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CONCERNING BINDING CATEGORIES

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A category A is binding if any algebraic category can be fully embedded into it (see [1]). By an algebraic category we mean in this paper any equationally definable category of algebras with finitary operations. J. SICHLER has found in [6] a finite category C such that a category A satisfying the following conditions (0)–(6) is binding if and only if C can be fully embedded into A:

(0) there exists a faithful functor $U : A \to Ens$ (it means that (A, U) is a concrete category),

(1) there are a class E of epis and a class M of monics of A such that A is a bicategory in the sense of Isbell with respect to these two classes,

(2) $U(m)$ is one-to-one mapping for every $m \in M$,

(3) for every object a of A and for every bijection $b : Ua \to x$ there is an isomorphism i of A such that $U(i) = b$,

(4) A has and U preserves equalizers,

(5) A is cocomplete,

(6) if $D : S \to A$ is a diagram and $a \in A$ is its colimit with the colimiting cone $\tau : D \dot{\to} a$, then
$$Ua = \bigcup_{s \in S} U(\tau_s)(UDs).$$

He has proved it in the following way. Let $\bar{G}$ be the category of all undirected graphs and their compatible mappings. Let $|\ |: \bar{G} \to Ens$ be the usual forgetful functor. Denote successively by $\underline{1}$, $\underline{2}$, $\underline{3}$ and $\underline{4}$ the full graph without diagonal having one, two, three and four vertices. An undirected graph is 3-colourable if it has a compatible mapping into $\underline{3}$. Let G be the full subcategory of $\bar{G}$ consisting of all 3-colourable graphs. The category G is binding. Let C be the full subcategory of $\bar{G}$ determined by graphs $\underline{1}$, $\underline{2}$ and $\underline{4}$. C is dense (left adequate) in $G \cup \{\underline{4}\}$ and cogenerates $G \cup \{\underline{4}\}$ because $\underline{4}$ cogenerates G itself. J. Sichler has shown that if C can be fully embedded into a category A satisfying (0)–(6), then there exists a full embedding $T: C \to A$ such that a left Kan extension L_0 of T is a full embedding of the binding category of all connected graphs from $G \cup \{\underline{4}\}$ into A.

On the other hand, let B be a category, B_0 a full subcategory of B which is small and cogenerates B, and A a cocomplete and co-well-powered category. If $T: B_0 \to A$ is a full embedding, then beginning with a left Kan extension L_0 of T we can transfinitely construct a functor $L_* : B \to A$ extending T such that whenever B_0 is dense in B and A has enough isomorphic copies of each of its objects, then L_* is a full embedding if and only if a full embedding extending T exists (see [5]).

If we take L_* instead of L_0 in the previous situation, we can enlarge the class of categories tested for bindability by a small category. First, we can show that any co-well-powered category A satisfying (0), (3), (5) and (6) is tested for bindability by C, again. For instance, such categories A cover all comonadic categories. Restrictive is the condition (6). However, if we replace C by a certain small category C_0, (6) can be weakened to a condition satisfied by any algebraic category. In this way we shall solve the problem set in [6] whether there is a small category testing the bindability of any algebraic category.

All necessary concepts of the category can be found in [2].

1. THE CONSTRUCTION

We shall describe the construction of L_* on objects. Let B_0 be a small full subcategory of B which cogenerates B, A a cocomplete co-well-powered category and $T: B_0 \to A$ a full embedding. Let $b \in B$ and denote by $P : (B_0 \downarrow b) \to B_0$ the projection of the comma category $(B_0 \downarrow b)$ into B_0. Then $L_0 b$ is a colimit of the functor

$$(B_0 \downarrow b) \xrightarrow{P} B_0 \xrightarrow{T} A\,.$$

Suppose that we have a functor $L_{\alpha-1}$ for an ordinal α. Then $L_\alpha b$ is a colimit of the following diagram.

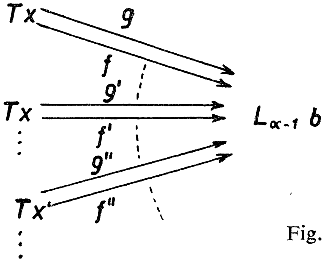

Fig. 1.

