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CONTINUOUS EXTENSION OF SEQUENTIALLY CONTINUOUS LINEAR FUNCTIONALS IN INDUCTIVE LIMITS OF NORMED SPACES

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INTRODUCTION

Let E be the inductive limit of a sequence E_n of normed linear spaces. Let M be a linear subspace of $E = \bigcup_{n=1}^{\infty} E_n$ and consider M as the inductive limit of the sequence $M_n = E_n \cap M$ of subspaces of M. It is well known that it is not always possible to extend every continuous linear functional on M to a continuous linear functional on E (see [7] and [8] for example). When it *is* possible, we say M has the Hahn-Banach Property (H.B.P.). If for each n, M_n is closed in E_n, M is said to be a sequentially closed subspace of E. In 1965, FOIAS and MARINESCU, [1], showed that every sequentially closed subspace of E has the H.B.P. if each E_n is a reflexive Banach space. This result has been extended to spaces with boundedness structures [2], [14]. In the case that the inductive limit is strict and each E_n is Banach, PTÁK [7] has given a necessary and sufficient condition for a sequentially closed subspace M to have the H.B.P. He calls a subspace satisfying the condition semiorthogonal.

In § 1 of this paper, we define the notion of a compressive subspace M of E and show that unless M fails in a trivial way to have the H.B.P., it has the H.B.P. if and only if it is compressive. Pták's assumptions of completeness and sequential closure are not required in his proof [7]. However, our result improves that of Pták by removing the necessity of assuming the strictness of the inductive limit. Our Lemma 3.1 shows that a subspace is compressive if and only if it is semiorthogonal, whenever the inductive limit *is* strict. Hence, the notion of compressive may be regarded as an extension of that of semiorthogonality to the non-strict environment.

In § 2, we derive some "concrete" conditions under which a subspace M will be compressive. For example, it is shown that if the spaces E_n are Banach spaces, and the inductive limit is strict, then a sequentially closed subspace is compressive if for each n, either M_n or M_n° is reflexive (M_n° the polar of M_n in E_n').

In § 3, some examples and remarks are given relating to developments in the first two sections. Here, we mention only that an example is given there in which every E_n

may be *non*-reflexive, but every sequentially closed subspace M of E has the H.B.P. Though this situation has been studied before [2], the simple proof given here shows the potential utility of the theory presented.

1. A NECESSARY AND SUFFICIENT CONDITION FOR THE H.B.P.

Let E_n be a sequence of normed linear spaces such that $E_1 \subseteq E_2 \subseteq E_3 \subseteq \ldots$, where the natural injections $i_n : E_n \to E_{n+1}$ are continuous. First observe that the norm on E_{n+1} may always be changed to an equivalent one so that

$$\|i_n\| \leqq 1 . \tag{1.1}$$

We shall therefore assume (1.1) holds. Let $E = \bigcup_{n=1}^{\infty} E_n$ be endowed with the inductive limit topology T, the finest locally convex topology under which the canonical injections $j_n : E_n \to E$ are continuous. For each n, denote by $i_n^* : E'_{n+1} \to E'_n$ the transpose of i_n and by j_n^* the transpose of j_n. If E_{n_k} is a subsequence of E_n, we will denote by I_{n_k} the map $i_{n_{k+1}-1} \circ i_{n_{k+1}-2} \circ \ldots \circ i_{n_k}$. We see then, that $\|i_n^*\| \leqq 1$ and $\|I_{n_k}^*\| \leqq 1$. If M is a subspace of E, denote $M \cap E_n$ by M_n. We will denote the polar of M_n in E'_n by M_n° and by M_n^θ the polar of M_n in E'_n *intersected* with $j_n^*(E')$. Notice that in case the inductive limit is strict, j_n^* is surjective and so $M_n^\circ = M_n^\theta$.

It is clear that every subspace M of E that has the H.B.P. must have the property that every linear functional on M continuous for the norm on each M_n is also continuous on each M_n regarded as a topological subspace of (E, T). Hence, only subspaces with this property are of interest for the problem under consideration. We shall refer to such subspaces as *compatible* subspaces. Moreover, for convenience, we shall call *sequentially continuous* any linear functional on M whose restriction to each M_n is norm continuous. We observe in passing that in case the inductive limit E is strict, every subspace is compatible. Subsequently, whenever an inductive limit of normed spaces is being considered, we shall adhere to the notations and terminology just introduced.

Definition 1.1. Let M be a subspace of an inductive limit E of normed spaces E_n. M is said to be *compressive* if there exists a subsequence E_{n_k} of E_n such that for each $k \geqq 1$, $I_{n_k}^* \circ I_{n_{k+1}}^*(M_{n_{k+2}}^\theta)$ is dense in $I_{n_k}^*(M_{n_{k+1}}^\theta)$ as subspaces of E'_{n_k}. If M is not compressive it will be called *noncompressive*. When the condition that M is compressive holds with $M_{n_{k+2}}^\theta$ and $M_{n_{k+1}}^\theta$ replaced by $M_{n_{k+2}}^\circ$ and $M_{n_{k+1}}^\circ$ respectively, we say that M is *strictly compressive*.

