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Czechoslovak Mathematical Journal, Vol. 25 (1975), No. 1, 110–133

Persistent URL: http://dml.cz/dmlcz/101298

Czechoslovak Mathematical Journal, 25 (100) 1975, Praha

ON A BOUNDARY VALUE PROBLEM FOR THE HEAT EQUATION

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(Received October 9, 1973)

1. THE FREDHOLM RADIUS OF ONE OPERATOR

The acquaintance with the paper „On a heat potential" (see [17]) is necessary for the understanding of the present paper. We will use most of the definitions and assertions from that article, especially those concerning properties of the parabolic variation V_K (see Def. 1.1 in [17]) and of the special type of the heat potential Tf (see Def. 2.1 in [17]).

Let $\langle a, b\rangle$ be a compact interval in $R^2(a < b)$, φ a continuous function of bounded variation on $\langle a, b\rangle$,

$$K = \{[\varphi(t), t];\ t \in \langle a, b\rangle\}\ .$$

Throughout this section we shall suppose

$$\sup\{V_K(\varphi(t), t);\ t \in \langle a, b\rangle\} < \infty\ . \tag{1.1}$$

Let $\mathscr{C}_0(\langle a, b\rangle)$ be the space of all continuous functions f on $\langle a, b\rangle$ such that $f(a) = 0$ endowed with the supremum norm (this space may be considered a space $\mathscr{C}_Q(\langle a, b\rangle)$ – see the definition at the beginning of Section 2 in [17] – where Q is a function on $\langle a, b\rangle$ for which $Q(a) = 0$ and $Q(t) = 1$ for $t \in (a, b\rangle$).

Let us define operators $\tilde{T}_+$ and $\tilde{T}_-$ on $\mathscr{C}_0(\langle a, b\rangle)$ by the following equalities:

$$\tilde{T}_+ f(t) = \lim_{\substack{[x',t'] \to [\varphi(t),t] \\ t' \in \langle a,b\rangle, x' > \varphi(t)}} Tf(x', t')\ , \tag{1.2}$$

$$\tilde{T}_- f(t) = \lim_{\substack{[x',t'] \to [\varphi(t),t] \\ t' \in \langle a,b\rangle, x' < \varphi(t)}} Tf(x', t') \tag{1.3}$$

for $f \in \mathscr{C}_0(\langle a, b\rangle)$, $t \in \langle a, b\rangle$. These limits exist according to Theorem 2.1 in [17]. It is easily seen that the functions $\tilde{T}_+ f$, $\tilde{T}_- f$ belong to $\mathscr{C}_0(\langle a, b\rangle)$ and hence one may consider $\tilde{T}_+$, $\tilde{T}_-$ linear operators on $\mathscr{C}_0(\langle a, b\rangle)$ which map $\mathscr{C}_0(\langle a, b\rangle)$ into $\mathscr{C}_0(\langle a, b\rangle)$.

We can slightly modify Remark 2.4 from [17] to get the following two equalities which hold for each $f \in \mathscr{C}_0(\langle a, b\rangle)$ and each $t \in (a, b\rangle$:

$$(1.4) \qquad \tilde{T}_+ f(t) = Tf(\varphi(t), t) + f(t)\left(2 - \frac{2}{\sqrt{\pi}}\, G(\alpha_{\varphi(t),t}(t))\right),$$

$$(1.5) \qquad \tilde{T}_- f(t) = Tf(\varphi(t), t) - \frac{2}{\sqrt{\pi}} f(t)\, G(\alpha_{\varphi(t),t}(t)) .$$

(For $\alpha_{\varphi(t),t}$ and G see respectively (1.1) and (1.15) in [17].) (It holds, of course, $\tilde{T}_+ f(a) = \tilde{T}_- f(a) = 0$ for any $f \in \mathscr{C}_0(\langle a, b\rangle)$.)

Let us further put

$$(1.6) \qquad \bar{T}f(t) = Tf(\varphi(t), t)$$

for $f \in \mathscr{C}_0(\langle a, b\rangle)$, $t \in \langle a, b\rangle$. For $f \in \mathscr{C}_0(\langle a, b\rangle)$ the function $\bar{T}f$ need not be continuous on $\langle a, b\rangle$. We look for a condition of continuity of $\bar{T}f$ for each $f \in \mathscr{C}_0(\langle a, b\rangle)$.

Lemma 1.1. *The set of all $t \in \langle a, b\rangle$ for which $\alpha_{\varphi(t),t}(t) = 0$ is dense in $\langle a, b\rangle$.*

Proof. We shall show that even $\alpha_{\varphi(t),t}(t) = 0$ for almost all $t \in (a, b\rangle$. It is well known that a function of bounded variation has finite derivative almost everywhere. Consider $t \in (a, b)$ and suppose $\varphi'(t) \in R^1$. Then

$$\alpha_{\varphi(t),t}(t) = \lim_{\tau \to t-} \frac{\varphi(t) - \varphi(\tau)}{2\sqrt{(t-\tau)}} = \lim_{\tau \to t-} \frac{\varphi(t) - \varphi(\tau)}{t - \tau} \tfrac{1}{2}\sqrt{(t-\tau)} = \varphi'(t) \, . \, 0 = 0 \, .$$

The assertion is proved.

Lemma 1.1, the equality (1.4) (or (1.5)) and the fact that $\tilde{T}_+$ (or $\tilde{T}_-$) maps $\mathscr{C}_0(\langle a, b\rangle)$ into $\mathscr{C}_0(\langle a, b\rangle)$ imply that $\bar{T}$ maps $\mathscr{C}_0(\langle a, b\rangle)$ into $\mathscr{C}_0(\langle a, b\rangle)$ if and only if $\alpha_{\varphi(t),t}(t) = 0$ for each $t \in (a, b\rangle$. If this condition is fulfilled and if I stands for the identity operator on $\mathscr{C}_0(\langle a, b\rangle)$ then

$$\tilde{T}_+ = \bar{T} + I \, , \quad \tilde{T}_- = \bar{T} - I$$

as follows from (1.4) and (1.5).

