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LEFT TRANSLATIONS OF COMPACT SEMILATTICES AND GENERALIZATIONS

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This paper shows that every left translation of a compact semilattice is continuous, and that the semilattice of all left translations with the compact-open topology is iseomorphic to the semilattice of all retract ideals with the hyperspace topology. A corollary to the latter result is that the semilattice of all left translations of a compact semilattice is compact. The first part of the paper will study the generalizations of these concepts in a larger class of partially ordered spaces.

1. IDEAL RETRACTIONS OF POSETS

A *poset* is a partially ordered set. A compact poset X is a compact Hausdorff space with a partial order $\leqq$ whose graph is closed in $X \times X$. For $x \in X$, $L(x) = \{y \in X : y \leqq x\}$ and $U(x) = \{y \in X : x \leqq y\}$. Most terminology on posets will be taken from [4] and most on semigroups from [5].

If X is a poset, a set $I \subset X$ is said to be an *ideal* if for $x \in I$ and $y \leqq x$, then $y \in I$. A map $f : X \to X$ is said to be an *ideal retraction* if for $y \leqq x \in X$, $f(y) \leqq f(x)$; for $x \in X, f(x) \leqq x; f^2 = f$; and $f(X)$ is an ideal. An ideal I of X is said to be a *retract ideal* if there exists an ideal retraction f of X such that $f(X) = I$. If X is a semilattice, then we define $x \geqq y$ if and only if $xy = y$. It will be seen in section 2 that the ideal retractions on a semilattice are precisely the left translations. Propositions 1.1 and 1.2 are a partial generalization of 2.2 which is taken from PETRICH [5] which summarizes material found in [6].

1.1. Proposition. *A subset I of a poset X is a retract ideal if and only if for any $x \in X$ there is an $\alpha \in I$ such that $I \cap L(x) = L(\alpha)$. Furthermore if f is an ideal retraction, $L(x) \cap f(X) = L(f(x))$.*

Proof. Let I be a retract ideal and f an ideal retraction such that $f(X) = I$. It will be shown that $L(x) \cap I = L(f(x))$. If $y \in L(x) \cap I$, then $y \leqq x$ which implies

$y = f(y) \leqq f(x)$. Thus $y \in L(f(x))$. Since $I = f(X)$ is an ideal, $L(f(x)) \subset I$, and since $f(x) \leqq x$, $L(f(x)) \subseteq L(x)$. Therefore, $L(f(x)) \subset I \cap L(x)$. This also proves the last statement in the proposition.

Clearly, any set I is an ideal if it has the property that for all $x \in X$, there is some y such that $L(x) \cap I = L(y)$. Define $f : X \to X$ by $I \cap L(x) = L(f(x))$. Note that for $a, b \in X$, $L(a) = L(b)$ implies $a = b$, so f is well defined. Now $f^2 = f$ and $f(x) \leqq x$. If $x \leqq y$, then $L(f(x)) = I \cap L(x) \subseteq I \cap L(y) = L(f(y))$ which implies $f(x) \leqq f(y)$.

1.2. Proposition. *For a poset X, the set $\Lambda(X)$ of ideal retractions with the operation of map composition forms a semilattice which is isomorphic, by the map $f \to f(X)$ (f an ideal retraction), to the semilattice R_X of retract ideals where the operation is set intersection.*

Proof. That the map $f \to f(X)$ is a one-to-one onto map from ideal retractions to retract ideals follows from 1.1. Let $g, f \in \Lambda(X)$. For $x \in X$,

$$(g \circ f)^2 (x) = (g \circ f) (g(f(x))) = g(g(f(x))) = g \circ f(x) .$$

Certainly, if $x \leqq z$, then $g \circ f(x) \leqq g \circ f(z)$, and if $z \leqq g \circ f(x)$, then $g \circ f(z) = g(z) = z$. Thus $g \circ f \in \Lambda(X)$.

