in A$ is an arbitrary element, $R(A, f) = |Z(A, f)|$. $Z(A, f)$ is called the *cycle* and $R(A, f)$ the *range* of (A, f).

2.8. Lemma. *Let (A, f) be a c-algebra, $x \in A$ arbitrary. Then*

(a) *$x \in Z(A, f)$ iff there is $n \in N - \{0\}$ such that $x \in \operatorname{dom} f^n$ and $f^n(x) = x$;*

(b) *$i, j \in N$, $i < j$, $x \in \operatorname{dom} f^j$, $f^i(x) = f^j(x)$ imply $f^i(x) \in Z(A, f)$.*

Proof of (a). If $D(A, f) = \emptyset$ then the assertion follows from [2], 1.5 (b). If $D(A, f) \neq \emptyset$ then $Z(A, f) = \emptyset$ by 2.5 and 2.7 and the assertion holds trivially.

Proof of (b). By 1.6 (d), we have $x \in \operatorname{dom} f^i$, $f^i(x) \in \operatorname{dom} f^{j-i}$ and $f^{j-i}(f^i(x)) = f^j(x) = f^i(x)$ which implies $f^i(x) \in Z(A, f)$ by (a).

2.9. Lemma. *Let (A, f) be a c-algebra. Then the following assertions hold:*

(a) *$D(A, f) \neq \emptyset$ iff $R(A, f) = 0$ and there is $x_0 \in A$ such that $|[x_0]_{(A,f)}| < \aleph_0$.*

(b) *$|[x]_{(A,f)}| < \aleph_0$ or $|[x]_{(A,f)}| \geqq \aleph_0$ for all $x \in A$ iff there is $x_0 \in A$ such that $|[x_0]_{(A,f)}| < \aleph_0$ or $|[x_0]_{(A,f)} \geqq \aleph_0$, respectively.*

(c) (A, f) *is complete iff either* $R(A, f) \neq 0$ *or there is* $x_0 \in A$ *such that* $|[x_0]_{(A,f)}| \geqq \aleph_0$.

Proof of (a). Let $D(A, f) \neq \emptyset$; then $Z(A, f) = \emptyset$ by 2.5 and 2.7 which implies $R(A, f) = 0$. Further, $|[d(A, f)]_{(A,f)}| = 1 < \aleph_0$.

On the other hand, suppose $R(A, f) = 0$ and the existence of $x_0 \in A$ such that $|[x_0]_{(A,f)}| < \aleph_0$.

(1) Then, for all $i, j \in N$ such that $i \neq j$, then conditions $x_0 \in \operatorname{dom} f^i$, $x_0 \in \operatorname{dom} f^j$ imply $f^i(x_0) \neq f^j(x_0)$. Indeed, if we had $f^i(x_0) = f^j(x_0)$ and, for example, $i < j$ then we should have, by 2.8 (b), $f^i(x_0) \in Z(A, f)$ which is a contradiction to $R(A, f) = 0$.

(2) Further, we put $m = |[x_0]_{(A,f)}|$. Then, by 1.7 and 1.6 (d), $x_0 \in \operatorname{dom} f^j$ for $j = 0, 1, \ldots, m - 1$. Further, $x_0 \notin \operatorname{dom} f^m$, because if we had $x_0 \in \operatorname{dom} f^m$ then we should have $i \in \{0, 1, \ldots, m - 1\}$ such that $f^m(x_0) = f^i(x_0)$ (because $|\{f^0(x_0), f^1(x_0), \ldots, f^{m-1}(x_0)\}| = m$ by (1)) which is a contradiction to (1). Hence $f^{m-1}(x_0) \notin$ $\notin \operatorname{dom} f$ by 1.6 (d) because $x_0 \in \operatorname{dom} f^{m-1}$. Thus $D(A, f) \neq \emptyset$.

Proof of (b). Clearly, the condition is necessary.

Let, on the other hand, $|[x_0]_{(A,f)}| < \aleph_0$ for $x_0 \in A$. Let $x \in A$ be arbitrary; then there exist $m, n \in N$ such that $x \in \operatorname{dom} f^m$, $x_0 \in \operatorname{dom} f^n$ and $f^m(x) = f^n(x_0)$. Hence $[f^m(x)]_{(A,f)} = [f^n(x_0)]_{(A,f)} \subseteq [x_0]_{(A,f)}$ which implies $|[f^m(x)]_{(A,f)}| < \aleph_0$. Further, $[x]_{(A,f)} = \{f(x), f^2(x), \ldots, f^{m-1}(x)\} \cup [f^m(x)]_{(A,f)}$ by 1.7 and 1.6 (d) and we obtain $|[x]_{(A,f)}| < \aleph_0$.
The second assertion is a consequence of the first one.

Proof of (c). The assertion follows from (a) and (b).

2.10. Lemma. *Let* (A, f) *be a c-algebra. Then* $(Z(A, f), f \mid Z(A, f))$ *is a subalgebra of* (A, f).

Proof. If $Z(A, f) = \emptyset$ then the assertion holds trivially. If $Z(A, f) \neq \emptyset$ then $R(A, f) \neq 0$ and (A, f) is complete by 2.9 (c). The assertion follows from [2], 1.4.

2.11. Lemma. *Let* (A, f) *be a c-algebra. Then the following assertions hold:*

(a) *If* $x \in Z(A, f)$ *is arbitrary then* $R(A, f) = \min \{n \in N - \{0\}; f^n(x) = x\}$;

(b) $R(A, f) < \aleph_0$.

Proof of (a). Since $R(A, f) \neq 0$ the c-algebra is complete by 2.9 (c) and the assertion follows from [2], 1.6 (a).

Proof of (b). If $D(A, f) = \emptyset$ then (A, f) is complete and the assertion follows from [2], 1.6 (b). If $D(A, f) \neq \emptyset$ then $R(A, f) = 0$ by 2.9 (a).

2.12. Lemma. *Let (A, f) be a c-algebra, $x \in Z(A, f)$ arbitrary. Then (A, f) is complete and the following assertions hold:*

(a) $f^{p.R(A,f)}(x) = x$ *for each* $p \in N$;

(b) $f^m(x) = x$ *iff* $R(A, f) \mid m$ *).

Proof. (A, f) is complete by 2.9 (c).

Proof of (a). The assertion follows from [2], 1.5 (a).

Proof of (b). Let $R(A, f) \mid m$; then there is $p \in N$ such that $m = R(A, f) . p$ which implies $f^m(x) = f^{p.R(A,f)}(x) = x$ by (a).

Let, on the other hand, $f^m(x) = x$ hold. Then $R(A, f) \leqq m$ by 2.11 (a) and there are $p, q \in N$ such that $m = p . R(A, f) + q$ and $0 \leqq q < R(A, f)$. By 1.6 (e), 2.10 and (a) we have $x = f^m(x) = f^{p.R(A,f)+q}(x) = f^{p.R(A,f)}(f^q(x)) = f^q(x)$. If $q \in N - \{0\}$ then $R(A, f) \leqq q$ by 2.11 (a) which is a contradiction to the definition of q. Hence $q = 0$ and we obtain $m = p . R(A, f)$. Thus, $R(A, f) \mid m$.

2.13. Notation. Let $\infty, \infty_1, \infty_2 \notin \mathrm{Ord}$.

If M is an arbitrary set of ordinals then we denote by $\leqq$ the order relation on $M \cup \{\infty_1, \infty_2\}$ such that its restriction $\leqq \cap M^2$ to M is the natural order relation of ordinals and that $\alpha < \infty_1 < \infty_2$ for each $\alpha \in M$.

2.14. Definition. Let (A, f) be a c-algebra. We put $A^\infty = \{x \in A$; there is a sequence $(x_i)_{i \in N}$ such that $x_i \in \mathrm{dom} f$ for each $i \in N - \{0\}$, $x_0 = x$ and $f(x_{i+1}) = x_i$ for each $i \in N\}$, $A^0 = \{x \in A; f^{-1}(x) = \emptyset\}$.

Let $\alpha \in \mathrm{Ord}$, $\alpha > 0$ and suppose that the sets $A^\varkappa$ have been defined for all $\varkappa \in W_\alpha$. Then we put $A^\alpha = \{x \in A - \bigcup_{\varkappa \in W_\alpha} A^\varkappa; f^{-1}(x) \subseteq \bigcup_{\varkappa \in W_\alpha} A^\varkappa\}$.