As we shall not suppose $\alpha_{\varphi(t),t}(t) = 0$ for each $t \in (a, b\rangle$ let us define an operator $\bar{T}_0$ on $\mathscr{C}_0(\langle a, b\rangle)$ by

$$(1.7) \qquad \bar{T}_0 = \tilde{T}_+ - I \, .$$

Considering (1.4) and (1.5) one can easily verify that

$$(1.8) \qquad \bar{T}_0 = \tilde{T}_- + I$$

and

$$(1.9) \qquad \bar{T}_0 f(t) = \bar{T} f(t) + f(t) \left(1 - \frac{2}{\sqrt{\pi}}\, G(\alpha_{\varphi(t),t}(t))\right)$$

for each $f \in \mathscr{C}_0(\langle a, b \rangle)$ and each $t \in (a, b\rangle$.

In connection with the boundary value problem for the heat equation it will be useful to know the Fredholm radius of the operator $\bar{T}_0$. Let us introduce the notation which we shall need in the sequel. We shall proceed similarly to [8].

If P is an operator which maps a Banach space B into B we denote the norm of P by $\|P\|$ or $\|P\|_B$; i.e., we put

$$\|P\| = \|P\|_B = \sup \{\|Pf\|;\ f \in B,\ \|f\| \leqq 1\},$$

where $\|\ldots\|$ stands for the norm on B. (In a Banach space we use actually the same symbol $\|\ldots\|$ for the norm on the space, for the norm of linear functionals on the space and for linear mappings from the space into itself; but certainly no confusion can arise.)

A linear operator $A : B \to B$ is said to be a compact operator on B if for each bounded set $M \subset B$ the set $A(M)$ is relatively compact in B.

Let $P : B \to B$ be a continuous linear operator. Put

$$(1.10) \qquad \omega P = \inf_A \|P - A\|,$$

where A runs over all compact linear operators on the space B. The value $(\omega P)^{-1}$ is called the Fredholm radius of the operator P (we take $0^{-1} = +\infty$).

The operator $\bar{T}_0$ is linear (and it follows from the condition (1.1) that it is also continuous) and maps $\mathscr{C}_0(\langle a, b \rangle)$ into $\mathscr{C}_0(\langle a, b \rangle)$. Similarly as in [14] (where the space $\mathscr{C}(\langle a, b \rangle)$ is dealt with) or [11] we can find out that each continuous linear operator P which maps $\mathscr{C}_0(\langle a, b \rangle)$ to $\mathscr{C}_0(\langle a, b \rangle)$ is of the form

$$(Pf)(t) = \int_a^b f(\tau)\, \mathrm{d}\lambda_t^P(\tau)$$

($f \in \mathscr{C}_0(\langle a, b \rangle)$, $t \in (a, b\rangle$) where for each $t \in (a, b\rangle$ the function λ_t^P is a function of bounded variation on $\langle a, b \rangle$ and

$$(1.11) \qquad \|P\| = \sup_{t \in (a,b\rangle} \operatorname{var} [\lambda_t^P; \langle a, b \rangle].$$

For each $t \in (a, b\rangle$ we define a function $\bar{\lambda}_t$ by

$$\bar{\lambda}_t(\tau) = \begin{cases} \dfrac{2}{\sqrt{\pi}}\, G(\alpha_{\varphi(t),t}(\tau)), & \tau \in \langle a, t) \\[2ex] \dfrac{2}{\sqrt{\pi}}\, G(\alpha_{\varphi(t),t}(t)), & \tau \in \langle t, b \rangle. \end{cases}$$

Then

$$(1.12) \qquad \bar{T}f(t) = Tf(\varphi(t), t) = \int_a^b f(\tau)\, \mathrm{d}\bar{\lambda}_t(\tau)$$

for each $f \in \mathscr{C}_0(\langle a, b\rangle)$, $t \in (a, b\rangle$ as for $t \in (a, b\rangle$ it holds

$$Tf(\varphi(t), t) = \frac{2}{\sqrt{\pi}} \int_a^t f(\tau) \exp\left(-\alpha^2_{\varphi(t),t}(\tau)\right) \mathrm{d}\alpha_{\varphi(t),t}(\tau)$$

and if $t_1 \in (a, b)$ then

$$\frac{2}{\sqrt{\pi}} \int_a^{\min\{t_1, t\}} \exp\left(-\alpha^2_{\varphi(t),t}(\tau)\right) \mathrm{d}\alpha_{\varphi(t),t}(\tau) = \int_a^{t_1} \mathrm{d}\bar{\lambda}_t(\tau)$$

according to Lemma 0.2 from [17].

Further we define for each $t \in (a, b\rangle$ a function λ_t^+ by

$$\lambda_t^+(\tau) = \begin{cases} \dfrac{2}{\sqrt{\pi}} G(\alpha_{\varphi(t),t}(\tau)), & \tau \in \langle a, t) \\ 2 & \tau \in \langle t, b\rangle . \end{cases}$$

From (1.4) we obtain

$$(1.13) \qquad \tilde{T}_+ f(t) = \int_a^b f(\tau)\, \mathrm{d}\lambda_t^+(\tau)$$

which holds for each $f \in \mathscr{C}_0(\langle a, b\rangle)$, $t \in (a, b\rangle$.

It is possible to define analogous functions λ_t^- for the operator $\tilde{T}_-$.

It holds

$$\operatorname{var}[\bar{\lambda}_t; \langle a, b\rangle] = \frac{2}{\sqrt{\pi}} V_K(\varphi(t), t), \quad (t \in (a, b\rangle),$$

$$\operatorname{var}[\lambda_t^+; \langle a, b\rangle] = \frac{2}{\sqrt{\pi}} V_K(\varphi(t), t) + 2 - \frac{2}{\sqrt{\pi}} G(\alpha_{\varphi(t),t}(t))$$

(it is now seen from the last equality and from the boundedness of the function G that the operator $\tilde{T}_+$ is continuous on $\mathscr{C}_0(\langle a, b\rangle)$).