Lemma 1.1. *A subspace M of E is noncompressive only if*

(1.2) *$\exists$ a subsequence E_{n_k} of E_n such that $\forall\ k \geqq 1$, $I_{n_1}^* \circ \ldots \circ I_{n_{k+1}}^*(M_{n_{k+2}}^\theta)$ is not norm dense in $I_{n_1}^* \circ \ldots \circ I_{n_k}^*(M_{n_{k+1}}^\theta)$ as subspaces of E'_{n_1}.*

Proof. Suppose M does not satisfy the condition (1.2) of the Lemma. Then we can show that

(1.3) for each n, $\exists N(n) > n$ such that $i_n^* \circ \ldots \circ i_{N(n)+k+1}^*(M_{N(n)+k+2}^\theta)$ is dense in $i_n^* \circ \ldots \circ i_{N(n)+k}^*(M_{N(n)+k+1}^\theta)$ for all $k \geqq 0$.

If this is *not* the case, then $\exists\, n$ so that for every $m > n$ $\exists\, k(m)$ such that $i_n^* \circ \ldots \ldots \circ i_{m+k(m)+1}^*(M_{m+k(m)+2}^\theta)$ is not dense in $i_n^* \circ \ldots \circ i_{m+k(m)}^*(M_{m+k(m)+1}^\theta)$. But if $n_1 = n$, $n_2 = (n_1 + 1) + k(n_1 + 1), \ldots, n_{j+1} = n_j + 1 + k(n_j + 1)$, the sequence E_{n_j} is seen to satisfy the condition of the lemma. Indeed, suppose to the contrary, $I_{n_1}^* \circ \ldots \ldots \circ I_{n_j}^*(M_{n_{j+1}}^\theta)$ is dense in $I_{n_1}^* \circ \ldots \circ I_{n_{j-1}}^*(M_{n_j}^\theta)$ for some j. Then,

$$I_{n_1}^* \circ \ldots \circ I_{n_j}^*(M_{n_{j+1}}^\theta) = I_{n_1}^* \circ \ldots \circ I_{n_{j-1}}^* \circ i_{n_j}^* \circ i_{n_{j+1}-1}^* \circ \ldots \circ i_{n_{j+1}-1}^* (M_{n_{j+1}}^\theta) \subseteq$$

$$\subseteq I_{n_1}^* \circ \ldots \circ I_{n_{j-1}}^* \circ i_{n_j+1}^*(M_{n_j+2}^\theta) \subseteq I_{n_1}^* \circ \ldots \circ I_{n_{j-1}}^* \circ i_{n_j}^*(M_{n_{j+1}}^\theta) \subseteq I_{n_1}^* \circ \ldots \circ I_{n_{j-1}}^*(M_{n_j}^\theta)\,.$$

Thus $i_{n_1}^* \circ \ldots \circ i_{n_j+1}^*(M_{n_j+2}^\theta)$ is dense in $i_{n_1}^* \circ \ldots \circ i_{n_j}^*(M_{n_{j+1}}^\theta)$, a contradiction. Now, then, let $n_1 = 1$, $n_{j+1} = N(n_j) + 2$ where $N(n_j)$ is chosen according to (1.3). The sequence E_{n_j} satisfies the condition for M to be compressive. For, $I_{n_j}^* \circ I_{n_{j+1}}^*(M_{n_{j+2}}^\theta)$ is by definition $i_{n_j}^* \circ i_{n_j+1}^* \circ \ldots \circ i_{n_{j+1}}^* \circ i_{n_{j+1}+1}^* \circ \ldots \circ i_{n_{j+2}-1}^*(M_{n_{j+2}}^\theta)$, which is dense in $i_{n_j}^* \circ \ldots \circ i_{n_{j+1}+1}^* \circ \ldots \circ i_{n_{j+2}-2}^*(M_{n_{j+2}-1}^\theta)$, which in turn is dense in $i_{n_j}^* \circ \ldots \ldots \circ i_{n_{j+1}+1}^* \circ \ldots \circ i_{n_{j+2}-3}^*(M_{n_{j+2}-2}^\theta)$. Continuing in this manner and using the fact that a dense subspace of a dense subspace is dense, we obtain $I_{n_j}^* \circ I_{n_{j+1}}^*(M_{n_{j+2}}^\theta)$ is dense in $I_{n_j}^*(M_{n_{j+1}}^\theta)$. We see then that M noncompressive implies the condition of the Lemma. ∎

In fact, the converse of the above Lemma is true as we shall see at the end of this section, providing that M is compatible.

Lemma 1.2. *Let F be a normed linear space and $F_1 \supseteq F_2 \supseteq \ldots$ a decreasing sequence of subspaces of F such that for all $k \geqq 1$, $\bar{F}_{k+1} \not\supseteq F_k$. Then there exists a sequence x_k in F_1 such that*

(1) $x_k \in F_k \setminus F_{k+1}$,

and

(2) $\|\sum_{i=1}^{k} x_i - y\| \geqq k - 1 \;\forall y \in F_{k+1}$.