2.15. Lemma. *Let (A, f) be a c-algebra. Then the following assertions hold:*

(a) $(A^\infty, f \mid A^\infty)$ *is a subalgebra of the c-algebra* (A, f);

(b) $Z(A, f) \subseteq A^\infty$.

Proof of (a). Let $x \in A^\infty$ be such that $x \in \mathrm{dom} f$. Then there is a sequence $(x_i)_{i \in N}$ such that $x_i \in \mathrm{dom} f$ for each $i \in N - \{0\}$, $x_0 = x$ and $f(x_{i+1}) = x_i$ for each $i \in N$. We put $f(x) = y_0$ and $y_{i+1} = x_i$ for all $i \in N$. Then $y_1 = x_0 = x \in \mathrm{dom} f$ and, for each $i \in N - \{0\}$, $y_{i+1} = x_i \in \mathrm{dom} f$. Thus $y_i \in \mathrm{dom} f$ for each $i \in N - \{0\}$. Further, $y_0 = f(x)$, $f(y_1) = f(x_0) = y_0$ and, for each $i \in N - \{0, 1\}$, we have $f(y_{i+1}) = f(x_i) = x_{i-1} = y_i$. Hence $f(x) \in A^\infty$ by 2.14.

*) $p \mid q$ for $p, q \in N$ means that p is a divisor of q.

Proof of (b). If $Z(A,f) = \emptyset$ then the assertion holds trivially. If $Z(A,f) \neq \emptyset$ then $R(A,f) \neq 0$ and (A,f) is complete by 2.9 (c). The assertion follows from [2], 1.15.

2.16. Definition. Let (A, f) be a c-algebra. Then we put $A^{\infty_1} = A^{\infty} - Z(A,f)$, $A^{\infty_2} = Z(A,f)$.

2.17. Lemma. *Let (A,f) be a c-algebra. Then*

(a) *if $x \in A^{\infty_1}$ then $f^{-1}(x) \cap A^{\infty_1} \neq \emptyset$;*

(b) *if $x \in A^{\infty_2}$ then $f^{-1}(x) \cap A^{\infty_2} \neq \emptyset$.*

Proof of (a). If $x \in A^{\infty_1}$ then $x \in A^{\infty}$ and there is a sequence $(x_i)_{i \in N}$ such that $x_i \in \in \operatorname{dom} f$ for each $i \in N - \{0\}$, $x_0 = x$ and $f(x_{i+1}) = x_i$ for each $i \in N$. Clearly, $x_1 \in A^{\infty}$. Further, $x_1 \notin Z(A,f)$ by 2.10 and we have $x_1 \in f^{-1}(x) \cap A^{\infty_1}$.

Proof of (b). If $x \in A^{\infty_2}$ then $x \in Z(A,f)$ and $f^{R(A,f)}(x) = x$ by 2.12 (a). Thus, $f^{R(A,f)-1}(x) \in f^{-1}(x)$ and $f^{R(A,f)-1}(x) \in Z(A,f) = A^{\infty_2}$ by 2.10.

2.18. Lemma. *Let (A,f) be a c-algebra, $\alpha, \beta \in \mathrm{Ord}$, $\alpha \neq \beta$. Then $A^{\alpha} \cap A^{\beta} = \emptyset$.*

Proof. If, for example, $\alpha < \beta$, then $A^{\beta} \cap A^{\alpha} \subseteq A^{\beta} \cap \bigcup_{\varkappa \in W_\beta} A^{\varkappa} = \emptyset$ because $A^{\beta} \subseteq \subseteq A - \bigcup_{\varkappa \in W_\beta} A^{\varkappa}$.

2.19. Lemma. *Let (A,f) be a c-algebra. Then:*

(a) *There is $\vartheta \in \mathrm{Ord}$ such that $A^{\vartheta} = \emptyset$.*

(b) *If $\vartheta \in \mathrm{Ord}$, $A^{\vartheta} = \emptyset$ then $A^{\lambda} = \emptyset$ for each $\lambda \in \mathrm{Ord}$ with the property $\lambda \geqq \vartheta$.*

Proof of (a). Let $\nu \in \mathrm{Ord}$ be an ordinal number such that $|A| \leqq \aleph_\nu$. Suppose $A^{\lambda} \neq \emptyset$ for each $\lambda \in W_{\omega_{\nu+1}}$. Then $\aleph_{\nu+1} \leqq \sum_{\lambda \in W_{\omega_{\nu+1}}} |A^{\lambda}| = |\bigcup_{\lambda \in W_{\omega_{\nu+1}}} A^{\lambda}| \leqq |A| \leqq \aleph_\nu$ by 2.18 which is a contradiction.

Thus, there is $\vartheta \in W_{\omega_{\nu+1}}$ such that $A^{\vartheta} = \emptyset$.

Proof of (b). We denote by $V(\lambda)$ the following assertion: $A^{\lambda} = \emptyset$. Then $V(\vartheta)$ holds.

Let $\beta \in \mathrm{Ord}$, $\vartheta < \beta$, suppose that $V(\lambda)$ holds for each $\lambda \in \mathrm{Ord}$ with the property $\vartheta \leqq \lambda < \beta$. Then $\bigcup_{\lambda \in W_\beta} A^{\lambda} = \bigcup_{\lambda \in W_\vartheta} A^{\lambda}$ which implies $A^{\beta} = \{x \in A - \bigcup_{\lambda \in W_\beta} A^{\lambda};\ f^{-1}(x) \subseteq \subseteq \bigcup_{\lambda \in W_\beta} A^{\lambda}\} = \{x \in A - \bigcup_{\lambda \in W_\vartheta} A^{\lambda}; f^{-1}(x) \subseteq \bigcup_{\lambda \in W_\vartheta} A^{\lambda}\} = A^{\vartheta} = \emptyset$.

The assertion follows by transfinite induction.

2.20. Definition. Let (A,f) be a c-algebra. Then we put $\vartheta(A,f) = \min \{\vartheta \in \mathrm{Ord}; A^{\vartheta} = \emptyset\}$.

2.21. Lemma. *Let (A, f) be a c-algebra. Then* $A^\infty = A - \bigcup_{\varkappa \in W_{\vartheta(A,f)}} A^\varkappa$.

Proof. If $x \in A - \bigcup_{\varkappa \in W_{\vartheta(A,f)}} A^\varkappa$ then there is an element $x' \in f^{-1}(x)$ such that $x' \in A - \bigcup_{\varkappa \in W_{\vartheta(A,f)}} A^\varkappa$. Indeed, if we had $f^{-1}(x) \subseteq \bigcup_{\varkappa \in W_{\vartheta(A,f)}} A^\varkappa$ then we should put $\vartheta = \min \{\lambda \in \mathrm{Ord}; f^{-1}(x) \subseteq \bigcup_{\varkappa \in W_\lambda} A^\varkappa\}$. Then $\vartheta \leqq \vartheta(A, f)$ and $x \in A^\vartheta$ by 2.14 which is a contradiction either to $A^{\vartheta(A,f)} = \emptyset$ (in the case $\vartheta = \vartheta(A, f)$) or to $x \in A - \bigcup_{\varkappa \in W_{\vartheta(A,f)}} A^\varkappa$ (in the case $\vartheta < \vartheta(A, f)$). Clearly, $x' \in \operatorname{dom} f$.

We put $x_0 = x$ and $x_{n+1} = x_n'$ for $n \in N$. Then $x_n \in \operatorname{dom} f$ for each $n \in N - \{0\}$ and $f(x_{n+1}) = x_n$ for each $n \in N$. Thus, $x \in A^\infty$ and $A - \bigcup_{\varkappa \in W_{\vartheta(A,f)}} A^\varkappa \subseteq A^\infty$.

Let us have, on the other hand, $x \in A^\infty \cap (\bigcup_{\varkappa \in W_{\vartheta(A,f)}} A^\varkappa)$. Then there exists a sequence $(x_i)_{i \in N}$ such that $x_i \in \operatorname{dom} f$ for each $i \in N - \{0\}$, $x_0 = x$ and $f(x_{i+1}) = x_i$ for each $i \in N$. By 2.18, there is precisely one $\varkappa_0 \in W_{\vartheta(A,f)}$ such that $x_0 \in A^{\varkappa_0}$.