Finally, we define functions $\bar{\lambda}_t^0$ by

$$\bar{\lambda}_t^0(\tau) = \begin{cases} \dfrac{2}{\sqrt{\pi}} G(\alpha_{\varphi(t),t}(\tau)), & \tau \in \langle a, t) \\ 1, & \tau \in \langle t, b\rangle \end{cases}$$

$(t \in (a, b\rangle)$. Then

$$(1.14) \qquad \bar{T}_0 f(t) = \int_a^b f(\tau)\, \mathrm{d}\bar{\lambda}_t^0(\tau)$$

for $f \in \mathscr{C}_0(\langle a, b\rangle)$, $t \in (a, b\rangle$ and

$$\operatorname{var}[\bar{\lambda}_t^0; \langle a, b\rangle] = \frac{2}{\sqrt{\pi}} V_K(\varphi(t), t) + \left|1 - \frac{2}{\sqrt{\pi}} G(\alpha_{\varphi(t),t}(t))\right| .$$

Hence

$$\|\bar{T}_0\| = \sup_{t \in (a,b\rangle} \left(\frac{2}{\sqrt{\pi}} V_K(\varphi(t), t) + \left|1 - \frac{2}{\sqrt{\pi}} G(\alpha_{\varphi(t),t}(t))\right|\right) . \tag{1.15}$$

Let ψ be a continuous function on $\langle a, b\rangle$. Given $r \geqq 0$, we define an operator ${}^rH^\psi = {}^rH_\varphi^\psi$ on the space $\mathscr{C}_0(\langle a, b\rangle)$ by

(1.16)

$${}^rH_\varphi^\psi f(t) = \begin{cases} 0 , & t \in \langle a, b\rangle , \quad t \leqq a + r \\ \dfrac{2}{\sqrt{\pi}} \displaystyle\int_a^{t-r} f(\tau) \exp\left(-\alpha_{\psi(t),t}^2(\tau)\right) \mathrm{d}\alpha_{\psi(t),t}(\tau) , & t \in \langle a, b\rangle , \quad t > a + r . \end{cases}$$

Further we put

$$H^\psi = H_\varphi^\psi = {}^0H_\varphi^\psi .$$

Lemma 1.2. *Let $r > 0$, $\psi \in \mathscr{C}(\langle a, b\rangle)$. Then ${}^rH_\varphi^\psi$ is a compact operator on $\mathscr{C}_0(\langle a, b\rangle)$.*

Remark 1.1. By (1.16) one may define an operator ${}^rH^\psi$ on the whole $\mathscr{C}(\langle a, b\rangle)$ and this operator is a compact operator on $\mathscr{C}(\langle a, b\rangle)$. But this is not necessary here.

Proof of Lemma 1.2. If $r \geqq b - a$ then the assertion is evident as ${}^rH_\varphi^\psi$ is the zero operator in this case.

Suppose $0 < r < b - a$. If we denote

$$\mathscr{B} = \{f; f \in \mathscr{C}_0(\langle a, b\rangle), \|f\| \leqq 1\}$$

then it suffices to show that ${}^rH_\varphi^\psi(\mathscr{B})$ is a relatively compact set in $\mathscr{C}_0(\langle a, b\rangle)$. Since $\mathscr{C}_0(\langle a, b\rangle)$ is a closed subspace of $\mathscr{C}(\langle a, b\rangle)$ it is sufficient to verify that ${}^rH_\varphi^\psi(\mathscr{B})$ is a set of equicontinuous and uniformly bounded functions.

First let us show that the functions belonging to ${}^rH_\varphi^\psi(\mathscr{B})$ are uniformly bounded. It holds ${}^rH_\varphi^\psi f(t) = 0$ for each $f \in \mathscr{C}_0(\langle a, b\rangle)$ and each $t \in \langle a, a + r\rangle$. If $t \in (a + r, b\rangle$, $f \in \mathscr{B}$ then

$$|{}^rH_\varphi^\psi f(t)| = \left|\frac{2}{\sqrt{\pi}} \int_a^{t-r} f(\tau) \exp\left(-\alpha_{\psi(t),t}^2(\tau)\right) \mathrm{d}\alpha_{\psi(t),t}(\tau)\right| \leqq$$

$$\leqq \frac{2}{\sqrt{\pi}} \|f\| \operatorname{var}\left[\frac{\psi(t) - \varphi(\tau)}{2\sqrt{(t - \tau)}} ; \langle a, t - r\rangle\right] \leqq$$

$$\leqq \frac{1}{\sqrt{(\pi r)}} \left(\operatorname{var}[\varphi; \langle a, b\rangle] + \sup_{\tau \in \langle a,b\rangle, \tau' \in \langle a,b\rangle} |\psi(\tau') - \varphi(\tau)|\right) < \infty .$$

The last term does not depend on $t \in (a + r, b\rangle$. Hence indeed, ${}^rH^{\psi}_{\varphi}(\mathscr{B})$ is a set of uniformly bounded functions.

Now it suffices to show that for each $t \in \langle a, b\rangle$

$$\lim_{\substack{s\to t\\ s\in\langle a,b\rangle}} \sup_{f\in\mathscr{B}} \left|{}^rH^{\psi} f(t) - {}^rH^{\psi} f(s)\right| = 0\,. \tag{1.17}$$

In virtue of the fact that ${}^rH^{\psi} f(t) = 0$ for each $t \in \langle a, a + r\rangle$ we may consider only $t, s \in \langle a + r, b\rangle$. Let t, s be such points and suppose, for instance, $t > s$. Then

$$\begin{aligned}
&\sup_{f\in\mathscr{B}} \left|{}^rH^{\psi} f(t) - {}^rH^{\psi} f(s)\right| = \\
&= \frac{2}{\sqrt{\pi}} \sup_{f\in\mathscr{B}} \left| \int_a^{t-r} f(\tau) \exp\left(-\alpha^2_{\psi(t),t}(\tau)\right) \mathrm{d}\alpha_{\psi(t),t}(\tau) - \right. \\
&\quad \left. - \int_a^{s-r} f(\tau) \exp\left(-\alpha^2_{\psi(s),s}(\tau)\right) \mathrm{d}\alpha_{\psi(s),s}(\tau) \right| \leqq \\
&\leqq \frac{2}{\sqrt{\pi}} \sup_{f\in\mathscr{B}} \left| \int_a^{s-r} f(\tau) \exp\left(-\alpha^2_{\psi(t),t}(\tau)\right) \mathrm{d}\alpha_{\psi(t),t}(\tau) - \right. \\
&\quad \left. - \int_a^{s-r} f(\tau) \exp\left(-\alpha^2_{\psi(s),s}(\tau)\right) \mathrm{d}\alpha_{\psi(s),s}(\tau) \right| + \\
&\quad + \frac{2}{\sqrt{\pi}} \sup_{f\in\mathscr{B}} \int_{s-r}^{t-r} f(\tau) \exp\left(-\alpha^2_{\psi(t),t}(\tau)\right) \mathrm{d}\alpha_{\psi(t),t}(\tau)\,.
\end{aligned} \tag{1.18}$$