Obviously, $f(X) \cap g(X) \subset g \circ f(X)$. Suppose $y \in g \circ f(X)$, then $y = g \circ f(z)$ for some $z \in X$. Since $g(f(z)) \leqq f(z)$, $f(y) = f(g \circ f(z)) = g \circ f(z) = y$. Therefore $g \circ f(X) \subset g(X) \cap f(X)$. It follows that $g \circ f(X) = g(X) \cap f(X)$ which implies R_X is a semilattice under set intersection and that the map $f \to f(X) : \Lambda(X) \to R_X$ is an isomorphism.

1.3. Definition. A topological poset X is said to be locally bounded directed (LBD) if for every $x \geqq y \in X$ and U a neighborhood of y, there exists a neighborhood V of x such that for $v \in V$ there exists a $z \in U$ such that $z \leqq v, y$. Such a neighborhood V is said to satisfy the LBD property with respect to y and U.

Note that every topological semilattice is a LBD poset.

1.4. Proposition. *For a compact LBD poset, every retract ideal is closed.*

Proof. Let I be a retract ideal and let x belong to the closure I^- of I. By [4, p. 48], the convex neighborhoods of x form a basis for the neighborhood system of x. Let U be a convex neighborhood of x. Let V be a neighborhood of x which satisfies the LBD property with respect to x and U. Since $x \in I^-$, there is a $z \in V \cap I$. By the choice of V there exists a $y \in U$ such that $y \leqq z, x$. By 1.1, there exists a $w \in I$ with $L(x) \cap I = L(w)$. Now $y \in L(x) \cap I = L(w)$ implies $y \leqq w \leqq x$, and since U is convex $w \in U$. Because U was picked arbitrarily from a neighborhood basis for x, it follows that $x = w \in I$.

1.5. Theorem. *Every ideal retraction of a compact LBD poset is continuous.*

Proof. Let f be an ideal retraction. By 1.1, $I = f(X)$ is a retract ideal, and by 1.4, it is compact. Let $y \in X$ and U be an open set of $f(y)$. By [4, p. 48] there exists an open increasing set V and an open decreasing set W of $f(y)$ such that $f(y) \in V \cap W \subset U$. Now $L(f(y)) \subset W$. By [4, p. 46], for every $z_1, z_2 \in X$ with $z_1 \not\geqq z_2$ there exists an open decreasing set D_1 of z_1 and an open increasing set D_2 of z_2 such that $D_1 \cap D_2 = \emptyset$. It follows that for $x \in X$, $L(x) = \bigcap\{Z : Z$ an open decreasing set of $x\}$. Thus $L(f(y)) = L(y) \cap I = \{U \cap I : U$ an open decreasing set containing $y\}$. By the compactness of I, there exists an open decreasing set U_0 of y such that $U_0 \cap I \subset W \cap I$. Let U_1 be an open neighborhood of y such that $U_1 \subset U_0$ and for $z \in U_1$, there exists a $z' \in V$ with $z' \leqq f(y), z$. Now $z' \leqq f(y), z$ implies $z' \in I \cap L(z) = L(f(z))$ which implies since V is upwards directed that $f(z) \in V$. Also $f(z) \leqq z$ implies $f(z) \in I \cap U_0 \subset W$. Thus $f(z) \in V \cap W \subset U$ which implies f is continuous.

Most of the following results on uniformities are drawn from [2, pp. 27–32]. If X is a compact space, the collection of all neighborhoods of the diagonal of $X \times X$ forms a system of entourages for a uniform structure on X that realizes the topology of X. Let $H(X)$ be the set of all non-empty closed subsets of X. If $\mathcal{U}$ is an entourage of X, define $\mathcal{U}^* = \{(A, B) \in H(X) \times H(X) : A \subset \mathcal{U}(B)$ and $B \subset \mathcal{U}(A)\}$. The collection of all $\mathcal{U}^*$ forms a basis for a uniformity on $H(X)$ whose associated topology is compact.