Suppose that we have constructed ordinals $\varkappa_0 > \varkappa_1 > \ldots > \varkappa_n$ with the property $x_i \in A^{\varkappa_i}$ for $i = 0, 1, \ldots, n$ where $n \in N$. Then $x_{n+1} \in f^{-1}(x_n) \subseteq \bigcup_{\varkappa \in W_{\varkappa_n}} A^\varkappa$ which implies the existence of $\varkappa_{n+1} < \varkappa_n$ such that $x_{n+1} \in A^{\varkappa_{n+1}}$. Thus, $(\varkappa_i)_{i \in N}$ is an infinite decreasing sequence of ordinals which is a contradiction.

Consequently, $A^\infty \subseteq A - \bigcup_{\varkappa \in W_{\vartheta(A,f)}} A^\varkappa$.

2.22. Theorem. *Let (A, f) be a c-algebra and put $W^* = W_{\vartheta(A,f)} \cup \{\infty_1, \infty_2\}$. Then $A = \bigcup_{\varkappa \in W^*} A^\varkappa$ with disjoint terms.*

Proof. The assertion is a consequence of 2.18, 2.21, 2.15 (6) and 2.16.

2.23. Definition. Let (A, f) be a c-algebra. We define a map $S(A, f) : A \to \mathrm{Ord} \cup \{\infty_1, \infty_2\}$ by the condition $S(A, f)(x) = \varkappa$ for each $x \in A^\varkappa$, $\varkappa \in W_{\vartheta(A,f)} \cup \{\infty_1, \infty_2\}$. $S(A, f)(x)$ is called the degree of x.

2.24. Notation. Let $\emptyset \neq M \subseteq \mathrm{Ord}$, $\alpha \in \mathrm{Ord}$. Then we put $M \leqq \alpha$ if $\beta \leqq \alpha$ for each $\beta \in M$.

2.25. Lemma. *Let (A, f) be a c-algebra, $\alpha \in \mathrm{Ord}$, $x \in A - A^\infty$. Then the following assertions hold:*

(a) *$S(A, f)(x) = \alpha$ iff $\alpha \leqq S(A, f)(x)$ and $S(A, f)(f^{-1}(x)) < \alpha$.*

(b) *If $S(A, f)(x) = \alpha$ then W_α is cofinal with $S(A, f)(f^{-1}(x))$.*

(c) *If $S(A, f)(f^{-1}(x)) < \alpha$ then $S(A, f)(x) \leqq \alpha$.*

Proof of (a). The assertion follows directly from 2.14 and 2.23 because $S(A,f)(x) = \alpha$ is equivalent to $x \in A - \bigcup_{\varkappa \in W_\alpha} A^\varkappa$, $f^{-1}(x) \subseteq \bigcup_{\varkappa \in W_\alpha} A^\varkappa$ which is equivalent to $\alpha \leqq S(A,f)(x)$, $S(A,f)(f^{-1}(x)) < \alpha$.

Proof of (b). Suppose $S(A,f)(x) = \alpha$ and, on the contrary, the existence of $\beta \in W_\alpha$ such that $\{\gamma; \beta \leqq \gamma < \alpha\} \cap S(A,f)(f^{-1}(x)) = \emptyset$. Then $S(A,f)(f^{-1}(x)) < \beta$ and, since $S(A,f)(x) = \alpha > \beta$, we obtain by (a) $S(A,f)(x) = \beta$ which is a contradiction.

Proof of (c). Suppose $S(A,f)(f^{-1}(x)) < \alpha$ and, on the contrary, $\alpha < S(A,f)(x)$. Then, by (b), there is $y \in f^{-1}(x)$ such that $S(A,f)(y) \geqq \alpha$ which is a contradiction to $S(A,f)(f^{-1}(x)) < \alpha$.

2.26. Lemma. *Let (A,f) be a c-algebra. Then the following assertions hold:*

(a) *If $x \in A - A^\infty$ and $n \in N$ are such that $x \in \operatorname{dom} f^n$ then $S(A,f)(f^n(x)) \geqq S(A,f)(x) + n$.*

(b) *If $x \in A$ is such that $x \in \operatorname{dom} f$ then $S(A,f)(f(x)) \geqq S(A,f)(x)$.*

(c) *If $D(A,f) \neq \emptyset$, $A^\infty = \emptyset$ then $\vartheta(A,f)$ is isolated and $S(A,f)(d(A,f)) = \vartheta(A,f) - 1$.*

(d) *If $D(A,f) \neq \emptyset$, $A^\infty \neq \emptyset$ then $S(A,f)(d(A,f)) = \infty_1$.*

Proof of (a). For an arbitrary $n \in N$, we denote by $V(n)$ the following assertion: if $x \in \operatorname{dom} f^n$ then $S(A,f)(f^n(x)) \geqq S(A,f)(x) + n$.

Clearly, $V(0)$ holds.

Let $n \in N - \{0\}$ and let $V(n-1)$ hold. Further, suppose $x \in \operatorname{dom} f^n$. If $S(A,f)(f^n(x)) \in \{\infty_1, \infty_2\}$ then $V(n)$ holds because $\{\infty_1, \infty_2\} > S(A,f)(x) + n$.

Suppose that $S(A,f)(f^n(x)) \in \mathrm{Ord}$. Since $x \in \operatorname{dom} f^n$ we have $x \in \operatorname{dom} f^{n-1}$ by 1.6 (d). Hence $S(A,f)(f^{n-1}(x)) \geqq S(A,f)(x) + n - 1$ by $V(n-1)$. Further, $f^{n-1}(x) \in f^{-1}(f^n(x))$; if we put $S(A,f)(f^n(x)) = \alpha$ then $f^{n-1}(x) \in \bigcup_{\varkappa \in W_\alpha} A^\varkappa$ and there is $\varkappa_0 \in W_\alpha$ such that $f^{n-1}(x) \in A^{\varkappa_0}$. Thus, $S(A,f)(f^{n-1}(x)) = \varkappa_0$ and we obtain $S(A,f)(f^n(x)) = \alpha \geqq \varkappa_0 + 1 = S(A,f)(f^{n-1}(x)) + 1 \geqq S(A,f)(x) + n - 1 + 1 = S(A,f)(x) + n$.

$V(n)$ holds.

Proof of (b). Let $x \in A$ be such that $x \in \operatorname{dom} f$.

If $S(A,f)(x) \in \mathrm{Ord}$ then $x \in A - A^\infty$ which implies $S(A,f)(f(x)) > S(A,f)(x)$ by (a).

If $S(A,f)(x) = \infty_1$ then $S(A,f)(f(x)) \in \{\infty_1, \infty_2\}$ and, finally, if $S(A,f)(x) = \infty_2$ then $S(A,f)(f(x)) = \infty_2$ by 2.15 (a) and 2.10.

Proof of (c). Let $D(A,f) \neq \emptyset$, $A^\infty = \emptyset$. Then $S(A,f)(d(A,f)) = \delta \in \mathrm{Ord}$ and $\delta < \vartheta(A,f)$. Let there be $\delta < \varepsilon < \vartheta(A,f)$ and $x \in A$ with the property $S(A,f)(x) =$

$= \varepsilon$. Then, by 2.3, there is $n \in N$ such that $x \in \operatorname{dom} f^n$ and $f^n(x) = d(A, f)$. Thus we obtain by (a) $\delta = S(A, f)(d(A, f)) = S(A, f)(f^n(x)) \geqq S(A, f)(x) + n = \varepsilon + n \geqq \varepsilon$ which is a contradiction to $\delta < \varepsilon$. Thus, $\vartheta(A, f)$ is isolated.

Further, $\delta + 1 = \vartheta(A, f)$ which implies $S(A, f)(d(A, f)) = \delta = \vartheta(A, f) - 1$.

Proof of (d). Let $D(A, f) \neq \emptyset$, $A^\infty \neq \emptyset$ and let $x \in A^\infty$ be arbitrary. Then, by 2.3, there is $n \in N$ such that $x \in \operatorname{dom} f^n$, $f^n(x) = d(A, f)$. Thus, by 2.15 (a), $d(A, f) \in A^\infty$. Further, $A^\infty = A^{\infty_1}$ because $D(A, f) \neq \emptyset$ and so $A^{\infty_2} = Z(A, f) = \emptyset$ by 2.5 and 2.7. Consequently, $S(A, f)(d(A, f)) = \infty_1$.

3. HOMOMORPHISMS OF c-ALGEBRAS

3.1. Lemma. *Let $(A, f), (B, g)$ be c-algebras and $F : (A, f) \to (B, g)$ a homomorphism. Then the following assertions hold:*

(a) *If $f^n(x) = x$ for $x \in Z(A, f)$ and $n \in N$, then $F(x) \in \operatorname{dom} g^n$ and $g^n(F(x)) = F(x)$.*

(b) $F(Z(A, f)) \subseteq Z(B, g)$.