We have

$$\sup_{f\in\mathscr{B}} \left| \int_{s-r}^{t-r} f(\tau) \exp\left(-\alpha^2_{\psi(t),t}(\tau)\right) \mathrm{d}\alpha_{\psi(t),t}(\tau) \right| \leqq \operatorname{var}\left[\alpha_{\psi(t),t};\ \langle s - r, t - r\rangle\right]. \tag{1.19}$$

Since $\operatorname{var}\left[\alpha_{\psi(t),t};\ \langle s - r, t - r\rangle\right] < \infty$ and since the function $\alpha_{\psi(t),t}$ is continuous at the point $t - r$ one sees that

$$\lim_{s\to t-} \operatorname{var}\left[\alpha_{\psi(t),t};\ \langle s - r, t - r\rangle\right] = 0\,. \tag{1.20}$$

Further we have

$$\begin{aligned}
&\sup_{f\in\mathscr{B}} \left| \int_a^{s-r} f(\tau) \exp\left(-\alpha^2_{\psi(t),t}(\tau)\right) \mathrm{d}\alpha_{\psi(t),t}(\tau) - \int_a^{s-r} f(\tau) \exp\left(-\alpha^2_{\psi(s),s}(\tau)\right) \mathrm{d}\alpha_{\psi(s),s}(\tau) \right| \leqq \\
&\quad \leqq \sup_{f\in\mathscr{B}} \left| \int_a^{s-r} f(\tau) \left(\exp\left(-\alpha^2_{\psi(t),t}(\tau)\right) - \exp\left(-\alpha^2_{\psi(s)s}(\tau)\right)\right) \mathrm{d}\alpha_{\psi(t),t}(\tau) \right| + \\
&\quad + \sup_{f\in\mathscr{B}} \left| \int_a^{s-r} f(\tau) \exp\left(-\alpha^2_{\psi(s),s}(\tau)\right) \mathrm{d}\left(\alpha_{\psi(t),t}(\tau) - \alpha_{\psi(s),s}(\tau)\right) \right| \leqq
\end{aligned} \tag{1.21}$$

$$\leqq \int_a^{s-r} |(\exp(-\alpha^2_{\psi(t),t}(\tau)) - \exp(-\alpha^2_{\psi(s),s}(\tau)))| \mathrm{d} \operatorname{var} \alpha_{\psi(t),t}(\tau) +$$

$$+ \operatorname{var} [\alpha_{\psi(t),t} - \alpha_{\psi(s),s};\ \langle a, s - r \rangle] = I(t, s) + II(t, s) .$$

It follows from the Lebesgue theorem (for $\operatorname{var}[\alpha_{\psi(t),t};\ \langle a, t - r\rangle] < \infty$) that

$$\lim_{s \to t-} I(t, s) = 0 . \tag{1.22}$$

We can easily verify that

$$\alpha_{\psi(t),t}(\tau) - \alpha_{\psi(s),s}(\tau) = \frac{\psi(t) - \varphi(\tau)}{2\sqrt{(t-\tau)}} - \frac{\psi(s) - \varphi(\tau)}{2\sqrt{(s-\tau)}} =$$

$$= (t - s) \frac{\varphi(\tau) - \psi(t)}{2\sqrt{[(t-\tau)(s-\tau)]}\,(\sqrt{(t-\tau)} + \sqrt{(s-\tau)})} + \frac{\psi(t) - \psi(s)}{2\sqrt{(t-\tau)}} =$$

$$= L^t_s(\tau) + M^t_s(\tau) .$$

Hence

$$II(t, s) \leqq \operatorname{var}[L^t_s;\ \langle a, s - r\rangle] + \operatorname{var}[M^t_s;\ \langle a, s - r\rangle]$$

and at the same time we find

$$\operatorname{var}[L^t_s;\ \langle a, s - r\rangle] \leqq (t - s)\, \tfrac{1}{4} r^{-3/2} (\operatorname{var}[\varphi;\ \langle a, b\rangle] + \sup_{\substack{\tau \in \langle a,b\rangle \\ \tau' \in \langle a,b\rangle}} |\varphi(\tau) - \psi(\tau')|) ,$$

$$\operatorname{var}[M^t_s;\ \langle a, s - r\rangle] \leqq |\psi(t) - \psi(s)| \frac{1}{2\sqrt{r}} .$$

Therefore

$$\lim_{s \to t-} II(t, s) = 0 \tag{1.23}$$

as the function ψ is continuous by the assumption. It follows from (1.23), (1.22), (1.20), (1.21), (1.19) and (1.18) that

$$\lim_{\substack{s \to t- \\ s \in \langle a,b\rangle}} \sup_{f \in \mathscr{B}} |{}^r H^\psi f(t) - {}^r H^\psi f(s)| = 0 .$$

Analogously in the case $s \to t+$. So we conclude that (1.17) is valid which completes the proof.

Remark 1.2. Let us note that throughout the proof just finished we did not use any assumption concerning the parabolic variation of the curve φ. Consequently, Lemma 1.2 is true even if the assumption (1.1) is omitted (it must be supposed only that φ is a continuous function of bounded variation on $\langle a, b\rangle$).