Proof. For each $k \geqq 1$, let $G_{k+1} = \bar{F}_{k+1} \cap F_k$, and note that $G_{k+1} \supseteq F_{k+1}$. By Riesz' Lemma, there is $x_1 \in F_1 \setminus G_2$ such that $\|x_1\| = 2$, and $\|x_1 - y\| \geqq \frac{3}{2}$ for all $y \in G_2$. Suppose $x_1, \ldots, x_n$ have been chosen so that (1) and (2) hold for $k = 1, 2, \ldots, n$. Again, by Riesz' Lemma, let $x_{k+1} \in F_{k+1} \setminus G_{k+2}$ be such that $\|x_{k+1}\| = 2\sum_{i=1}^{k} \|x_i\|$ and $\|x_{k+1} - y\| \geqq \frac{3}{2}\sum_{i=1}^{k} \|x_i\|$ for all $y \in G_{k+2}$. Then

$$\|\sum_{i=1}^{k+1} x_i - y\| \geqq \|x_{k+1} - y\| - \|\sum_{i=1}^{k} x_i\| \geqq \tfrac{3}{2}\sum_{i=1}^{k} \|x_i\| - \sum_{i=1}^{k} \|x_i\| = \tfrac{1}{2}\sum_{i=1}^{k} \|x_i\| \geqq k$$

for all $y \in G_{k+2}$. The result follows by induction. ∎

Theorem 1.1. *If M is a noncompressive subspace of E, then there exists a sequentially continuous linear functional f on M which for each n is continuous on M_n for the topology induced by T but which has no continuous linear extension to E; that is, M non-trivially does not have the H.B.P.*

Proof. By Lemma 1.1, we may assume that the defining sequence E_n for E is such that $i_1^* \circ \ldots \circ i_{n+1}^*(M_{n+2}^\theta)$ is *not* dense in $i_1^* \circ \ldots \circ i_n^*(M_{n+1}^\theta)$ by considering E_n to be itself the subsequence whose existence is asserted there. Let $F_n = i_1^* \circ \ldots \circ i_{n-1}^*(M_n^\theta)$, $n = 2, 3, \ldots$, and $F_1 = M_1^\theta$. Applying Lemma 1.2 to F_n with $F = E_1'$, there is a sequence x_n' of functionals in M_1^θ so that $x_n' \in F_n \setminus F_{n+1}$ and $\| \sum_{i=1}^{n} x_i' - y' \| > n - 1$ for all $y' \in F_{n+1}$. Now, since $M_n^\theta = M_n^\circ \cap j_n^*(E')$, for each $m \geqq 2$ there is a $h_{m+1} \in E'$ such that $x_m' = i_1^* \circ \ldots \circ i_{m-1}^* \circ j_m^*(h_{m+1})$ and $j_m^*(h_{m+1}) \in M_m^\circ$, where $x_1' = j_1^*(h_2)$. Let $f_1(x) = 0$ and

$$f_n(x) = \sum_{m=1}^{n-1} j_n^*(h_{m+1})(x)\,, \quad x \in E_n\,.$$

It is immediate to verify that if $x \in M_n$, $f_{n+1}(x) = f_n(x)$ since $M_n \subseteq M_{n+1}$ and $j_{n+1}^*(h_{n+1}) \in M_n^\circ$. For each n, it is manifest that f_n is continuous for the norm topology as well as the topology induced by T on M_n. Define the linear functional f on M by $f(x) = f_n(x)$, $x \in M_n$. Suppose φ *were* a sequentially continuous extension of f to E and let $\varphi_n = \varphi|_{E_n}$. Then, noting that $\varphi_n = (\varphi_n - f_n) + f_n$ and that $\varphi_n - f_n \in M_n^\theta$, we have

$$\varphi_1 = i_1^* \circ \ldots \circ i_{n-1}^*(\varphi_n - f_n) + i_1^* \circ \ldots \circ i_{n-1}^*(f_n) = \bar{y} + \sum_{m=1}^{n-1} x_m'\,, \quad \text{where} \quad \bar{y} \in F_n\,.$$

But then,

$$\|\varphi_1\| = \| \sum_{m=1}^{n} x_m' + \bar{y} \| \geqq n - 2 \quad \text{for all} \quad n\,.$$

This contradicts the assumption that φ is sequentially continuous. Thus, the functional f satisfies the conclusion of the Theorem. ∎

The essential content of the following lemma is found in Proposition 1.1 of [7]. In addition, the construction used in Theorem 1.2 closely parallels that of Theorem 2.1 of [7]. However, we will include the proofs here for completeness since there are considerable notational changes.

Lemma 1.3. *Let $F_3 \supseteq F_2 \supseteq F_1$ be locally convex linear topological spaces, N_3 a subspace of F_3, and $N_i = F_i \cap N_3$, $i = 1, 2$. Let the canonical injections $i_j : F_j \to F_{j+1}$, $j = 1, 2$, be continuous and denote by N_i° the polar of N_i in F_i', $i = 1, 2, 3$. Suppose F_1 is normed. Then $i_1^* \circ i_2^*(N_3^\circ)$ is a norm-dense subspace of $i_1^\circ(N_2^\circ)$ in F_1' if and only if $\forall f \in F_2'$, $\varphi \in N_3'$ and $\varepsilon > 0$ such that $f|_{N_2} = \varphi|_{N_2}$, there exists $\psi \in F_3'$ such that $\psi|_{M_3} = \varphi$ and $\|i_1^* \circ i_2^*(\psi) - i_1^*(f)\| < \varepsilon$.*