(c) *If $R(B, g) = 0$ then $R(A, f) = 0$.*

(d) *If $R(B, g) \neq 0$ then $R(B, g) \mid R(A, f)$.*

Proof of (a) follows immediately from 1.8.

Proof of (b). Let $y \in F(Z(A, f))$ be arbitrary. Then there is $x \in Z(A, f)$ such that $F(x) = y$. Thus, (A, f) is complete and there is $n \in N - \{0\}$ such that $f^n(x) = x$ by 2.8. Hence $y \in \operatorname{dom} g^n$, $g^n(y) = y$ by (a). We have, by 2.8, $y \in Z(B, g)$.

Proof of (c). If $R(B, g) = 0$ then $Z(B, g) = \emptyset$. If we had $R(A, f) \neq 0$ then we should have $Z(A, f) \neq \emptyset$ and, by (b), $\emptyset \neq F(Z(A, f)) \subseteq Z(B, g) = \emptyset$ which is a contradiction.

Proof of (d). Clearly, the assertion holds for $R(A, f) = 0$. Further, let $R(A, f) \neq 0$ and $x \in Z(A, f)$. Then, by 2.12 (a), $f^{R(A,f)}(x) = x$ and, by (a), $F(x) \in \operatorname{dom} g^{R(A,f)}$, $g^{R(A,f)}(F(x)) = F(x)$. By 2.12 (b) we obtain $R(B, g) \mid R(A, f)$.

3.2. Lemma. *Let $(A, f), (B, g)$ be c-algebras, $F : (A, f) \to (B, g)$ a homomorphism. Then $F(A^\infty) \subseteq B^\infty$.*

Proof. Let $x \in A^\infty$. Then there exists a sequence $(x_i)_{i \in N}$ such that $x_i \in \operatorname{dom} f$ for each $i \in N - \{0\}$, $x_0 = x$ and $f(x_{i+1}) = x_i$ for each $i \in N$. For each $i \in N$ we put $y_i = F(x_i)$. Then $y_i \in \operatorname{dom} g$ for each $i \in N - \{0\}$ by 1.3. Further, $y_0 = F(x_0) = F(x)$; finally, if $i \in N$ then $g(y_{i+1}) = g(F(x_{i+1})) = F(f(x_{i+1})) = F(x_i) = y_i$. Thus, $F(x) \in B^\infty$.

3.3. Lemma. *Let (A, f), (B, g) be c-algebras, $F : (A, f) \to (B, g)$ a homomorphism, $x \in A$ arbitrary. Then the following assertions hold:*

(a) $S(A, f)(x) \leqq S(B, g)(F(x))$.

(b) *If $n \in N$, $x \in \operatorname{dom} f^n$ then $F(x) \in \operatorname{dom} g^n$ and $S(A, f)(f^n(x)) \leqq S(B, g)(g^n(F(x))$.*

Proof of (a). (1) Clearly, if $S(A, f)(x) = 0$ then the assertion holds.

Let $0 < \alpha < \vartheta(A, f)$, $S(A, f)(x) = \alpha$ and suppose that the assertion holds for each $y \in A$ with the property $S(A, f)(y) < \alpha$.

Clearly, if $S(B, g)(F(x)) \in \{\infty_1, \infty_2\}$ then the assertion holds by 2.13.

Thus, suppose that $S(B, g)(F(x)) \in \mathrm{Ord}$. Let $y \in f^{-1}(x)$ be arbitrary. Then $y \in \in \operatorname{dom} f$ and, by 1.3, $F(y) \in \operatorname{dom} g$. Further, $g(F(y)) = F(f(y)) = F(x)$ which implies $S(B, g)(g(F(y)) = S(B, g)(F(x)) \in \mathrm{Ord}$. We obtain $S(B, g)(F(y)) \leqq S(B, g)(g(F(y)) \in \in \mathrm{Ord}$ by 2.26 (b) and hence $F(y) \in B - B^\infty$. We have, by 2.26 (a), $S(B, g)(F(y)) < < S(B, g)(g(F(y))$. We obtain by the induction hypothesis $S(A, f)(y) \leqq S(B, g)(F(y)) < < S(B, g)(g(F(y)) = S(B, g)(F(x))$.

Thus, $S(A, f)(f^{-1}(x)) < S(B, g)(F(x))$ because $y \in f^{-1}(x)$ was arbitrary. We conclude $S(A, f)(x) \leqq S(B, g)(F(x))$ by 2.25 (c).

(2) Suppose that $S(A, f)(x) = \infty_1$; then $x \in A^\infty$ and $F(x) \in B^\infty$ by 3.2; thus, $S(B, g)(F(x)) \in \{\infty_1, \infty_2\}$ and the assertion holds.

(3) If $S(A, f)(x) = \infty_2$ then $x \in Z(A, f)$ and $F(x) \in Z(B, g)$ by 3.1 (b); thus, $S(B, g)(F(x)) = \infty_2$ and the assertion holds.

Proof of (b). Let $x \in \operatorname{dom} f^n$. Then $F(x) \in \operatorname{dom} g^n$ and $F(f^n(x)) = g^n(F(x))$ by 1.8. Thus, $S(A, f)(f^n(x)) \leqq S(B, g)(F(f^n(x)) = S(B, g)(g^n(F(x))$ by (a).

3.4. Definition. Let (A, f), (B, g) be c-algebras. Then $x \in A$, $x' \in B$ are said to be a *pair of h-elements* of (A, f) and (B, g) if, for each $n \in N$, $x \in \operatorname{dom} f^n$ implies $x' \in \operatorname{dom} g^n$ and $S(A, f)(f^n(x)) \leqq S(B, g)(g^n(x'))$.

3.5. Definition. Let (A, f), (B, g) be c-algebras. Then (B, g) is said to be *admissible* for (A, f) if the following conditions hold:

(a) if $R(B, g) \neq 0$ then $R(B, g) \mid R(A, f)$;

(b) if $R(B, g) = 0$ then $R(A, f) = 0$ and there exists a pair of h-elements of (A, f) and (B, g).

3.6. Lemma. *Let (A, f), (B, g) be c-algebras such that (B, g) is admissible for (A, f). Then,*

(a) *if $D(B, g) \neq \emptyset$ then $D(A, f) \neq \emptyset$,*

(b) *if (A, f) is complete then (B, g) is complete.*

Proof of (a). Let $D(B, g) \neq \emptyset$. Then, by 2.9 (a), (b), $R(B, g) = 0$ and, for each $y \in B$, $|[y]_{(B,g)}| < \aleph_0$. Thus, by 3.5 (b), $R(A, f) = 0$ and there is a pair of h-elements $x \in A$, $x' \in B$ of (A, f) and (B, g). We obtain that, for each $n \in N$, $x \in \operatorname{dom} f^n$ implies $x' \in \operatorname{dom} g^n$. Since $|[x']_{(B,g)}| < \aleph_0$ we have $|[x]_{(A,f)}| < \aleph_0$.

Indeed, let, on the contrary, $|[x]_{(A,f)}| \geqq \aleph_0$; then $x \in \operatorname{dom} f^n$ for each $n \in N$. Thus, $x' \in \operatorname{dom} g^n$ for each $n \in N$ and there are $i, j \in N$, $i < j$, such that $g^i(x') = g^j(x')$ because $|[x']_{(B,g)}| < \aleph_0$. Hence $Z(B, g) \neq \emptyset$ by 2.8 (b) which is a contradiction to $R(B, g) = 0$.

We see that $R(A, f) = 0$ and $|[x]_{(A,f)}| < \aleph_0$ which implies $D(A, f) \neq \emptyset$ by 2.9 (a).

Proof of (b). If (A, f) is complete then $D(A, f) = \emptyset$ which implies $D(B, g) = \emptyset$ by (a). Thus, (B, g) is complete.

3.7. Lemma. *Let (A, f), (B, g) be c-algebras such that (B, g) is admissible for (A, f). Then there is a pair of h-elements of (A, f) and (B, g).*

Proof. Let $R(B, g) \neq 0$. We take $x' \in Z(B, g)$ arbitrary. Since (B, g) is complete by 2.9 (c), it is $x' \in \operatorname{dom} g^n$ for each $n \in N$. Let $x \in A$ be arbitrary. Then for each $n \in N$ such that $x \in \operatorname{dom} f^n$ we have $S(B, g)(g^n(x')) = \infty_2 \geqq S(A, f)(f^n(x))$ by 2.10. Thus, $x \in A$, $x' \in B$ is a pair of h-elements of (A, f) and (B, g).