Corollary 1.1. *Let ψ be a continuous function on $\langle a, b\rangle$ and suppose that $\psi(t) \neq \varphi(t)$ for each $t \in \langle a, b\rangle$. Then H_φ^ψ is a compact operator on $\mathscr{C}_0(\langle a, b\rangle)$.*

Proof. Since for any $r > 0$ the operator ${}^rH_\varphi^\psi$ is compact and it is well known that a limit of a sequence of compact operators is a compact operator, too (the limit being considered in the norm topology in the space of linear continuous operators), it suffices to show that

$$\lim_{r\to 0+} \left\|H_\varphi^\psi - {}^rH_\varphi^\psi\right\| = 0 . \tag{1.24}$$

It holds for $f \in \mathscr{C}_0(\langle a, b\rangle)$, $t \in \langle a, b\rangle$, $r \in (0, b - a)$ that

$$H_\varphi^\psi f(t) - {}^rH_\varphi^\psi f(t) = \frac{2}{\sqrt{\pi}} \int_{\max\{a,t-r\}}^t f(\tau) \exp\left(-\alpha^2_{\psi(t),t}(\tau)\right) \mathrm{d}\alpha_{\psi(t),t}(\tau)$$

and it follows from (1.11) that

$$\left\|H_\varphi^\psi - {}^rH_\varphi^\psi\right\| = \frac{2}{\sqrt{\pi}} \sup_{t\in(a,b\rangle} \int_{\max\{a,t-r\}}^t \exp\left(-\alpha^2_{\psi(t),t}(\tau)\right) \mathrm{d}\operatorname{var} \alpha_{\psi(t),t}(\tau) . \tag{1.25}$$

Since the functions φ, ψ are continuous and $\varphi(t) \neq \psi(t)$ for each $t \in \langle a, b\rangle$ there are $r_0 \in (0, b - a)$, $c > 0$ such that

$$|\varphi(\tau) - \psi(\tau')| \geqq c$$

for any $\tau, \tau' \in \langle a, b\rangle$ for which $|\tau - \tau'| < r_0$. Then for $t \in (a, b\rangle$, $\tau \in \langle a, b\rangle \cap (t - r_0, t)$ we have

$$|\alpha_{\psi(t),t}(\tau)| = \frac{|\psi(t) - \varphi(\tau)|}{2\sqrt{(t - \tau)}} \geqq \frac{c}{2\sqrt{(t - \tau)}}$$

and hence

(1.26)

$$\int_{\max\{a,t-r\}}^t \exp\left(-\alpha^2_{\psi(t),t}(\tau)\right) \mathrm{d}\operatorname{var} \alpha_{\psi(t),t}(\tau) \leqq \int_{\max\{a,t-r\}}^t \exp\left(-\frac{c^2}{4(t-\tau)}\right) \mathrm{d}\operatorname{var} \alpha_{\psi(t),t}(\tau)$$

for each $r \in (0, r_0)$ (and $t \in (a, b\rangle$).

It is easily seen that there is $h \in R^1$ such that for each $t \in (a, b\rangle$, n positive integer

$$\operatorname{var}\left[\alpha_{\psi(t),t};\left\langle a, \max\left\{a, t - \frac{1}{n}\right\}\right\rangle\right] \leqq h\sqrt{n}$$

(it is possible to put, for instance,

$$h = \operatorname{var}\left[\varphi; \langle a, b\rangle\right] + \sup_{\tau\in\langle a,b\rangle,\tau'\in\langle a,b\rangle} |\varphi(\tau) - \psi(\tau')|) .$$

For $r \in (0, r)$ let $n(r)$ stand for the least positive integer n such that $1/n < r$. Let $t \in (a, b\rangle$, $r \in (0, r_0)$, $r \leqq t - a$. Then we obtain from the preceding

$$\int_{t-r}^{t} \exp\left(-\frac{c^2}{4(t-\tau)}\right) \mathrm{d}\, \mathrm{var}\, \alpha_{\psi(t),t}(\tau) =$$
$$= \int_{t-r}^{t-1/n(r)} \exp\left(-\frac{c^2}{4(t-\tau)}\right) \mathrm{d}\, \mathrm{var}\, \alpha_{\psi(t),t}(\tau) +$$
$$+ \sum_{n=n(r)}^{\infty} \int_{t-1/n}^{t-1/(n+1)} \exp\left(-\frac{c^2}{4(t-\tau)}\right) \mathrm{d}\, \mathrm{var}\, \alpha_{\psi(t),t}(\tau) \leqq$$
$$\leqq \sum_{n=n(r)-1} h\sqrt{(n+1)}\, e^{-nc^2/4}.$$

Hence and from (1.26), (1.25) we get immediately that for $r \in (0, r_0)$

$$\|H_\varphi^\psi - {}^r H_\varphi^\psi\| \leqq \frac{2}{\sqrt{\pi}} \sum_{n=n(r)-1} h\sqrt{(n+1)}\, e^{-nc^2/4}.$$

Since $n(r) \to \infty$ as $r \to 0+$, the equality (1.24) follows which completes the proof.

In the following we suppose again that the condition (1.1) is fulfilled. Then the value $\alpha_{\varphi(t),t}(t)$ is defined for each $t \in (a, b\rangle$ and we are justified to define a function α_K on $(a, b\rangle$ by

$$\alpha_K(t) = \left|1 - \frac{2}{\sqrt{\pi}} G(\alpha_{\varphi(t),t}(t))\right|. \tag{1.27}$$

Let us prove the following assertion.

Lemma 1.3. *For any given $r > 0$ the function*

$$\frac{2}{\sqrt{\pi}} V_K(r; \varphi(t), t) + \alpha_K(t), \quad (t \in (a, b\rangle)$$

is lower-semicontinuous on $(a, b\rangle$. If we put

$$\overline{T}_r f = {}^r H_\varphi^\psi f$$

for each $f \in \mathscr{C}_0(\langle a, b\rangle)$ and if $\mathscr{B}$ stands for the unit ball in $\mathscr{C}_0(\langle a, b\rangle)$ then

$$\frac{2}{\sqrt{\pi}} V_K(r; \varphi(t), t) + \alpha_K(t) = \sup_{f \in \mathscr{B}} (\overline{T}_0 f(t) - \overline{T}_r f(t)), \quad (t \in (a, b\rangle).$$

Proof. Since the least upper bound of a family of continuous functions is a lower-semicontinuous function and for $f \in \mathscr{C}_0(\langle a, b\rangle)$ the function $(\overline{T}_0 f - \overline{T}_r f)$ belongs to $\mathscr{C}_0(\langle a, b\rangle)$ it suffices to prove that (1.34) is valid.