1.6. Proposition. *If X is a compact LBD poset, then R_X is a compact subset of $H(X)$.*

Proof. Let $\{I_\alpha\}$, $\alpha \in \Gamma$, be a net in R_X that converges to a closed set I. Let $x \in X$. For each $\alpha \in \Gamma$, there exists a $y_\alpha \in I_\alpha$ such that $I_\alpha \cap L(x) = L(y_\alpha)$. Since X is compact it may be assumed that the net $\{y_\alpha\}$, $\alpha \in \Gamma$, converges to an element y. By proposition 45 of [2, p. 29], $y \in I$. For each $\alpha \in \Gamma$, $y_\alpha \leqq x$ which implies since the graph of $\leqq$ is closed that $y \leqq x$. If $z \in L(y)$ and U is an open set of z, there exists a neighborhood V of y that satisfies the LBD property for z and U. Again by 45 of [2], there exists an I_α, $\alpha \in \Gamma$, such that $I_\alpha \cap V \neq \emptyset$ which implies $I_\alpha \cap U \neq \emptyset$. Thus z is a cluster point of $\{I_\alpha\}$, $\alpha \in \Gamma$, which implies $z \in I$. Thus $L(y) \subset I \cap L(x)$.

Let $w \in I \cap L(x)$. Since the space $H(X)$ is compact, we may assume that the net $\{L(y_\alpha)\}$, $\alpha \in \Gamma$, converges to a closed set L. Clearly $L \subset L(y)$. Let U be an open neighborhood of w, and let U_1 be a neighborhood of w which satisfies the LBD property with respect to w and U. Since $w \in I$, there exists for each $\alpha \in \Gamma$ a $\beta \geqq \alpha$ such that $I_\beta \cap U_1 \neq \emptyset$. Let $t \in I_\beta \cap U_1$. Then there exists a $t' \in U$ such that $t' \leqq t$, which implies that $t' \in I_\beta \cap L(w) \subset I_\beta \cap L(x) = L(y_\beta)$. Thus $t' \in L(y_\beta) \cap U$ which implies that U meets $\{L(y_\alpha)\}$ for a cofinal set of indices α. This implies that $w \in L \subset L(y)$. Therefore, $I \cap L(x) \subset L(y)$.

Remark. Having shown that R_X is a compact space and an algebraic semilattice, the natural conjecture is that R_X is a compact semilattice. We have also shown that R_X

is algebraically isomorphic to $\Lambda(X)$. Now $\Lambda(X)$ is a topological semilattice with the compact-open topology which leads to the conjecture that $\Lambda(X)$ is iseomorphic to R_X. We will now show that when X is a compact semilattice these conjectures are true; it is then natural to ask for what class of compact posets are the conjectures true.

2. LEFT TRANSLATIONS OF COMPACT SEMILATTICES

A topological semigroup is a semigroup S equipped with a Hausdorff topology for which the operation of multiplication is continuous as a map from $S \times S$ into S. A topological semilattice is a meet semilattice together with a Hausdorff topology for which the meet operation is continuous, or equivalently, a commutative topological semigroup in which every element is idempotent.

If S is a semigroup, a left translation of S is a map λ of S into S such that $\lambda(xy) = (\lambda x)\, y$ for all $x, y \in S$. The set $\Lambda(S)$ of all left translations forms a semigroup under function composition. Furthermore, if S is locally compact, $\Lambda(S)$ equipped with the compact-open topology is a topological semigroup, and the evaluation map $(\lambda, s) \to \lambda s : \Lambda(S) \times S \to S$ is continuous. For each $s \in S$, let λ_s be the left translation $\lambda_s t = st$ for all $t \in S$. The canonical map $\pi : S \to \Lambda(S)$ defined by $\pi(s) = \lambda_s$ is a continuous homomorphism.