Proof. First, assume $i_1^* \circ i_2^*(N_3^\circ)$ is dense in $i_1^*(N_2^\circ)$. If $g \in F_3'$ is an extension of φ to F_3, then $f - i_2^* g$ is in N_2°, and therefore there exists $h \in N_3^\circ$ such that $\|i_1^*(f - i_2^*(g)) - i_1^* \circ i_2^*(h)\| < \varepsilon$. That is, $\|i_1^*(f) - i_1^* \circ i_2^*(g + h)\| < \varepsilon$. Set $\psi = g + h$. Conversely, if $f \in N_2^\circ$, let $\varphi = 0$ on N_3. Then there exists $\psi \in F_3'$ such that $\psi|_{N_3} = \varphi = 0$ (hence $\psi \in N_3^\circ$) and $\|i_1^* \circ i_2^*(\psi) - i_1^*(f)\| < \varepsilon$. Thus, $i_1^* \circ i_2^*(N_3^\circ)$ is dense in $i_1^*(N_2^\circ)$. ∎

Theorem 1.2. *Let (E, T) be an inductive limit of normed spaces. Then a subspace M of E has the H.B.P. if etiher*

a) *M is compatible and compressive*

or

b) *M is strictly compressive.*

Proof. In the proof, E_n' denotes the dual of E_n under the norm topology as usual. Suppose M is compressive and compatible, and f is a sequentially continupus linear functional on M. There is no loss in assuming that the subsequence in the definition of compressive (Definition 1.1) is E_n itself. Set $f_n = f|_{M_n}$. Consider the spaces E_n and M_n with their inherited topologies $T|_{E_n}$ and $T|_{M_n}$ denoted both by T_n. Let $\varphi_1 \in E_1'$ be a T_1-continuous extension of f_1, and $\varphi_2 \in E_2'$ a T_2-continuous extension of f_2. Set $\hat{\varphi}_1 = i_1^*(\varphi_2)$. By Lemma 1.3, choose $\varphi_3 \in E_3'$ such that $\varphi_3|_{M_3} = f_3$, φ_3 is T_3-continuous and $\|i_1^* \circ i_2^*(\varphi_3) - i_1^*(\varphi_2)\| = \|i_1^* \circ i_2^*(\varphi_3) - \hat{\varphi}_1\| < \frac{1}{2}$. For $n = 2, 3, 4, \ldots$, set

$$\hat{\varphi}_n = i_n^*(\varphi_{n+1})$$

and choose $\varphi_{n+2} \in E_{n+2}'$ such that $\varphi_{n+2}|_{M_{n+2}} = f_{n+2}$, φ_{n+2} is T_{n+2}-continuous, and

$$\|i_n^* \circ i_{n+1}^*(\varphi_{n+2}) - \hat{\varphi}_n\| < (\tfrac{1}{2})^n .$$

Consider the sequence h_j in E_k' given by

$$h_j = i_k^* \circ i_{k+1}^* \circ \ldots \circ i_{k+j}^*(\hat{\varphi}_{k+j+1}) .$$

We have

$$\|i_k^* \circ \ldots \circ i_{k+j+1}^*(\hat{\varphi}_{k+j+2}) - i_k^* \circ \ldots \circ i_{k+j}^*(\hat{\varphi}_{k+j+1})\| =$$
$$= \|i_k^* \circ \ldots \circ i_{k+j}^* \circ i_{k+j+1}^* \circ i_{k+j+2}^*(\varphi_{k+j+3}) - i_k^* \circ \ldots \circ i_{k+j}^*(\hat{\varphi}_{k+j+1})\| \leqq$$
$$\leqq \|i_k^* \circ \ldots \circ i_{k+j}^*\| \cdot \|i_{k+j+1}^* \circ i_{k+j+2}^*(\varphi_{k+j+3}) - \hat{\varphi}_{k+j+1}\| \leqq (\tfrac{1}{2})^{k+j+1} ,$$

recalling that $\|i_n\| = \|i_n^*\| \leqq 1$ for all n. Thus

$$\|h_{j+m} - h_j\| \leqq \sum_{l=0}^{m-1} \|h_{j+l+1} - h_{j+l}\| \leqq \sum_{l=0}^{m-1} (\tfrac{1}{2})^{j+l+1} ,$$

which implies h_j is Cauchy. Since E_k' is complete, h_j converges to a linear functional

$g_k \in E_k'$. Let g be defined on E by $g(x) = g_k(x)$ if $x \in E_k$. Since

$$g_k = \lim_j i_k^* \circ \dots \circ i_{k+j+1}^*(\hat{\varphi}_{k+j+2}) = i_k^*(\lim_j i_{k+1}^* \circ \dots \circ i_{k+j+1}^*(\hat{\varphi}_{k+j+2})) = i_k^*(g_{k+1}),$$

it is clear that g is well defined. Thus, g is a continuous linear functional on E. Moreover if $x \in M_k$,

$$\begin{aligned} g(x) = g_k(x) &= \lim_j [i_k^* \circ \dots \circ i_{k+j}^*(\hat{\varphi}_{k+j+1})](x) = \\ &= \lim_j [i_k^* \circ \dots \circ i_{k+j+1}^*(\varphi_{k+j+2})](x) = \\ &= \lim_j [\varphi_{k+j+2} \circ i_{k+j+1} \circ \dots \circ i_k(x)] = \\ &= \lim_j \varphi_{k+j+2}(x) = \lim_j f_{k+j+2}(x) = \lim_j f_k(x) = f_k(x). \end{aligned}$$