If $R(B, g) = 0$ then the assertion holds in virtue of 3.5 (b).

3.8. Definition. Let (A, f) be a c-algebra, $x \in A$ arbitrary. We put $P_0(x) = [x]_{(A,f)}$, $P_1(x) = f^{-1}(P_0(x)) - P_0(x)$. Let $n \in N - \{0\}$ and suppose that the sets $P_0(x), P_1(x), \ldots, P_n(x)$ have been defined. Then we put $P_{n+1}(x) = f^{-1}(P_n(x))$.

3.9. Lemma. *Let (A, f) be a c-algebra and $x \in A$ arbitrary. Then the following assertions hold:*

(a) $Z(A, f) \subseteq P_0(x)$;

(b) *if* $D(A, f) \neq \emptyset$ *then* $d(A, f) \in P_0(x)$ *and* $\bigcup_{k=1}^{\infty} P_k(x) \subseteq \operatorname{dom} f$;

(c) $A = \bigcup_{k=0}^{\infty} P_k(x)$ *with disjoint terms.*

Proof of (a). $Z(A, f) = Z(x) \subseteq [x]_{(A,f)} = P_0(x)$ by 2.4.

Proof of (b). By 2.3, there is $n \in N$ such that $x \in \operatorname{dom} f^n$ and $f^n(x) = d(A, f)$. Thus, $d(A, f) \in [x]_{(A,f)} = P_0(x)$ and $\bigcup_{k=1}^{\infty} P_k(x) \subseteq \operatorname{dom} f$.

Proof of (c). By 3.8 and (b) we have: if $k \in N$, $y \in P_k(x)$ and $n \in N$ are arbitrary then $n < k$ implies $y \in \operatorname{dom} f^n$ and $f^n(y) \in P_{k-n}(x)$ and $n \geqq k$, $y \in \operatorname{dom} f^n$ implies $f^n(y) \in P_0(x)$.

Now, let $k, l \in N$, $k \neq l$; then $P_k(x) \cap P_l(x) = \emptyset$. Indeed, if we had $y \in P_k(x) \cap P_l(x)$ and, for example, $k > 1$ then we should have $f^{k-1}(y) \in P_1(x)$ because

$y \in P_k(x)$ and $k - 1 < k$ and $f^{k-1}(y) \in P_0(x)$ because $y \in P_l(x)$ and $k - 1 \geqq l$; thus, $f^{k-1}(y) \in P_1(x) \cap P_0(x)$ which is a contradiction to 3.8.

It holds $A = \bigcup_{k=0}^{\infty} P_k(x)$.

Let, on the contrary, $y \in A - \bigcup_{k=0}^{\infty} P_k(x)$. Then $y \in \operatorname{dom} f$ because $d(A, f) \in P_0(x)$. Hence $f(y) \in A - \bigcup_{k=0}^{\infty} P_k(x)$ by 3.8. We obtain by induction that $y \in \operatorname{dom} f^n$ implies $f^n(y) \in A - \bigcup_{k=0}^{\infty} P_k(x)$. Further, there exist $p, q \in N$ such that $y \in \operatorname{dom} f^p$, $x \in \operatorname{dom} f^q$ and $f^p(y) = f^q(x) \in P_0(x)$ which is a contradiction.

3.10. Lemma. *Let (A, f), (B, g) be c-algebras. Then the following assertions hold:*

(a) *Let $y \in A$, $y' \in B$ be such that $S(A, f)(y) \leqq S(B, g)(y')$. Then for each $x \in f^{-1}(y)$ there exists $x' \in g^{-1}(y')$ such that $S(A, f)(x) \leqq S(B, g)(x')$.*

(b) *Let $x_0 \in A$, $n \in N$ be arbitrary. Let a map $F : P_n(x_0) \to B$ be defined such that, for each $y \in P_n(x_0)$, $S(A, f)(y) \leqq S(B, g)(F(y))$. Then, for each $x \in P_{n+1}(x_0)$, there exists $x' \in g^{-1}(F(f(x)))$ such that $S(A, f)(x) \leqq S(B, g)(x')$.*

Proof of (a). Suppose that $S(A, f)(y) \leqq S(B, g)(y')$ holds for $y \in A$, $y' \in B$. Let $x \in f^{-1}(y)$ be arbitrary.

If $S(B, g)(y') = \infty_2$ then, by 2.17 (b), there is $x' \in g^{-1}(y')$ such that $S(B, g)(x') = \infty_2$. Thus, $S(A, f)(x) \leqq S(A, f)(y) \leqq S(B, g)(y') = \infty_2$ which implies $S(A, f)(x) \leqq S(B, g)(x')$.

Similarly, if $S(B, g)(y') = \infty_1$ then, by 2.17 (a), there is $x' \in g^{-1}(y')$ such that $S(B, g)(x') = \infty_1$ and $S(A, f)(x) \leqq S(B, g)(x')$.

Finally, let $S(B, g)(y') \in \mathrm{Ord}$. Then $S(A, f)(y) \in \mathrm{Ord}$ and $S(A, f)(x) < S(A, f)(y)$ by 2.26 (a). Therefore $S(A, f)(x) < S(B, g)(y')$ and, by 2.25 (b), there is $x' \in g^{-1}(y')$ with the property $S(A, f)(x) \leqq S(B, g)(x') < S(B, g)(y')$.

Proof of (b). Let $x_0 \in A$, $n \in N$ be arbitrary. Suppose that, for each $y \in P_n(x_0)$, we have $F(y) \in B$ such that $S(A, f)(y) \leqq S(B, g)(F(y))$. Let $x \in P_{n+1}(x_0)$ be arbitrary. Then $f(x) \in P_n(x_0)$ by 3.8 and $S(A, f)(f(x)) \leqq S(B, g)(F(f(x)))$. Since $x \in f^{-1}(f(x))$, there is, by (a), $x' \in g^{-1}(F(f(x)))$ such that $S(A, f)(x) \leqq S(B, g)(x')$.

3.11. Definition. Let (A, f), (B, g) be c-algebras such that (B, g) is admissible for (A, f). We define a map $F : A \to B$ in the following way:

(i) We take a pair of h-elements $x_0 \in A$, $x'_0 \in B$ of (A, f) and (B, g) (see 3.7). Then we put, for each $f^n(x_0) \in P_0(x_0)$, $F(f^n(x_0)) = g^n(x'_0)$.

(ii) Let $n \in N - \{0\}$. Suppose that, for each $x \in \bigcup_{k=0}^{n-1} P_k(x_0)$, we have defined $F(x)$ in such a way that $S(A, f)(x) \leqq S(B, g)(F(x))$.

Let $x \in P_n(x_0)$ be arbitrary. We take $x' \in g^{-1}(F(f(x)))$ such that $S(A, f)(x) \leqq$ $\leqq S(B, g)(x')$ (see 3.10 (b)). Then we put $F(x) = x'$.

Then we say that the map $F : A \to B$ has been defined by the *construction c-K* (with respect to (A, f) and (B, g)).

3.12. Theorem. *Let (A, f), (B, g) be c-algebras and $F : (A, f) \to (B, g)$ a homomorphism. Then the following assertions hold:*

(a) *(B, g) is admissible for (A, f).*

(b) *The map $F : A \to B$ is defined by the construction c-K.*

Proof of (a). The property (a) in 3.5 follows from 3.1 (d). The property (b) in 3.5 follows from 3.1 (c) and 3.3 (b) where we take an arbitrary $x \in A$ and put $x' = F(x)$.

Proof of (b). By (a), (B, g) is admissible for (A, f). Let $x_0 \in A$ be arbitrary. We put $x'_0 = F(x_0)$. Then, by 3.3 (b), $x_0 \in A$, $x'_0 \in B$ is a pair of h-elements of (A, f) and (B, g).

Thus, for $f^n(x_0) \in P_0(x)$ we have $F(f^n(x_0)) = g^n(F(x_0)) = g^n(x'_0)$.

Further, let $n \in N - \{0\}$, $x \in P_n(x_0)$. Putting $x' = F(x)$ we have $S(A, f)(x) \leqq$ $\leqq S(B, g)(x')$. Since, by 3.9 (b), $x \in \operatorname{dom} f$ we have $x' \in \operatorname{dom} g$ and $F(f(x)) =$ $= g(F(x)) = g(x')$. Thus, $x' \in g^{-1}(F(f(x)))$.