Arrows of this diagram are all arrows of A with the domain in TB_0 and the codomain $L_{\alpha-1}b$. Arrows $f, g : Tx \to L_{\alpha-1}b$ have the same domain in this diagram if and only if $L_{\alpha-1}(h) . f = L_{\alpha-1}(h) . g$ for every arrow $h : b \to y$ and every $y \in B_0$. Denote by $\lambda_b^{\alpha-1,\alpha}$ the component of the colimiting cone with the domain $L_{\alpha-1}b$. Further, let α be limit and consider the diagram

$$(**) \qquad L_0 b \xrightarrow{\lambda_b^{0,1}} L_1 b \xrightarrow{\lambda_b^{1,2}} \dots L_\beta b \xrightarrow{\lambda_b^{\beta,\beta+1}} L_{\beta+1} b \xrightarrow{\lambda_b^{\beta+1,\beta+2}} \dots$$

having objects $L_\beta b$ and arrows $\lambda_b^{\beta,\beta+1}$ for $\beta < \alpha$. Then $L_\alpha b$ is defined to be a colimit of this diagram and by $\lambda_b^{\beta,\alpha} : L_\beta b \to L_\alpha b$ we denote components of the colimiting cone. This process stops at some ordinal γ and we put $L_* b = L_\gamma b$. We get functors L_β and L_* extending T. Moreover, our λ's determine natural transformations $\lambda^{\beta,\alpha} : L_\beta \dot{\to} L_\alpha$ and $\lambda^\alpha : L_\alpha \dot{\to} L_*$ for any $\beta < \alpha$. It holds $\lambda^{\beta,\alpha} . \lambda^{\delta,\beta} = \lambda^{\delta,\alpha}$ for any $\delta < \beta < \alpha$ and therefore $L_\alpha b$ is a colimit of a slight modified diagram $(**')$, which we obtain from $(**)$ taking all $\lambda_b^{\delta,\beta}$, $\delta < \beta < \alpha$ as arrows.

The detailed description of this construction can be found in [5].

Lemma 1. *Let $x \in B_0$, $b \in B$ and $f, g : L_* x \to L_* b$ in A such that $L_*(h) f = L_*(h) g$ for any $h : b \to y$ and any $y \in B_0$. Then $f = g$.*

Proof immediately follows from the construction of $L_* b$.

Lemma 2. *In addition, let B_0 generate B. Then L_* is faithful.*

Proof. Let $x \in B_0$, $b \in B$ and $f \neq g : x \to b$. Since B_0 cogenerates B, there exist $y \in B_0$ and $h : b \to y$ such that $hf \neq hg$. Since T is faithful, $L_*(hf) \neq L_*(hg)$ and therefore $L_*(f) \neq L_*(g)$. Now, faithfulness of L_* follows similarly from the fact that B_0 generates B (see [5] Prop. 1).

Now, we are going to give some sufficient conditions for L_* to be a full embedding. A subcategory B_0 of a concrete category $(B, |\ |)$ projectively generates B if for any $b, c \in B$ and any mapping $f : |b| \to |c|$, $f = |f_1|$ for an arrow $f_1 : b \to c$ of B if and only if for every $x \in B_0$ and every arrow $h : c \to x$ of B there is an arrow $h' : b \to x$ of B such that $|h'| = |h| . f$. Choose the following classes of small categories: $\mathscr{C}_1$ is the class of all well-ordered sets without the greatest element taken as categories, $\mathscr{C}_2$ consists of all connected small categories S containing an object t such that any non-identity arrow of S has the codomain t and finally $\mathscr{C}_3(B_0, B)$ consists of all comma categories $(B_0 \downarrow b)$ for $b \in B$. Put $\mathscr{C}(B_0, B) = \mathscr{C}_1 \cup \mathscr{C}_2 \cup \mathscr{C}_3(B_0, B)$. Clearly any colimit needed in the construction of L_* is a colimit of a diagram $D : S \to A$ with $S \in \mathscr{C}(B_0, B)$. Namely, $(*)$ for $S \in \mathscr{C}_2$ and $(**')$ for $S \in \mathscr{C}_1$.

The crucial part of the following proof, the proof of fulness of L_* with respect to arrows of A having a domain in $L_* B_0$, is a modification of Lemma 4 from [6].