If $t \in (a, b\rangle$, $t > a + r$ (we consider this case only if $r < b - a$), $f \in \mathscr{C}_0(\langle a, b\rangle)$ we have (see (1.14), (1.16))

$$\bar{T}_0 f(t) - \bar{T}_r f(t) = \int_a^b f(\tau)\, \mathrm{d}\bar{\lambda}_t^0(\tau) - \int_a^{t-r} f(\tau)\, \mathrm{d}\bar{\lambda}_t^0(\tau) = \int_{t-r}^b f(\tau)\, \mathrm{d}\bar{\lambda}_t^0(\tau)\,.$$

Hence it follows (see the definition of the function $\bar{\lambda}_t^0$)

$$\sup_{f \in \mathscr{B}} (\bar{T}_0 f(t) - \bar{T}_r f(t)) = \operatorname{var}\,[\bar{\lambda}_t^0;\, \langle t - r, b\rangle] = = \frac{2}{\sqrt{\pi}} V_K(r; \varphi(t), t) + \alpha_K(t)\,.$$

Consider $t \in (a, a + r\rangle$. Then $\bar{T}_r f(t) = 0$ for each $f \in \mathscr{C}_0(\langle a, b\rangle)$ and thus

$$\sup_{f \in \mathscr{B}} (\bar{T}_0 f(t) - \bar{T}_r f(t)) = \sup_{f \in \mathscr{B}} \bar{T}_0 f(t) = \operatorname{var}\,[\bar{\lambda}_t^0;\, \langle a, b\rangle] = = \frac{2}{\sqrt{\pi}} V_K(\varphi(t), t) + \alpha_K(t) = \frac{2}{\sqrt{\pi}} V_K(r; \varphi(t), t) + \alpha_K(t)\,.$$

The lemma is proved.

For the curve φ we shall write further

$$\mathscr{F}_r K = \sup_{t \in (a,b\rangle} \frac{2}{\sqrt{\pi}} V_K(r; \varphi(t), t)$$

and

$$\mathscr{F} K = \lim_{r \to 0+} \mathscr{F}_r K$$

(this limit exists since $\mathscr{F}_r K$ is a non-decreasing function with respect to r).

Lemma 1.4. *For each $r > 0$ the following equality is valid:*

(1.28) $$\mathscr{F}_r K = \sup_{t \in (a,b\rangle} \left(\frac{2}{\sqrt{\pi}} V_K(r; \varphi(t), t) + \alpha_K(t) \right).$$

Moreover,

$$\omega \bar{T}_0 \leqq \mathscr{F} K\,.$$

Proof. The set

$$N_0 = \{t;\, t \in (a, b\rangle,\, \alpha_K(t) = 0\}$$

is dense in $\langle a, b\rangle$ since $\alpha_K(t) = 0$ iff $\alpha_{\varphi(t),t}(t) = 0$ and this is true almost everywhere in $\langle a, b\rangle$ (see Lemma (1.1)). As the function $(2/\sqrt{\pi})\, V_K(r; \varphi(t), t) + \alpha_K(t)$ is lower-semicontinuous on $(a, b\rangle$ we have

$$\sup_{t \in (a,b\rangle} \left(\frac{2}{\sqrt{\pi}} V_K(r; \varphi(t), t) + \alpha_K(t) \right) = \sup_{t \in N_0} \left(\frac{2}{\sqrt{\pi}} V_K(r; \varphi(t), t) + \alpha_K(t) \right) = = \sup_{t \in N_0} \frac{2}{\sqrt{\pi}} V_K(r; \varphi(t), t) = \mathscr{F}_r K$$

(the last equality is valid since the function $V_K(r; \varphi(\cdot), \cdot)$ is lower-semicontinuous on $(a, b\rangle$, too) so that (1.29) is proved.

(1.28) implies (see also (1.11)) that

$$\|\bar{T}_0 - \bar{T}_r\| = \sup_{t \in (a,b\rangle} \left(\frac{2}{\sqrt{\pi}} V_K(r; \varphi(t), t) + \alpha_K(t) \right) = \mathscr{F}_r K$$

for each $r > 0$. In virtue of Lemma 1.2 the operator $\bar{T}_r = {}^r H_\varphi^\psi$ $(r > 0)$ is a compact operator on $\mathscr{C}_0(\langle a, b\rangle)$ and thus it follows from the definition of the value $\omega \bar{T}_0$ that

$$\omega \bar{T}_0 \leqq \inf_{r > 0} \|\bar{T}_0 - \bar{T}_r\| = \lim_{r \to 0+} \mathscr{F}_r K = \mathscr{F} K .$$

This completes the proof.

We shall show in the sequel that even $\omega \bar{T}_0 = \mathscr{F}K$. Similarly to [8] we shall define $\mathfrak{V}$ to be the set of all operators $V : \mathscr{C}_0(\langle a, b\rangle) \to \mathscr{C}_0(\langle a, b\rangle)$ of the form

$$Vf(t) = \sum_{i=1}^{n} f_i(t) \int_a^b f(\tau) \, \mathrm{d}g_i(\tau) \tag{1.29}$$

($f \in \mathscr{C}_0(\langle a, b\rangle)$, $t \in \langle a, b\rangle$) where $f_i \in \mathscr{C}_0(\langle a, b\rangle)$ and g_i $(i = 1, 2, \ldots, n)$ are functions of bounded variation on $\langle a, b\rangle$. If P is a linear operator on $\mathscr{C}_0(\langle a, b\rangle)$ then

$$\omega P = \inf \{\|P - V\|; \ V \in \mathfrak{V}\} \tag{1.30}$$

which follows from the fact that any compact operator can be approximated by operators of the form (1.29) (see, for instance, [14] or [11] where the space $\mathscr{C}(\langle a, b\rangle)$ is considered, but the argument can be modified without great difficulty to fit our case).