Hence g extends f. The proof of b) is essentially identical, the only change being to omit the conclusion that the choices of the functionals φ_n will be T_n-continuous. We leave this modification to the reader. ∎

We remark that in Theorem 1.1, it is actually proved that the condition (1.2) implies that M does not have the H.B.P. By Theorem 1.2, it is therefore evident that M being noncompressive is, in fact, *equivalent* to (1.2) for compatible subspaces M. Combining the results of Theorem 1.1 and Theorem 1.2, we have

Theorem 1.3. *A subspace M of an inductive limit E of normed spaces E_n has the H.B.P. if and only if*

i) *M is compatible*

and

ii) *M is compressive.*

Corollary 1.3.1. *If M is a subspace of a strict inductive limit of normed spaces, then M has the H.B.P. if and only if M is strictly compressive.*

Proof. The assertion follows from the Theorem with the observation that in a *strict* inductive limit every subspace is compatible, and compressive is equivalent to strictly compressive. ∎

2. VERY COMPRESSIVE SUBSPACES

We begin by defining a condition somewhat stronger than strictly compressive, but easier to deal with.

Definition 2.1. Let M be a subspace of an inductive limit E of normed spaces. M is said to be very compressive if there exists a subsequence E_{n_k} of E_n such that for each $k \geqq 1$, $I_{n_k}^*(M_{n_{k+1}}^\circ)$ is dense in $M_{n_k}^\circ$.

Clearly, a very compressive subspace is strictly compressive. The theorem of Foias and Marinescu [1] hinges on the fact proved in that paper as a lemma that if each E_n is reflexive and each M_n is closed, then M is very compressive. An example of a compressive but not very compressive subspace will be given in § 3. The relationship of the notions of strictly compressive and very compressive to the extension results of Pták ([6], Corollary 2.2, and [7]) in the case the spaces E_n are Banach, is revealed in Lemma 1.3. When $F_1 = F_2$, we see thereby that M is very compressive if and only if there is a subsequence E_{n_k} of E_n so that for each $k \geqq 1$, if $f \in E'_{n_k}$, $\varphi \in M'_{n_{k+1}}$ and $\varepsilon > 0$ are given, then there exists an extension $\psi \in E'_{n_{k+1}}$ of φ so that $\|I^*_{n_k}\psi - f\| < \varepsilon$. Stated in this way, it is easy to compare the condition of every compressive subspace with the stronger condition of orthogonal subspace as stated in 1° of Proposition 1.1 of [6]. On the other hand, from Lemma 1.3 as stated, it is immediate that the property of being compressive is equivalent to the condition of semiorthogonality given in [7] when the inductive limit is strict.

It is the primary purpose of this section to provide interesting sufficient conditions on M under which it is very compressive regardless of whether or not the underlying spaces E_n are reflexive.

If F is a locally convex linear topological space, and A is a subset of F or F', A° denotes its polar in the system (F, F'). If A is a subset of F' or F'', $A^\bullet$ denotes it polar in the system (F', F''), where F'' is the strong dual of F'. Beyond this convention, we will notationally follow Horvath [3].

Before proceeding, we present the following useful lemma. The proof is elementary and is left to the reader.

Lemma 2.1. *Let E and F be two locally convex linear topological spaces, and $J : E \to F$ a continuous linear mapping. Let be a subspace of E and N a closed subspace of F such that $J^{-1}(N) \subseteq M$. Then $J^*[N^\circ]$ is $\sigma(E', E)$ dense in M°, where J^* denotes the transpose of J.*

Theorem 2.1. *Let $E = \bigcup_{n=1}^{\infty} E_n$ be an inductive limit of normed spaces. Then a subspace M of E is very compressive if for each n, $\sigma(E'_n, E''_n)$ and $\sigma(E'_n, E_n)$ coincide on M°_n, and M_n is closed in E_n.*

Proof. The above Lemma implies that $i^*_n(M^\circ_{n+1})$ is $\sigma(E'_n, E_n)$-dense in M°_n. But, since $\sigma(E'_n, E_n)$ and $\sigma(E'_n, E''_n)$ coincide on M°_n, $i^*_n(M^\circ_{n+1})$ is strongly dense in M°_n. That is, M is very compressive. ∎

In the following discussion, we call a locally convex topological vector space, F, semi-reflexive if the natural injection from F into its double dual F'' is a surjection. We also make use of the fact that F is semi-reflexive if and only if every $\sigma(F, F')$-bounded set is $\sigma(F, F')$-relatively compact.