Theorem 1. *Let $(B, |\ |)$ be a concrete category and B_0 a small dense full subcategory of B which cogenerates and projectively generates B. Let B_0 contains an object e such that* card $|e| = 1$ *and for any $x \in B$ and any mapping $u' : |e| \to |x|$*

there exists $u : e \to x$ in B_0 with $|u| = u'$. For any $z \in B_0$, $b \in B$ and $g : z \to b$ in B let there exist $d \in B_0$ with the following properties:

a) *d is a cogenerator of B,*

b) *for any permutation s' of $|d|$ interchanging two elements of $|d|$ there exists $s : d \to d$ in B with $|s| = s'$,*

c) *there is an $h_0 : b \to d$ in B such that for any $h : b \to d$ in B, an $s : d \to d$ in B with $hg = sh_0g$ can be found,*

d) card $(|d| - |h_0g|(|z|)) > 1$.

Let (A, U) be a co-well-powered concrete category having colimits of functors $D : S \to A$ for $S \in \mathscr{C}(B_0, B)$, satisfying (6) for these colimits and fulfilling the condition (3). Let $T : B_0 \to A$ be a full embedding.

Then $L_* : B \to A$ is a full embedding.

Proof. By the previous lemma and Corollary 2 from [5] it suffices to prove that for any $y \in B_0$, $b \in B$ and $f : L_*y \to L_*b$ in A there is an arrow $f' : y \to b$ in B with $L_*(f') = f$.

Denote by $u_{x,i}$ the arrow $u_{x,i} : e \to x$ of B for which $|u_{x,i}| (|e|) = \{i\}$, where $x \in B$ and $i \in |x|$. Let us have a $d \in B_0$ satisfying a)–d). Let $p \in UL_*e$ such that $UL_*(u_{d,i})(p) = UL_*(u_{d,j})(p)$ for some $i, j \in |d|$, $i \neq j$. Take $k \in |d|$, $i \neq k \neq j$ and $s : d \to d$ such that $|s|$ is the permutation of $|d|$ interchanging i and k. We get $UL_*(u_{d,k})(p) = UL_*(su_{d,i})(p) = UL_*(su_{d,j})(p) = UL_*(u_{d,j})(p)$. Thus there exists $p \in UL_*e$ such that $UL_*(u_{d,i})(p) \neq UL_*(u_{d,j})(p)$ for any $i, j \in |d|$, $i \neq j$. Suppose that $UL_*(u_{x,i})(p) = UL_*(u_{x,j})(p)$ for some $x \in B_0$ and $i, j \in |x|$, $i \neq j$. Since d is a cogenerator, there exists $h : x \to d$ in B such that $u_{d,|h|(i)} = hu_{x,i} \neq hu_{x,j} = u_{d,|h|(j)}$. Further, $UL_*(u_{d,|h|(i)})(p) = UL_*(u_{d,|h|(j)})$, which is a contradiction.

Let $b \in B$ and $f : L_*e \to L_*b$. Since L_*b is defined by colimits of functors $D : S \to A$, where $S \in \mathscr{C}(B_0, B)$ and any $\lambda^{\beta,\alpha}$ is a natural transformation, (6) enables us to find $z \in B_0$, $g : z \to b$ in B and $q \in UL_*z$ such that $U(f)(p) = UL_*(g)(q)$. Take d for this g. We shall denote $u_{d,i}$ for d just taken briefly by u_i.

Consider h_0 from c). We can find u_k such that $L_*(h_0) . f = L_*(u_k)$. Suppose that $k \notin |h_0g| (|z|)$. Let $s : d \to d$ be an arrow in B such that $|s|$ is the permutation of $|d|$ interchanging k with an element $i \in |h_0g| (|z|)$. Then there exists $u_{z,n} : e \to z$ in B with $u_k = sh_0gu_{z,n}$. It holds $UL_*(u_k)(p) = U(L_*(h_0)f)(p) = UL_*(h_0g)(q)$. By d) and b) there exists an $s' : d \to d$ such that $|s'|$ is the permutation of $|d|$ interchanging i with some $j \in |d| - (|h_0g| (|z|) \cup \{k\})$. We have $UL_*(u_k)(p) = UL_*(s'u_k)(p) = UL_*(s'h_0g)(q)$ and therefore $UL_*(u_i)(p) = UL_*(su_k)(p) = UL_*(ss'h_0g)(q) = UL_*(s'h_0g)(q) = UL_*(u_k)(p)$, which is a contradiction. Thus $k \in |h_0g| (|z|)$ and there exists a $u_{z,n} : e \to z$ such that $u_k = h_0gu_{z,n}$. Put $f' = gu_{z,n}$.