Let $\mathfrak{W} \subset \mathfrak{V}$ be the set of all operators of the form (1.29) where, in addition, the functions g_i are supposed to be continuous on $\langle a, b\rangle$.

Lemma 1.5. *It holds*

$$\omega \bar{T}_0 = \inf \{\|\bar{T}_0 - W\|; \ W \in \mathfrak{W}\} .$$

Proof. The proof is quite similar to that of Lemma 1.18 in [8] (analogously the proof of the following theorem is similar to that of Theorem 1.19 in [8]). We only recapitulate it here.

Let $V \in \mathfrak{V}$ be of the form (1.29), where $f_i \in \mathscr{C}_0(\langle a, b\rangle)$, g_i have finite variation on $\langle a, b\rangle$ $(i = 1, 2, \ldots, n)$. For each $t \in (a, b\rangle$ we put

$$h(t) = \operatorname{var}_\tau \left[\bar{\lambda}_t^0(\tau) - \sum_{i=1}^{n} f_i(t) \, g_i(\tau); \ \langle a, b\rangle\right] . \tag{1.31}$$

$\mathscr{B}$ being the unit ball in $\mathscr{C}_0(\langle a, b\rangle)$, we have

$$h(t) = \sup_{f \in \mathscr{B}} (\bar{T}_0 - V) f(t) . \tag{1.32}$$

For each $f \in \mathscr{C}_0(\langle a, b\rangle)$ the function $(\bar{T}_0 - V) f$ is continuous on $\langle a, b\rangle$ which implies (by virtue of (1.32)) that the function h is lower-semicontinuous on $(a, b\rangle$. Since the set

$$N_0 = \{t \in (a, b\rangle;\ \alpha_K(t) = 0\}$$

is dense in $\langle a, b\rangle$ we get

$$\|\bar{T}_0 - V\| = \sup_{t \in (a,b\rangle} h(t) = \sup_{t \in N_0} h(t)\,. \tag{1.33}$$

Given $i = 1, 2, \ldots, n$ we can write

$$g_i = q_i + s_i$$

where q_i is a continuous function on $\langle a, b\rangle$ (of bounded variation) and s_i is a saltus-function. We put

$$Q f(t) = \sum_{i=1} f_i(t) \int_a^b f(\tau)\, \mathrm{d}q_i(\tau) \tag{1.34}$$

for each $f \in \mathscr{C}_0(\langle a, b\rangle)$, $t \in \langle a, b\rangle$. Then $Q \in \mathfrak{W}$. In exactly the same way as we showed that the function h is lower-semicontinuous on $(a, b\rangle$ one can verify that the function p defined on $(a, b\rangle$ by

$$p(t) = \operatorname{var}_\tau \left[\bar{\lambda}_t^0(\tau) - \sum_{i=1}^{n} f_i(t)\, q_i(\tau);\ \langle a, b\rangle\right]$$

is lower-semicontinuous on $(a, b\rangle$. Hence

$$\|\bar{T}_0 - Q\| = \sup_{t \in N_0} p(t)\,. \tag{1.35}$$

Considering $t \in N_0$ we have

$$\lim_{\tau \to t-} \bar{\lambda}_t^0(\tau) = \lim_{\tau \to t-} \frac{2}{\sqrt{\pi}}\, G(\alpha_{\varphi(t),t}(\tau)) = \frac{2}{\sqrt{\pi}}\, G(0) = 1$$

and it is seen from the definition of the function $\bar{\lambda}_t^0$ that for such t the function $\bar{\lambda}_t^0$ is continuous on $\langle a, b\rangle$. Since the functions q_i are continuous and s_i are saltus-functions we get, taking into account that the variation of a sum of a continuous function and a saltus-function equals the sum of the variations of those functions,

$$\begin{aligned} h(t) &= \operatorname{var}_\tau \left[\bar{\lambda}_t^0(\tau) - \sum_{i=1}^{n} f_i(t)\, q_i(\tau) - \sum_{i=1}^{n} f_i(t)\, s_i(\tau);\ \langle a, b\rangle\right] = \\ &= \operatorname{var}_\tau \left[\bar{\lambda}_t^0(\tau) - \sum_{i=1}^{n} f_i(t)\, q_i(\tau);\ \langle a, b\rangle\right] + \\ &\quad + \operatorname{var}_\tau \left[\sum_{i=1}^{n} f_i(t)\, s_i(\tau);\ \langle a, b\rangle\right] \geqq p(t) \end{aligned}$$

for each $t \in N_0$.

Hence

$$\|\overline{T}_0 - V\| \geqq \|\overline{T}_0 - Q\| \tag{1.36}$$

by (1.33) and (1.35). In other words, given $V \in \mathfrak{V}$ there is a $Q \in \mathfrak{W}$ such that (1.36) is valid. This implies

$$\inf_{V\in\mathfrak{V}} \|\overline{T}_0 - V\| \geqq \inf_{Q\in\mathfrak{W}} \|\overline{T}_0 - Q\| . \tag{1.37}$$

However, $\mathfrak{W} \subset \mathfrak{V}$ so that the inequality (1.37) turns to the equality. Hence (making use of (1.30) where we write $\overline{T}_0$ instead of P) the assertion follows and the proof is complete.

Theorem 1.1. *It holds*

$$\omega\overline{T}_0 = \mathscr{F}K = \lim_{r\to 0+} \sup_{t\in(a,b\rangle} \left(\frac{2}{\sqrt{\pi}} V_K(r; \varphi(t), t) + \alpha_K(t) \right).$$

Proof. It is easily verified that

$$\frac{2}{\sqrt{\pi}} V_K(r; \varphi(t), t) + \alpha_K(t) = \operatorname{var}\,[\bar{\lambda}_t^0; \langle \max\{a, t - r\}, b\rangle] .$$

According to Lemmas 1.5 and 1.4 it is seen that it suffices to show that

$$\|\overline{T}_0 - W\| \geqq \lim_{r\to 0+} \sup_{t\in(a,b\rangle} \operatorname{var}\,[\bar{\lambda}_t^0; \langle \max\{a, t - r\}, b\rangle] \tag{1.38}$$

for each $W \in \mathfrak{W}$.