Theorem 2.2. *Let F a Fréchet space, F' its dual, and F'' its strong double dual. If N is a subspace of F, the following are equivalent:*

(1) *N° is a semireflexive subspace of F',*

(2) $\sigma(F', F)$ and $\sigma(F', F'')$ *coincide on the $\sigma(F', F'')$-closed, $\sigma(F', F'')$-bounded subsets of N°,*

(3) *The $\sigma(F', F)$-closed, convex subsets and the $\sigma(F', F'')$-closed, convex subsets of N° coincide,*

(4) *$\sigma(F', F)$ and $\sigma(F', F'')$ coincide on N°.*

Proof. We will prove (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1). Suppose N° is semireflexive. On N° the topology $\sigma(F', F'')$ coincides with the topology $\sigma(N^\circ, N^{\circ\prime})$. Thus, if B is any $\sigma(F', F'')$-closed, bounded subset of N°, it is $\sigma(F', F'')$-compact. Since the identity map i from B under $\sigma(F', F'')$ to B under $\sigma(F', F)$ is clearly continuous, and $\sigma(F', F)$ is Hasudorff, i is a homeomorphism. That is, (1) $\Rightarrow$ (2). Suppose now A is a $\sigma(F', F'')$-closed and convex subset of N°. Let B be any balanced, convex, $\sigma(F', F)$-closed, equicontinuous subset of F'. B and N° are surely $\sigma(F', F'')$-closed, and so $A \cap B$ is a $\sigma(F', F'')$-closed set. Moreover, since B is equicontinuous, it is $\sigma(F', F'')$-bounded, and hence $\sigma(F', F'')$ and $\sigma(F', F)$ coincide on $B \cap N^\circ$. Since $A \cap B \subseteq$ $\subseteq B \cap N^\circ$, it is then $\sigma(F', F)$-closed. By the Krein-Smulian Theorem, A is $\sigma(F', F)$-closed. Hence (2) $\Rightarrow$ (3). Now assume (3). Under either topology, N° has the same closed subspaces, hence the same continuous linear functionals. It follows that (3) $\Rightarrow$ (4). To show (4) $\Rightarrow$ (1), let A be a $\sigma(N^\circ, N^{\circ\prime})$-closed and bounded set of N°. Since $\sigma(F', F)$ and $\sigma(F', F'')$ coincide on N° and $\sigma(N^\circ, N^{\circ\prime}) = \sigma(F', F'')|_{N^\circ}$, A is $\sigma(F', F)$-closed and bounded. But F is barrelled, and hence A is equicontinuous. It follows that A is $\sigma(F', F)$-compact and therefore $\sigma(F', F'')$-compact. Since $\sigma(F', F'')|_{N^\circ} = \sigma(N^\circ, N^{\circ\prime})$, A is $\sigma(N^\circ, N^{\circ\prime})$-compact. Thus, N° is semireflexive. ■

The following corollary is immediate.

Corollary 2.2.1. *Let F be a Fréchet space, and N a subspace of F. If N has a reflexive topological direct summand, then $\sigma(F', F)$ and $\sigma(F', F'')$ coincide on N°.*

Theorem 2.3. *Let F be a locally convex linear topological space with dual F' and strong double dual F'', and N a subspace of F'. Then $F + N^\bullet = F''$ algebraically if and only if $\sigma(F', F)$ and $\sigma(F', F'')$ coincide on N.*

Proof. Assume first that $F + N^\bullet = F''$ algebraically. Then if $x \in F''$, $x = y + z$, where $y \in F$ and $z \in N^\bullet$. For rach $w \in N$, $\langle y + z, w\rangle = \langle y, w\rangle + \langle z, w\rangle = \langle y, w\rangle$. Thus $\{x\}^\circ$ and $\{y\}^\circ$ coincide on N.

Now suppose $\sigma(F', F)|_N = \sigma(F', F'')|_N$. Let $x \in F''$. Then $x|_N$ is $\sigma(F', F)|_N$ continuous and thus there exists $y \in F$ such that $x|_N = y|_N$. But then $x - y \in N^\bullet$. Hence $F'' = F + N^\bullet$ algebraically. ■

The following theorem provides an equivalent way of viewing compressive and very compressive subspaces and is very useful in distinguishing between the two types.

Theorem 2.4. *Let $F_1 \subseteq F_2 \subseteq F_3$ be locally convex linear topological vector spaces with duals F'_k and strong double duals F''_k, $k = 1, 2, 3$. Let N_3 be a subspace of F_3, and $N_k = F_k \cap N_3$, $k = 1, 2$. Denote by i and j the canonical injections of F_1 into F_2 and F_2 into F_3 respectively, and $\boldsymbol{i}$ and $\boldsymbol{j}$ the unique weakly continuous extensions of i and j to F''_1 and F''_2 respectively. Then $i^* \circ j^*(N_3^\circ)$ is strongly dense in $i^*(N_2^\circ)$ if and only if $\boldsymbol{i}^{-1}(N_2^{\circ\bullet}) = (\boldsymbol{j} \circ \boldsymbol{i})^{-1}(N_3^{\circ\bullet})$, which we will write as $N_2^{\circ\bullet} \cap F''_1 = N_3^{\circ\bullet} \cap F''_1$.*