Suppose that there exists $x \in B_0$ and $h : b \to x$ in B such that $L_*(h)f \neq L_*(hf')$. Let $u : e \to x$ with $L_*(h)f = L_*(u)$. There is $h' : x \to d$ with the property $h'u \neq h'hf'$ and $s : d \to d$ such that $h'hg = sh_0g$ (see c)). Thus $h'hf = sh_0f'$. Hence

$UL_*(su_k)(p) = UL_*(sh_0f')(p) = UL_*(h'hf')(p) \neq UL_*(h'u)(p) = U(L_*(h'h)f)(p) = UL_*(h'hg)\ (q) = UL_*(sh_0g)\ (q) = UL_*(su_k)\ (p)$, a contradiction. Therefore $L_*(h)f = L_*(h)\,L_*(f')$ for any $x \in B_0$ and any $h : b \to x$ and by Lemma 1, $f = L_*(f')$.

Now, let $y \in B_0$, $b \in B$ and let $f : L_*y \to L_*b$ be an arrow in A. Define $\bar{f} : /y/ \to /b/$ by $fL_*(u_{y,i}) = L_*(u_{b,\bar{f}(i)})$ for $i \in /y/$. Let $x \in B_0$ and $h : b \to x$. Then $L_*(h)f = L_*(t)$ for some $t : y \to x$ and we have $L_*(tu_{y,i}) = L_*(h)fL_*(u_{y,i}) = L_*(hu_{b,\bar{f}(i)})$. Thus $/tu_{y,i}/ = /h/\bar{f}/u_{y,i}/$ for any $i \in /y/$. Hence $/h/\bar{f} = /t/$ and $\bar{f} = /f'/$ for an arrow $f' : y \to b$ because B_0 projectively generates B. Moreover, $L_*(hf') = L_*(t) = L_*(h)f$ and Lemma 1 yields $L_*(f') = f$. The proof is complete.

Note. Using p from the previous proof we may define a natural monotransformation $\alpha : /\ / \dot{\to} UL_*$ by $\alpha_b(i) = UL_*(u_{b,i})\,(p)$. If we want to avoid the axiom of choice for classes, which is used in the proof of Corollary 2 of [5], we can suppose that $/t/ = id_{/b/}$ implies $t = id_b$ for any isomorphism $t : b \to b$ in B and apply Lemma 1.5 of [4].

2. TESTING CATEGORIES

It is easy to see that $B = G \cup \{\underline{4}\}$ and $B_0 = C$ fulfil all suppositions of Theorem 1. Indeed, $e = \underline{1}, \underline{4}$ is the only d and $\underline{4}$ projectively generates $G \cup \{\underline{4}\}$ by Lemma 1 of [6].

Theorem 2. *A co-well-powered category A satisfying* (0), (3), (5) *and* (6) *is binding if and only if C can be fully embedded into it.*

We shall define a small category C_0 testing bindability of any algebraic category. Let $\underline{\aleph}_0$ be the full graph with the diagonal having countably many vertices. Let C_0 be the full subcategory of $\bar{G}$ containing $\underline{\aleph}_0$, $\underline{4}$ and all graphs decomposing into a finite number of components of the form $\underline{1}$ or $\underline{2}$.

Lemma 3. *A comma category $(C_0 \downarrow x)$ is filtered for any graph $x \in G$.*

Proof. Let $x \in G$ and put $S = (C_0 \downarrow x)$. We have to prove that any diagram in S of the form

$$\bullet \qquad \bullet \qquad \text{or} \qquad \bullet \rightrightarrows \bullet$$

is a base for a cone.

Let $f, f' \in S$. It means that $f : z \to x$ and $f' : z' \to x$ are compatible mappings and $z, z' \in C_0$. Since x is 3-colourable, $\underline{4} \neq z,\ z' \neq \underline{\aleph}_0$. Let z'' be the coproduct of z and z' in G with injections $i : z \to z''$, $i' : z' \to z''$ and $f'' : z'' \to x$ the unique arrow of G such that $f''i = f$ and $f''i' = f'$. Since $/z''/ = /z/ \cup /z'/$ and any edge of z'' is an edge of z or z', we have $z'' \in C_0$. Therefore $f'' \in S$ and $i : f \to f''$, $i' : f' \to f''$ are arrows of S.

Let $f, g \in S$, $g : z \to x$ and let $i, j : f \to g$ be a parallel pair of arrows of S. Define a compatible mapping $k : z \to z$ such that two components of z have the same image in k if and only if they have the same image in g. There is an $h : z \to x$ such that $g = hk$, which means that $k : g \to h$ is an arrow in S and moreover, $ki = kj$ holds.