3.13. Theorem. *Let (A, f), (B, g) be c-algebras and $F : A \to B$ a map defined by the construction c-K. Then $F : (A, f) \to (B, g)$ is a homorphism.*

Proof. Let a map $F : A \to B$ be defined by the construction *c-K* as in 3.11. Then $x_0 \in A$, $x'_0 \in B$ is a pair of h-elements of (A, f) and (B, g).

Let $x \in P_0(x_0)$ be an arbitrary element and let $x = f^n(x_0)$. Then $F(x) \in \operatorname{dom} g^n$ and $F(x) = g^n(x'_0)$. If $x = d(A, f)$ then in virtue of 1.3 we have nothing to prove. Thus, let $x \neq d(A, f)$. Then $F(x) \neq d(B, g)$ because, for $n \neq 0$, we have $F(x) \in$ $\in \operatorname{dom} g^n \subseteq \operatorname{dom} g$ by 1.6 (b) and, for $n = 0$, we obtain $x = x_0$ and $x_0 = x \neq$ $\neq d(A, f)$ implies $F(x) = F(x_0) \in \operatorname{dom} g$ by 3.4.

We see that $x \in \operatorname{dom} f$ implies $F(x) \in \operatorname{dom} g$; further, we conclude $F(f(x)) =$ $= F(f^{n+1}(x_0)) = g^{n+1}(x'_0) = g(F(x))$.

Suppose $x \in \bigcup_{k=1}^{\infty} P_k(x_0)$. Then $x \in \operatorname{dom} f$ by 3.9 (b). Since F is defined by the construction *c-K* we have $F(x) \in g^{-1}(F(f(x)))$ by 3.11 (ii). Thus, $F(x) \neq d(B, g)$ and $F(x) \in \operatorname{dom} g$. Finally, $g(F(x)) = F(f(x))$.

The map $F : A \to B$ is a homomorphism $F : (A, f) \to (B, g)$.

3.14. Theorem. *Let (A, f), (B, g) be c-algebras, $F : A \to B$ a map. Then $F : (A, f) \to$ $\to (B, g)$ is a homomorphism if and only if F is defined by the construction c-K with respect to (A, f) and (B, g).*

Proof is a consequence of 3.12 and 3.13.

4.1. Lemma. *Let (A, f) be a unary algebra and let $\varrho(A, f)$ be defined by* 1.9. *Then $\varrho(A, f)$ is an equivalence on A.*

Proof. $\varrho(A, f)$ is reflexive because, for each $x \in A$, $x \in \operatorname{dom} f^0$ and $x = f^0(x)$. Clearly, $\varrho(A, f)$ is symmetric. Further, let $x, y, z \in A$ and $(x, y) \in \varrho(A, f)$, $(y, z) \in \in \varrho(A, f)$. Then there are $m, n, n', p \in N$ such that $x \in \operatorname{dom} f^m$, $y \in \operatorname{dom} f^n$, $y \in \in \operatorname{dom} f^{n'}$, $z \in \operatorname{dom} f^p$ and we have $f^m(x) = f^n(y)$, $f^{n'}(y) = f^p(z)$. We suppose that, for example, $n \leqq n'$. Then $f^n(y) \in \operatorname{dom} f^{n'-n}$ by 1.6 (d) and this implies $f^m(x) \in \in \operatorname{dom} f^{n'-n}$. Thus, by 1.6 (e), we obtain $f^{m+n'-n}(x) = f^{n'-n}(f^m(x)) = f^{n'-n}(f^n(y)) = = f^{n'}(y) = f^p(z)$. Hence $(x, z) \in \varrho(A, f)$ and $\varrho(A, f)$ is transitive.

4.2. Definition. Let (A, f) be a unary algebra. Then we denote $\Theta(A, f) = A/\varrho(A, f)$.

4.3. Lemma. *Let (A, f) be a unary algebra and let $T \in \Theta(A, f)$. Then*

(a) $(T, f \mid T)$ *is a subalgebra of* (A, f);

(b) $(T, f \mid T)$ *is a c-algebra.*

Proof of (a). If $x \in T$ is such that $x \in \operatorname{dom} f$ then $(x, f(x)) \in \varrho(A, f)$ because $x \in \operatorname{dom} f$, $f(x) \in \operatorname{dom} f^0$ and $f(x) = f^0(f(x))$. Thus, $f(x) \in T$.

Proof of (b). The assertion follows from (a) and 4.2.

4.4. Lemma. *Let (A, f), (B, f) be unary algebras, $F : (A, f) \to (B, g)$ a homomorphism. Then, for each $T \in \Theta(A, f)$, there is $T' \in \Theta(B, g)$ such that $F(T) \subseteq T'$.*

Proof. Let $x', y' \in F(T)$ be arbitrary. Then there are $x, y \in T$ such that $F(x) = x'$, $F(y) = y'$. Thus, $(x, y) \in \varrho(A, f)$ and there are $m, n \in N$ such that $x \in \operatorname{dom} f^m$, $y \in \operatorname{dom} f^n$ and $f^m(x) = f^n(y)$. It follows, by 1.3, $x' \in \operatorname{dom} g^m$, $y' \in \operatorname{dom} g^n$ and $g^m(x') = g^m(F(x)) = F(f^m(x)) = F(f^n(y)) = g^n(F(y)) = g^n(y')$. Thus, $x', y' \in \varrho(B, g)$ and there is $T' \in \Theta(B, g)$ such that $F(T) \subseteq T'$.

4.5. Definition. Let (A, f), (B, g) be unary algebras. We define a map $F : A \to B$ in this way:

(i) We take a map $\Phi : \Theta(A, f) \to \Theta(B, g)$ such that, for each $T \in \Theta(A, f)$, $(\Phi(T), g \mid \Phi(T))$ is admissible for the c-algebra $(T, f \mid T)$. For each $T \in \Theta(A, f)$, we define a map $F_T : T \to \Phi(T)$ by the construction *c-K*.

(ii) We put $F = \bigcup_{T \in \Theta(A, f)} F_T$.

Then we say that the map $F : A \to B$ has been defined by the *construction K* (with respect to (A, f) and (B, g)).

4.6. Theorem. *Let (A, f), (B, g) be unary algebras, $F : (A, f) \to (B, g)$ a homomorphism. Then the map $F : A \to B$ is defined by the construction K.*

Proof. Let $T \in \Theta(A, f)$ be arbitrary. Then there is (precisely one) $T' \in \Theta(B, g)$ such that $F(T) \subseteq T'$ by 4.4. We put $\Phi(T) = T'$ and $F_T = F \mid T$. Then $(T, f \mid T)$, $(T', g \mid T')$ are c-algebras by 4.3 (b) and $F_T : (T, f \mid T) \to (T', g \mid T')$ is a homomorphism. Consequently, by 3.12 (a), $(T', g \mid T')$ is admissible for $(T, f \mid T)$ and $F_T : A \to B$ is a map defined by the construction *c-K* by 3.12 (b).

Further, clearly $F = \bigcup_{T \in \Theta(A, f)} F_T$.

4.7. Theorem. *Let (A, f), (B, g) be unary algebras, $F : A \to B$ a map defined by the construction K. Then $F : (A, f) \to (B, g)$ is a homomorphism.*

Proof. Let $F : A \to B$ be defined by the construction K and let $x \in A$ be such that $x \in \operatorname{dom} f$. Then there is $T \in \Theta(A, f)$ such that $x \in T$. By 4.3 (b), $(T, f \mid T)$, $(\Phi(T), g \mid \Phi(T))$ are c-algebras. Thus, $f(x) \in T$ and $F(x) = F_T(x)$, $F(f(x)) = F_T(f(x))$ where $F_T : T \to \Phi(T)$ is a map defined by the construction *c-K*. Thus, $F_T : (T, f \mid T) \to (\Phi(T), g \mid \Phi(T))$ is a homomorphism by 3.13. We obtain $F(x) = F_T(x) \in \operatorname{dom} g$ and $g(F(x)) = g(F_T(x)) = F_T(f(x)) = F(f(x))$.

4.8. Main Theorem. *Let (A, f), (B, g) be unary algebras, $F : A \to B$ a map. Then $F : (A, f) \to (B, g)$ is a homomorphism if and only if F is defined by the construction K with respect to (A, f) and (B, g).*

Proof is a consequence of 4.6 and 4.7.

5. COROLLARIES

Some corollaries for complete unary algebras can be found in [5].