Proof. $i^* \circ j^*(N_3^\circ)$ is strongly dense in $i^*(N_2^\circ)$ if and only if $(i^* \circ j^*(N_3^\circ))^\bullet = (i^*(N_2^\circ))^\bullet$. But, $z \in (i^* \circ j^*(N_3^\circ))^\bullet$ if and only if $|\langle z, y\rangle| \leqq 1$ for all y in $i^* \circ j^*(N_3^\circ)$ if and only if $|\langle \boldsymbol{j} \circ \boldsymbol{i}(z), w\rangle| \leqq 1$ for all $w \in N_3^\circ$. Thus, $(i^* \circ j^*(N_3^\circ))^\bullet = N_3^{\circ\bullet} \cap F''_1$. Similarly, $(i^*(N_2^\circ))^\bullet = N_2^{\circ\bullet} \cap F''_1$, and the result follows. ∎

Corollary 2.4.1. *Under the same hypotheses as Theorem 2.4, $j^*(N_3^\circ)$ is strongly dense in N_2° if and only if $\boldsymbol{j}^{-1}(N_3^{\circ\bullet}) = N_2^{\circ\bullet}$. We write this last equality as $N_3^{\circ\bullet} \cap F_2^{\circ\bullet} = N_2^{\circ\bullet}$.*

Proof. Set $F_1 = F_2$ in Theorem 2.4. ∎

Corollary 2.4.2. *Let $F_1 \subseteq F_2$ be Banach spaces with duals F'_k, $k = 1, 2$, and suppose the canonical injection $i : F_1 \to F_2$ is continuous. Let F_1 be weakly closed in F_2, and N_1, N_2 be reflexive subspaces of F_1 and F_2 respectively such that $N_2 \cap F_1 = N_1$. Then $i^*(N_2^\circ)$ is strongly dense in N_1°.*

Proof. Let F''_1 and F''_2 denote the double duals of F_1 and F_2 respectively, and consider F_1 and F_2 as subspaces of F''_2. F''_1 is also considered as a subspace of F''_2 via the extension $\boldsymbol{i}$ of i to F''_1. Let $x \in F_2 \cap F''_1$. If $x \notin F_1$, there exists $\varphi \in F'_2$ such that $F_1 \subseteq \ker \varphi$ and $\varphi(x) = 1$. Thus $\varphi|_{F_1} \equiv 0$, and since F_1 is weakly dense in F''_1, $\varphi|_{F''} \equiv 0$. This implies $\varphi(x) = 0$, a contradiction. Thus, $F_2 \cap F''_1 \subset F_1$. This fact combined with the reflexivity of N_1 and N_2 yields

$$N_2^{\circ\bullet} \cap F''_1 = N_2 \cap F''_1 = N_2 \cap F_1 = N_1 = N_1^{\circ\bullet}.$$

Here, we use the fact that $N_k^{\circ\bullet} = \boldsymbol{i}(N''_k)$ in F''_k, $k = 1, 2$. ∎

Theorem 2.5. *Let $E = \bigcup_{n=1}^{\infty} E_n$ be a strict inductive limit of Banach spaces. A subspace M of E is very compressive if for each n, $M_n = M \cap E_n$ is closed in E_n, and either M_n° is reflexive or M_n is reflexive.*

Proof. It follows immediately from the hypotheses that there exists a defining subsequence E_{n_k} such that either M_{n_k} is reflexive for all k or $M_{n_k}^\circ$ is reflexive for all k. The result now follows from Corollary 2.4.2, Theorem 2.1, and Theorem 2.2. ∎

In [5], we have defined the notion of a nearly closed subspace of a convergence vector space. For the purposes of this paper, we give the following simplified version.

Definition 2.2. Let $E = \bigcup_{n=1}^{\infty} E_n$ be a *strict* inductive limit of Banach spaces. A subspace M of E is said to be *nearly closed* if for each n there exists N such that $(\bigcup_{k=1}^{\infty} \overline{M}_k^k) \cap E_n \subset \overline{M}_N^N \cap E_n$. Here, $\overline{M}_k^k$ denotes the closure of M_k in E_k.

Theorem 2.6. *Let $E = \bigcup_{n=1}^{\infty} E_n$ be a strict inductive limit of Banach spaces. A subspace M of E has the H.B.P. if*

(1) *M is nearly closed, and*
(2) *for each n, either $\overline{M}_n^n$ is reflexive, or M_n° is reflexive.*

Proof. In [5] we showed that if M is nearly closed, then every sequentially continuous linear functional has a sequentially continuous linear extension to $\overline{M} = \bigcup_{k=1}^{\infty} \overline{M}_k^k$, and $\overline{M} \cap E_n$ is closed in E_n for all n. As a consequence of (2), we may assume (by passing to a subsequence if necessary) that either $\overline{M}_n^n$ is reflexive for all n, or M_n° is reflexive for all n. Suppose first that all $\overline{M}_n^n$ are reflexive. Since $(\overline{M})_n = \overline{M} \cap E_n = \overline{M}_N^N \cap E_n$ for some $N \geqq n$, and since E_N induces the original topology on E_n, it follows that $(\overline{M})_n$ is a closed subspace of the reflexive space $\overline{M}_N^N$. Thus $(\overline{M})_n$ is reflexive for all n, and the result now follows from Theorem 2.5.