Theorem 3. *Let A be a co-well-powered category satisfying (0), (3) which has colimits of functors $D : S \to A$ for S filtered or $S \in \mathscr{C}_2$ and fulfils (6) for these colimits. Then A is binding if and only if C_0 can be fully embedded into it.*

Proof. Put $B = G \cup \{\underline{4}, \underline{\aleph_0}\}$ and $B_0 = C_0$. Then all suppositions of Theorem 1 are satisfied. Namely, $e = \underline{1}$ and $\aleph_0$ is the only d. Since any category belonging to $\mathscr{C}_1$ is filtered and by Lemma 3 the same holds for $\mathscr{C}_3(B_0, B)$, Theorem 3 follows from Theorem 1.

Lemma 4. *Let (A, U) be a concrete category having kernel pairs and let Uf be epi for any coequalizer f in A. Then A satisfies (6) for any $S \in \mathscr{C}_2$.*

Proof. Let $S \in \mathscr{C}_2$ and let $a \in A$ be a colimit of a functor $D : S \to A$ with the colimiting cone τ. Denote by f the component of τ with the domain t. Let $f_1, f_2 : a_0 \to$ $\to t$ be a kernel pair of f. We shall prove that f is a coequalizer of f_1 and f_2. Suppose that $gf_1 = gf_2$ for an arrow g in A. Let $h_1, h_2 : s \to t$ be a parallel pair of arrows of S. Since $f \,.\, D(h_1) = f \,.\, D(h_2)$, there is a unique arrow $k : D(s) \to a_0$ in A such that $f_i k = D(h_i)$ for $i = 1, 2$. Therefore $g \,.\, D(h_1) = g \,.\, D(h_2)$ and g determines a cone from the base D. Thus, there is a unique arrow k' in A with $k'f = g$. Hence f is a coequalizer and Uf is epi. But it means that (6) is satisfied.

Corollary 1. *An algebraic category A is binding if and only if C_0 can be fully embedded into it.*

Proof. The usual forgetful functor $U : A \to Ens$ preserves filtered colimits (see [2, p. 209]). Further, A is cocomplete, complete and Uf is epi for any coequalizer f in A. The last fact can be found in [3], Lemma 6. The result follows from Lemma 4 and Theorem 3.

The infiniteness of C_0 plays no role because we can take instead of C_0 the full subcategory C' of $\bar{G}$ having objects $\underline{1}, \underline{4}, \underline{\aleph_0}$ and the graph having countably many copies of $\underline{1}$ and countably many copies of $\underline{2}$ as components. In this case we have to use only the graphs from G having at least one edge instead of the whole G. After computation that C' is dense in $G \cup \{\underline{4}, \underline{\aleph_0}\}$ nad that $(C' \downarrow x)$ is filtered for any $x \in G$ having at least one edge, we obtain that this four-object category C' tests bindability of any algebraic category. It seems to be an interesting problem to find such a testing category with the smallest possible number of objects.

The method presented does not work for categories of algebras of arbitrary arities because categories from $\mathscr{C}_1$ needed for the construction of L_* are not $\mathfrak{m}$-filtered for a cardinal $\mathfrak{m} > \aleph_0$.

Added in proof. This difficulty is avoided and further results are given in the following author's papers: On extensions of full embeddings and binding categories (Comm. Math. Univ. Carol. *15* (1974), 631–653) and Codensity and binding categories (to appear in Comm. Math. Univ. Carol.).

References

[1] *Z. Hedrlín, A. Pultr* and *V. Trnková:* Concerning a categorical approach to topological and algebraic theories, Proc. of the Second Prague Topological Symposium, 1966, Prague 1967, 176–181.

[2] *S. Mac Lane:* Categories for the Working Mathematician, Springer-Verlag, New York 1971.

[3] *F. E. J. Linton:* Applied functorial semantics, II., Seminar on Triples and Categorial Homology Theory, Lecture Notes 80, 1969.

[4] *A. Pultr:* On full embeddings of concrete categories with respect to forgetful functors, Comment. Math. Univ. Carolinae *9* (1968), 281–304.

[5] *J. Rosický:* Full embeddings with a given restriction, Comment. Math. Univ. Carolinae *14* (1973), 518–540.

[6] *J. Sichler:* Testings categories and strong universality, Can. Jour. of Math., *XXV* (1973), 370–386.

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