Let A, B be sets, $\alpha \subseteq A \times B$ arbitrary. Then α is said to be a correspondence from A to B. If α is a correspondence from A to B then we put

$$\operatorname{dom} \alpha = \{x \in A;\ \text{there is } y \in B \text{ such that } (x, y) \in \alpha\},$$
$$\operatorname{Im} \alpha = \{y \in B;\ \text{there is } x \in A \text{ such that } (x, y) \in \alpha\}.$$

If α is a correspondence from A to B, $A \supseteq C \supseteq \operatorname{dom} \alpha$, $B \supseteq D \supseteq \operatorname{Im} \alpha$ then $\alpha \cap (C \times D)$ is a correspondence from C to D. Further, if α_i is a correspondence from A_i to B_i for $i \in I$ then $\bigcup_{i \in I} \alpha_i$, $\bigcap_{i \in I} \alpha_i$ are correspondences from $\bigcup_{i \in I} A_i$ to $\bigcup_{i \in I} B_i$. Finally, if α is a correspondence from A to B, $\beta \subseteq \alpha$ then β is a correspondence from A to B. Clearly, the correspondence α from A to B is a partial map from A into B if (x, y_1), $(x, y_2) \in \alpha$ implies $y_1 = y_2$.

The partial map α from A into B is said to be injective if (x_1, y), $(x_2, y) \in \alpha$ implies $x_1 = x_2$.

The map $\alpha : A \to B$ is said to be surjective if $\operatorname{Im} \alpha = B$ and bijective if it is injective and surjective.

If $\varphi : A \to B$ is a map, $n \in N - \{0\}$ arbitrary then we put, for each $(x_1, x_2, \ldots, x_n) \in A^n$, $\varphi^n(x_1, x_2, \ldots, x_n) = (\varphi(x_1), \varphi(x_2), \ldots, \varphi(x_n))$; it is a map $\varphi^n : A^n \to B^n$.

5.1. Definition. (a) Let $(A, \mathscr{F})$ be a complete universal algebra, $n \in N - \{0\}$ arbitrary. Then we put $\mathscr{F}(0) = \mathscr{F} \cap A$, $\mathscr{F}(n) = \{f \in \mathscr{F}; f : A^n \to A\}$.

(b) Let $(A, \mathscr{F})$, $(B, \mathscr{G})$ be complete universal algebras. Then $(A, \mathscr{F})$, $(B, \mathscr{G})$ are said to be similar if there is a bijection α: $\mathscr{F} \to \mathscr{G}$ such that, for each $n \in N$, $\alpha(\mathscr{F}(n)) = \mathscr{G}(n)$ and $\alpha \mid \mathscr{F}(0) \cap A \cap B = \mathrm{id}_{\mathscr{F}(0) \cap A \cap B}$ and, for each $n \in N - \{0\}$, $f \in \mathscr{F}(n)$, $f \mid A^n \cap B^n = \alpha(f) \mid A^n \cap B^n$.

5.2. Problem. *Let A, B be sets, Φ a set of maps $A \to B$. Construct a system $\mathscr{F}$ of complete operations on A and a system $\mathscr{G}$ of complete operations on B in such a way that $(A, \mathscr{F})$, $(B, \mathscr{G})$ are similar universal algebras and that each $\varphi \in \Phi$ is a homomorphism of $(A, \mathscr{F})$ into $(B, \mathscr{G})$.*

5.3. Lemma. *Let A_1, A_2, B_1, B_2 be sets, f a partial map from A_1 into A_2, g a partial map form B_1 into B_2. Let $F_i : A_i \to B_1 \cup B_2$ $(i = 1, 2)$ be maps such that $F_1 \mid A_1 \cap A_2 = F_2 \mid A_1 \cap A_2$. Then $F_i(A_i) \subseteq B_i$ $(i = 1, 2)$ and, for each $x \in \operatorname{dom} f$, $F_2(f(x)) = g(F_1(x))$ iff $F_1(A_1 - \operatorname{dom} f) \subseteq B_1$, $F_2(A_2 - \operatorname{Im} f) \subseteq B_2$ and $F_1 \cup F_2 : (A_1 \cup A_2, f) \to (B_1 \cup B_2, g)$ is a homomorphism.*

Proof. The condition is necessary: We have $F(A_1 - \operatorname{dom} f) \subseteq F(A_1) \subseteq B_1$, $F(A_2 - \operatorname{Im} f) \subseteq F(A_2) \subseteq B_2$. Further, let $x \in A_1 \cup A_2$ and let $x \in \operatorname{dom} f$. Then $F_2(f(x))$ is defined and $F_2(f(x)) = g(F_1(x))$. Thus, $F_1(x) \in \operatorname{dom} g$. Since $x \in \operatorname{dom} f \subseteq A_1$ we obtain $(F_1 \cup F_2)(x) = F_1(x)$ and since $f(x) \in A_2$ we have $(F_1 \cup F_2)(f(x)) = F_2(f(x))$. Thus, $(F_1 \cup F_2)(x) \in \operatorname{dom} g$ and $(F_1 \cup F_2)(f(x)) = F_2(f(x)) = g(F_1(x)) = g((F_1 \cup F_2)(x))$. $F_1 \cup F_2$ is a homomorphism.

The condition is sufficient: Let $x \in \operatorname{dom} f$. Then $(F_1 \cup F_2)(x) \in \operatorname{dom} g$ and $(F_1 \cup F_2)(f(x)) = g((F_1 \cup F_2)(x))$. Further, $x \in A_1$ and $f(x) \in \operatorname{Im} f \subseteq A_2$ which implies $(F_1 \cup F_2)(x) = F_1(x)$ and $(F_1 \cup F_2)(f(x)) = F_2(f(x))$. Hence $F_2(f(x)) = (F_1 \cup F_2)(f(x)) = g(F_1 \cup F_2)(x) = g(F_1(x))$.

Further, $F_1(x) = (F_1 \cup F_2)(x) \in \operatorname{dom} g \subseteq B_1$ and we have $F_1(\operatorname{dom} f) \subseteq B_1$. Thus, $F_1(A_1) = F_1(\operatorname{dom} f) \cup F_1(A_1 - \operatorname{dom} f) \subseteq B_1$.

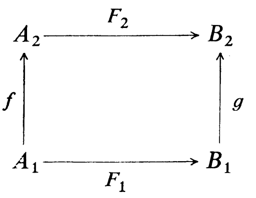
Finally, let $y \in \operatorname{Im} f$ be arbitrary. Suppose, without loss of generality, that $f(x) = y$. Then $F_2(y) = F_2(f(x)) = g(F_1(x)) \in \operatorname{Im} g \subseteq B_1$. Thus, $F_2(\operatorname{Im} f) \subseteq B_2$ which implies $F_2(A_2) = F_2(\operatorname{Im} f) \cup F_2(A_2 - \operatorname{Im} f) \subseteq B_2$.

5.4. Theorem. *Let A_1, A_2, B_1, B_2 be sets, f a partial map from A_1 into A_2, g a partial map from B_1 into B_2. Let $F_i : A_i \to B_i$ $(i = 1, 2)$ be maps such that $F_1 \mid A_1 \cap A_2 = F_2 \mid A_1 \cap A_2$. Then, for each $x \in \operatorname{dom} f$, $F_2(f(x)) = g(F_1(x))$ if and only if $F_1 \cup F_2 : (A_1 \cup A_2, f) \to (B_1 \cup B_2, g)$ is a homomorphism.*

Proof. The theorem is a corollary of 5.3.

5.5. Theorem. *Let A_1, A_2, B_1, B_2 be sets, f a partial map from A_1 into A_2, g a partial map from B_1 into B_2. Let $F_i : A_i \to B_i$ $(i = 1, 2)$ be maps such that $F_1 \mid A_1 \cap A_2 = F_2 \mid A_1 \cap A_2$. Then the following three conditions are equivalent:*

(A) *The diagram*



is commutative.

(B) *$F_1 \cup F_2 : (A_1 \cup A_2, f) \to (B_1 \cup B_2, g)$ is a homomorphism.*

(C) *The map $F_1 \cup F_2 : A_1 \cup A_2 \to B_1 \cup B_2$ is defined by the construction K with respect to $(A_1 \cup A_2, f)$ and $(B_1 \cup B_2, g)$.*

Proof. (A) and (B) are equivalent by 5.4, (B) and (C) are equivalent by 4.8.