Now suppose that all M_n° are reflexive. Since $M_n \subset (\overline{M})_n$, we have $M_n^\circ \supset (\overline{M})_n^\circ$. Thus, $(\overline{M})_n^\circ$ is a closed subspace of a reflexive space, and hence, is reflexive. Again the result follows via Theorem 2.5. ∎

3. REMARKS AND EXAMPLES

We first observe that if $E = \bigcup_{n=1}^{\infty} E_n$ is an inductive limit of reflexive Banach spaces, and M is a subspace of E such that for all n $M_n = M \cap E_n$ is closed in each E_n, then M is very compressive as a result of Theorems 2.1 and 2.2, and hence by Theorem 1.2 has the H.B.P. This is the theorem of Foias and Marinescu [1]. We remark that the fact that M_n is closed in E_n is not explicitly stated in [1] but is used in the proof given there.

Theorem 2.6 shows readily that if $E = \bigcup_{n=1}^{\infty} E_n$ is a strict inductive limit of Banach spaces and M is a nearly closed subspace of E such that for each n, $\overline{M}_n^n$ is of finite codimension in E_n, then M has the H.B.P. Hence, in particular if M_n itself is closed and of finite codimension in E_n, M has the H.B.P. Moreover, if M is of *countable* codimension in E and nearly closed, then it must have the H.B.P., since a simple application of the Baire Category Theorem then shows that $\overline{M}_n^n$ is of *finite* codimension in E_n for each n. In addition to these more or less immediate examples, since reflexive subspaces of non-reflexive Banach spaces are not unusual, Theorem 2.6 should apply to many situations where previous results would not.

The examples indicated above are based on criteria for a subspace M to very compressive. One might wonder if every compressive subspace is very compressive (the converse being obvious). Indeed this is *not* the case, as we now show.

Let E be the inductive limit of a sequence E_n of normed spaces and M a subspace of E such that for all n $M_n = M \cap E_n$ is closed in E_n. For each n, let v_n be the canonical homomorphism of E_n onto E_n/M_n. The map $j_n : E_n/M_n \to E_{n+1}/M_{n+1}$ defined by $j_n([x]) = [i_n(x)]$ is readily verified to be continuous and injective. With this notation, we have the following commutative diagram for each n.

$$
\begin{array}{ccccccc}
E'_n & & & \xleftarrow{\quad i_n^* \quad} & & & E'_{n+1} \\
\uparrow & & & & & & \uparrow \\
 & & E_n & \xrightarrow{\quad i_n \quad} & E_{n+1} & & \\
v_n^* & & \downarrow v_n & & \downarrow v_{n+1} & & v_{n+1}^* \\
 & & E_n/M_n & \xrightarrow[\quad j_n \quad]{} & E_{n+1}/M_{n+1} & & \\
 & & & & & & \\
(E_n/M_n)' & & & \xleftarrow[\quad j_n^* \quad]{} & & & (E_{n+1}/M_{n+1})'
\end{array}
$$

Since v_n^* is a homeomorphism of $(E_n/M_n)'$ onto $M_n^\circ \subseteq E'_n$, each under the norm topologies, it is evident that $i_n^*(M_{n+1}^\circ)$ is norm dense in M_n° if and only if $j_n^*((E_{n+1}/M_{n+1})')$ is norm dense in $(E_n/M_n)'$.

With this notation, we can state:

Proposition 3.1. *If M is a sequentially closed subspace of E, then the trivial subspace, 0, is a very compressive subspace of E/M if and only if M is a very compressive subspace of E.*

Proof. The proof is an immediate consequence of the above observation and Definition 2.1. ∎

Pták [7] has given an example of a sequentially closed subspace, M, of an inductive limit of Banach spaces which is not compressive, and hence is not very compressive. For his example, by the above proposition, we see that the subspace $\{0\}$ is *not* very compressive in E/M. However, by Theorem 1.1, it is compressive. Thus we have an example of a sequentially closed subspace which is compressive, but not very compressive.

As an example where the criterion of very compressive does not seem to be adequate, while that of compressive does, we give a new proof of the known result (Hogbe-Nlend [2]):

Let E be an inductive limit of a sequence E_n of normed spaces, and $i_n : E_n \to E_{n+1}$ be the natural injections. If for each n, the image $i_n(B_n)$ of the unit ball is relatively $\sigma(E_{n+1}, E'_{n+1})$-compact in E_{n+1}, then every sequentially closed subspace of E has the H.B.P.

Proof. Since $M_{n+2} = M \cap E_{n+2}$ is closed in E_{n+2}, it follows that $i^*_{n+1}(M^\circ_{n+2})$ is $\sigma(E'_{n+1}, E_{n+1})$-dense in M°_{n+1}. Thus, the $\tau(E'_{n+1}, E_{n+1})$-closure of $i^*_{n+1}(M^\circ_{n+2})$ contains M°_{n+1}. But $i_n(B_n)$ being relatively $\sigma(E_{n+1}, E'_{n+1})$ compact implies i^*_n is $\tau(E'_{n+1}, E_{n+1})$, $\beta(E'_n, E_n)$-continuous. Thus, with τ and β referring to these topologies, $i^*_n(M^\circ_{n+1}) \subset i^*_n\left[\overline{i^*_{n+1}(M^\circ_{n+2})}^{\tau}\right] \subset \overline{i^*_n \circ i^*_{n+1}(M^\circ_{n+2})}^{\beta}$, and the result follows from Theorem 1.2.

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