2.1. Proposition. *Let S be a locally compact ideal of a topological semigroup V; then the map $\tau : s \to \lambda_s \mid S$ of V into $\Lambda(S)$ is a continuous homomorphism. Furthermore, if S is a semilattice, then τ is the unique extension of the canonical homomorphism of S into $\Lambda(S)$.*

Proof. The proof that τ is a continuous homomorphism is standard, so it will be omitted. The rest of the proposition follows from III 1.12 and V 6.5 of [5].

If S is a semilattice, then an ideal I of S is called a retract ideal if there exists an endomorphism α of S such that $\alpha(S) \subset I$ and $\alpha(x) = x$ for all $x \in I$. Such a mapping α is called an I-endomorphism. The following proposition is taken from [5, p. 168].

2.2. Proposition. *Let Y be a semilattice. Then left translations of Y coincide with I-endomorphisms of Y where I is an ideal of Y. The set R_Y of retract ideals coincides with the set of all ideals I having the property that for every $\alpha \in Y$, $I \cap J(\alpha)$ is a principal ideal. The mapping Ψ defined by:*

$$\Psi : \lambda \to \lambda(Y) \quad (\lambda \in \Lambda(Y))$$

is an isomorphism of $\Lambda(Y)$ onto R_Y where multiplication in the latter is the set theoretical intersection. Moreover, for $I \in R_Y$ the corresponding left translation is given by: $I \cap J(\alpha) = J(\lambda\alpha)$ for every $\alpha \in Y$.

Remark. It becomes clear from 2.2 that an ideal retraction on a semilattice is just a left translation. It is also evident that the two uses of retract ideal coincide on semilattices. Using the results of section 1, and in addition 7.1 of [1, p. 47] in part (b), we have the following theorem.

2.3. Theorem. *For a compact semilattice X the following are true.*

(a) *Every retract ideal is closed.*
(b) *R_X is a compact semilattice of $H(X)$.*
(c) *Every left translation of X is continuous.*

If X is a compact semilattice, R_X will always be understood to have the subspace topology of $H(X)$, and $\Lambda(X)$ the compact-open topology.
The proof of the next theorem was suggested to me by P. Bacon.

2.4. Theorem. *If X is a compact semilattice, the map $\Psi : \lambda \to \lambda(X)$ is an iseomorphism of $\Lambda(X)$ onto R_X.*

Proof. Let ε denote the evaluation map of $\Lambda(X) \times X \to X$, $\mathscr{L} = \varepsilon \circ (\text{id}\, x\Psi^{-1})$. Let P_X denote the set of principal ideals of X in the hoperspace topology, and let $\mathscr{P}$ denote the map from $X \to P$ defined by $\mathscr{P}(t) = XtX$. For X a compact semilattice $\mathscr{P}$ is an isomorphism. The map $\mathscr{L}$ can be factored as

$$R_X \times X \xrightarrow{\text{id}\, x\mathscr{P}} R_X \times P_X \xrightarrow{\cap} P_X \xrightarrow{\mathscr{P}^{-1}} X$$

where each map is continuous. Since by 2.2 Ψ is an isomorphism, the Ψ^{-1} image of the topology of R_X on $\Lambda(X)$ is a compact Hausdorff topology for which the evaluation map ε is continuous. The compact open topology on $\Lambda(X)$ is the minimal Hausdorff topology for which the evaluation map is continuous, and a compact Hausdorff topology is a minimal Hausdorff topology. Hence the two topologies are equal and Ψ is an iseomorphism.

Corollary. *For a compact semilattice X, the semilattice $\Lambda(X)$ of left translations is a compact semilattice.*

Remark. The translational hull of a semigroup is defined in [5, p. 63]. For a semilattice, the translational hull is the same as the semilattice of left translations [5, p. 171]. So we may restate the corollary: If X is a compact semilattice, the translational hull of X is a compact semilattice.
We now give an application of the above theorem.