5.6. Definition. Let A, B be sets, Φ a set of maps $A \to B$.

(i) We put $\beta_0^\varphi = \{(x, \varphi(x)); x \in A$ such that $x \in A \cap B$ implies $\varphi(x) = x\}$ for each $\varphi \in \Phi$.

(ii) If $n \in N - \{0\}$, $\varphi \in \Phi$ are arbitrary then we put $\beta_n^\varphi = \{(f, g); f : A^n \to A$, $g : B^n \to B, f \cup g$ is a map defined by the construction K with respect to $(A^n \cup B^n, \varphi^n)$ and $(A \cup B, \varphi)\}$.

(iii) We put $\beta_n = \bigcap_{\varphi \in \Phi} \beta_n^\varphi$ for each $n \in N$.

(iv) We take $\alpha \subseteq \bigcup_{n=0}^{\infty} \beta_n$ such that α is an injective partial map (from dom $\bigcup_{n=0}^{\infty} \beta_n$ into Im $\bigcup_{n=0}^{\infty} \beta_n$). Then we put $\mathscr{F} = \operatorname{dom} \alpha$, $\mathscr{G} = \operatorname{Im} \alpha$.

Then we say that $(A, \mathscr{F})$, $(B, \mathscr{G})$ is a pair of complete universal algebras defined by the *construction $A-K$* with respect to Φ.

5.7. Theorem. *Let A, B be sets, Φ a set of maps $A \to B$. Then $(A, \mathscr{F})$, $(B, \mathscr{G})$ are similar complete universal algebras and $\varphi : (A, \mathscr{F}) \to (B, \mathscr{G})$ a homomorphism for each $\varphi \in \Phi$ if and only if $(A, \mathscr{F})$, $(B, \mathscr{G})$ is a pair of complete universal algebras defined by the construction $A-K$ with respect to Φ.*

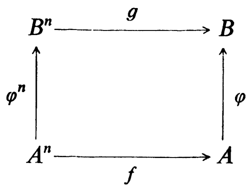
Proof. The condition is necessary:

Let $(A, \mathscr{F})$, $(B, \mathscr{G})$ be similar complete universal algebras and let $\alpha : \mathscr{F} \to \mathscr{G}$ be a bijection such that $\alpha(\mathscr{F}(n)) = \mathscr{G}(n)$ for each $n \in N$ and $\alpha \mid \mathscr{F}(0) \cap A \cap B =$

$= \mathrm{id}_{\mathscr{F}(0)\cap A\cap B}$ and, for each $n \in N - \{0\}$, $f \in \mathscr{F}(n)$, $g = \alpha(f)$ implies $f \mid A^n \cap B^n = g \mid A^n \cap B^n$.

We put $\alpha_n = \alpha \mid \mathscr{F}(n)$ for each $n \in N$.

Let $\varphi \in \Phi$ be arbitrary. Let $(f, g) \in \alpha_0$. Then $f \in \mathscr{F}(0) \subseteq A$; further, we have $g = \varphi(f)$ because φ is a homomorphism and $f \in \mathscr{F}(0) \cap A \cap B$ implies $g = f$. Thus, $(f, g) \in \beta_0^\varphi$. Further, let $(f, g) \in \alpha_n$ for an arbitrary $n \in N - \{0\}$. Then, for each $(x_1, x_2, \ldots, x_n) \in A^n$, we have $\varphi(f(x_1, x_2, \ldots, x_n)) = g(\varphi(x_1), \varphi(x_2), \ldots, \varphi(x_n)) = g(\varphi^n(x_1, x_2, \ldots, x_n))$ (because φ is a homomorphism). Thus, the diagram



is commutative. Further, $f \mid A^n \cap B^n = g \mid A^n \cap B^n$ which implies that the map $f \cup g$ is defined by the construction K with respect to $(A^n \cup B^n, \varphi^n)$ and $(A \cup B, \varphi)$ by 5.5. Thus, $(f, g) \in \beta_n^\varphi$.

We obtain $\alpha_n \subseteq \beta_n^\varphi$ for each $\varphi \in \Phi$ and each $n \in N$. This implies $\alpha_n \subseteq \bigcap_{\varphi\in\Phi} \beta_n^\varphi = \beta_n$ for each $n \in N$.

Finally, $\alpha = \bigcup_{n=0}^{\infty} \alpha_n \subseteq \bigcup_{n=0}^{\infty} \beta_n$ and $\mathrm{dom}\,\alpha = \mathscr{F}$, $\mathrm{Im}\,\alpha = \mathscr{G}$.

The condition is sufficient:

Let $(A, \mathscr{F})$, $(B, \mathscr{G})$ be a pair of complete universal algebras defined by the construction $A - K$ (with respect to Φ) where $\mathscr{F} = \mathrm{dom}\,\alpha$, $\mathscr{G} = \mathrm{Im}\,\alpha$ for an $\alpha \subseteq \bigcup_{n=0}^{\infty} \beta_n$ by 5.6. We put $\alpha_n = \alpha \cap \beta_n$ for each $n \in N$. Then $\alpha = \bigcup_{n=0}^{\infty} \alpha_n$ with disjoint terms because β_n are mutually disjoint.

Further, $\mathrm{dom}\,\alpha_0 \subseteq \mathrm{dom}\,\alpha \cap \mathrm{dom}\,\beta_0 \subseteq \mathscr{F} \cap A$, $\mathrm{Im}\,\alpha_0 \subseteq \mathrm{Im}\,\alpha \cap \mathrm{Im}\,\beta_0 \subseteq \mathscr{G} \cap B$ and, for each $n \in N - \{0\}$, $\mathrm{dom}\,\alpha_n \subseteq \mathrm{dom}\,\alpha \cap \mathrm{dom}\,\beta_n \subseteq \mathscr{F} \cap \{f; f : A^n \to A\}$, $\mathrm{Im}\,\alpha_n \subseteq \mathrm{Im}\,\alpha \cap \mathrm{Im}\,\beta_n \subseteq \mathscr{G} \cap \{g; g : B^n \to B\}$. Thus, $\mathrm{dom}\,\alpha_n = \mathscr{F}(n)$, $\mathrm{Im}\,\alpha_n = \mathscr{G}(n)$ for each $n \in N$. α is an injective partial map (from $\mathrm{dom} \bigcup_{n=0}^{\infty} \beta_n$ into $\mathrm{Im} \bigcup_{n=0}^{\infty} \beta_n$) by 5.6. Then $\alpha : \mathscr{F} \to \mathscr{G}$ is a surjective (complete) map because $\mathrm{dom}\,\alpha = \mathscr{F}$, $\mathrm{Im}\,\alpha = \mathscr{G}$. Thus, $\alpha : \mathscr{F} \to \mathscr{G}$ is bijective.

Further, $\alpha(\mathscr{F}(n)) = \alpha_n(\mathscr{F}(n)) = \alpha_n(\mathrm{dom}\,\alpha_n) = \mathrm{Im}\,\alpha_n = \mathscr{G}(n)$ for each $n \in N$.

Finally, $\alpha \mid \mathscr{F}(0) \cap A \cap B = \mathrm{id}_{\mathscr{F}(0)\cap A\cap B}$ by 5.6 (i) and if $f \in \mathscr{F}(n)$ for each $n \in N - \{0\}$ and $g = \alpha(f)$ then $f \mid A^n \cap B^n = g \mid A^n \cap B^n$ because $f \cup g$ is a map $A^n \cup B^n \to A \cup B$ by 5.6 (ii).

Thus, $(A, \mathscr{F})$, $(B, \mathscr{G})$ are similar complete universal algebras.

Further, let $\varphi \in \Phi$ be arbitrary. Let $f \in \mathscr{F}$, $g = \alpha(f)$.

If $f \in \mathscr{F}(0)$ then $g \in \mathscr{G}(0)$ and $(f, g) \in \alpha_0 \subseteq \beta_0 \subseteq \beta_0^{\varphi}$ which implies $\varphi(f) = g$ by 5.6 (i).

Suppose $n \in N - \{0\}$. If $f \in \mathscr{F}(n)$ then $g \in \mathscr{G}(n)$ and $(f, g) \in \alpha_n \subseteq \beta_n \subseteq \beta_n^{\varphi}$. We have, for each $(x_1, x_2, \ldots, x_n) \in A^n$, $\varphi(f(x_1, x_2, \ldots, x_n)) = g(\varphi^n(x_1, x_2, \ldots, x_n)) = g(\varphi(x_1), \varphi(x_2), \ldots, \varphi(x_n))$ by 5.6 (ii).

Thus, φ is a homomorphism.

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