2.5. Theorem. *For a compact semilattice X the following statements are equivalent.*

(a) *X may be embedded as an ideal in a connected semigroup with identity;*

(b) $\Lambda(X)$ *is connected;*

(c) *for every non-zero retract ideal* R *of* X *and every open set* $\mathscr{U}$ *of the diagonal in* $X \times X$, *there exists a retract ideal* $T \subset R$, $T \neq R$ *and* $T \subset \mathscr{U}^*(R)$.

Proof. (a) $\Rightarrow$ (b). If X is embedded as an ideal in a connected semigroup T with identity, we may without loss of generality assume that X is an ideal of T. If τ is the map defined in 2.1, then $T' = \tau(T)$ is a connected subsemilattice of $\Lambda(X)$ containing the identity and zero of $\Lambda(X)$. Observe that $\Lambda(X) = \bigcup \lambda T'$, $\lambda \in \Lambda(X)$. Thus $\Lambda(X)$ is a union of connected sets with non-empty intersection which implies $\Lambda(X)$ is connected.

(b) $\Rightarrow$ (a). Since X is compact, the map $\pi : s \to \lambda_s : X \to \Lambda(X)$ is an embedding, and by III 1.7 [5, p. 64], $\pi(X)$ is an ideal of $\Lambda(X)$.

(b) $\Rightarrow$ (c). By 2.4, $\Lambda(X)$ connected implies R_X is connected, which implies that each non-zero element R of R_X is connected to the retract ideal $\{0\}$ by a connected totally ordered semilattice I of R_X, with $R = \sup I$. If $\mathscr{U}$ is an open set of the diagonal in $X \times X$, then $\mathscr{U}^*(R)$ is an open neighborhood of R. Since I is connected, there exists a $T \in \mathscr{U}^*(R) \cap I$ such that $T \subset R$, $T \neq R$.

(c) $\Rightarrow$ (b). Let I_0 be the zero of R_X. Condition (c) implies that condition (2) of theorem 2 in [4] is satisfied for the open set $R_X \setminus \{I_0\}$. Also condition (c) implies $R_X \setminus \{I_0\}$ is not closed. Therefore, by theorem 2 of [4] every retract ideal R belongs to a connected chain C in R_X that also contains I_0. Thus R_X is a union of connected sets with non-empty intersection, and so R_X is connected. By 2.4, $\Lambda(X)$ is connected.

Example. Let $S = \{(x, y) : 0 \leqq x \leqq 1 \text{ and } 0 \leqq y \leqq 1\}$ with the operation of min multiplication in each coordinate. Let X be the subsemilattice $\{(x, y) : 0 \leqq x \leqq 1$ and $0 \leqq y \leqq 1 - x\}$. For each $(a, b) \in S$, let $I(a, b) = \{(x, y) : 0 \leqq x \leqq a$ and $0 \leqq y \leqq b\} \cap X$. Then $R_X = \{I(a, b) : (a, b) \in S\}$. The map $(a, b) \to I(a, b)$ is an iseomorphism of S onto R_X. The corresponding iseomorphism of S onto $\Lambda(X)$ is given by $(a, b) \to \lambda_{(a,b)} \mid X$.

We now wish to conclude the paper with some comments.

(1) A Lawson semilattice is a topological semilattice with enough homomorphisms into the unit interval with the min multiplication to separate points. From the above example a natural question arises: Does the semilattice of left translations of a compact Lawson semilattice form a Lawson semilattice?

(2) The author gives an inexact statement of a question raised by R. Koch. How large is the semilattice of left translations compared to the original semilattice?

(3) Since there is a great similarity between the theory of ideal retractions on a compact locally bounded directed poset and the theory of left translations on a compact semilattice, it is natural to conjecture that in the more general case the retract ideals do indeed form a compact semilattice which is iseomorphic to the semilattice of ideal retractions.

(4) Using the remark after theorem 2.3, that the translational hull of a compact semilattice is a compact semilattice, and V 6.4 of [5], one should be able to give an explicit description of the translational hull of a compact semigroup which is a semilattice of groups. This paper, in fact, was originally motivated by the author's desire to begin a study of topologizing the translational hull of compact semigroups.

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