Let $W \in \mathfrak{W}$ be of the form

$$Wf(t) = \sum_{i=1}^{n} f_i(t) \int_a^b f(\tau)\, \mathrm{d}g_i(\tau) , \quad (f \in \mathscr{C}_0(\langle a, b\rangle), t \in \langle a, b\rangle)$$

where $f_i \in \mathscr{C}_0(\langle a, b\rangle)$ and g_i are continuous functions of bounded variation ($i = 1, 2, \ldots, n$). Then

$$\|\overline{T}_0 - W\| = \sup_{t\in\langle a,b\rangle} \operatorname{var}\,[\bar{\lambda}_t^0 - \sum_{i=1}^{n} f_i(t)\, g_i; \langle a, b\rangle] . \tag{1.39}$$

As the function $\bar{\lambda}_t^0$ is constant on $\langle t, b\rangle$ and the functions g_i are continuous on $\langle a, b\rangle$ we obtain for $t \in (a, b\rangle$, $r > 0$ that

$$\operatorname{var}\,[\bar{\lambda}_t^0 - \sum_{i=1}^{n} f_i(t)\, g_i; \langle a, b\rangle] \geqq$$

$$\geqq \operatorname{var}\,[\bar{\lambda}_t^0 - \sum_{i=1}^{n} f_i(t)\, g_i; \langle \max\{a, t - r\}, b\rangle] \geqq$$

$$\geqq \operatorname{var}[\bar{\lambda}_t^0; \langle \max\{a, t-r\}, b\rangle] - \operatorname{var}[\bar{\lambda}_t^0; (t, b\rangle] -$$

$$- \operatorname{var}[\sum_{i=1}^n f_i(t)\, g_i; \langle \max\{a, t-r\}, t\rangle] = \operatorname{var}[\bar{\lambda}_t^0; \langle \max\{a, t-r\}, b\rangle] -$$

$$- \operatorname{var}[\sum_{i=1}^n f_i(t)\, g_i; \langle \max\{a, t-r\}, t\rangle].$$

Taking $t = a$ we have, of course,

$$\operatorname{var}[\bar{\lambda}_a^0 - \sum_{i=1}^n f_i(a)\, g_i; \langle a, b\rangle] \geqq \operatorname{var}[\bar{\lambda}_a^0; \langle a, b\rangle]$$

($\bar{\lambda}_a^0$ is a saltus-function, g_i are continuous).

In order to prove (1.38) it will then suffice to verify that

$$\lim_{r\to 0+} \sup_{t\in\langle a,b\rangle} \operatorname{var}[\sum_{i=1}^n f_i(t)\, g_i; \langle \max\{a, t-r\}, t\rangle] = 0. \tag{1.40}$$

Writing

$$c = \sup\{|f_i(t)|;\ t \in \langle a, b\rangle,\ i = 1, 2, \dots, n\}$$

we have

$$\operatorname{var}[\sum_{i=1}^n f_i(t)\, g_i; \langle \max\{a, t-r\}, t\rangle] \leqq \tag{1.41}$$

$$\leqq c \sum_{i=1}^n \operatorname{var}[g_i; \langle \max\{a, t-r\}, t\rangle] = h_r(t)$$

for each $t \in \langle a, b\rangle$, $r > 0$. The functions h_r so defined are continuous (because g_i are continuous) and it is seen that

$$h_{r_1}(t) \geqq h_{r_2}(t), \quad (t \in \langle a, b\rangle)$$

whenever $r_1 \geqq r_2 > 0$. Further it follows from the continuity of the functions g_i that

$$\lim_{r\to 0+} h_r(t) = 0$$

for each $t \in \langle a, b\rangle$. It is well known that a sequence which converges pointwise to zero on a compact interval and is non-decreasing converges also uniformly on that interval (the Dini theorem).

Thus

$$\lim_{r\to 0+} \sup_{t\in\langle a,b\rangle} h_r(t) = 0.$$

Using (1.41) we conclude that (1.40) is valid and so is (1.38). The assertion is proved.

First we shall deal with the following boundary value problem. Let $\langle a, b\rangle$ be a compact interval in R^1, φ a continuous function of bounded variation on $\langle a, b\rangle$. Put

$$K = \{[\varphi(t), t];\ t \in \langle a, b\rangle\}\,, \quad M = \{[x, t];\ t \in (a, b),\ x > \varphi(t)\}\,, \tag{2.1}$$

$$B = K \cup \{[x, a];\ x \geqq \varphi(a)\}\,.$$

Let F be a continuous bounded function on B. We look for a function G on M which solves the heat equation on M and satisfies

$$\lim_{\substack{[x,t]\to[x_0,t_0]\\ [x,t]\in M}} G(x, t) = F(x_0, t_0) \tag{2.2}$$

for each point $[x_0, t_0] \in B$.

Let us suppose

$$\sup_{t\in\langle a,b\rangle} V_K(\varphi(t), t) < \infty\,. \tag{2.3}$$

Let $\overline{T}_0$ have the same meaning as in Section 1 and suppose

$$\omega \overline{T}_0 \leqq 1\,. \tag{2.4}$$

We shall show that in this case there is a solution of the problem formulated above and it may be expressed in an integral form.

Let $\tilde{T}_+$, $\tilde{T}_-$ denote the same as above, i.e., let

$$\tilde{T}_+ f(t) = Tf(\varphi(t), t) + f(t)\left(2 - \frac{2}{\sqrt{\pi}}\, G(\alpha_{\varphi(t),t}(t))\right),$$

$$\tilde{T}_- f(t) = Tf(\varphi(t), t) - \frac{2}{\sqrt{\pi}} f(t)\, G(\alpha_{\varphi(t),t}(t))$$

for each $f \in \mathscr{C}_0(\langle a, b\rangle)$, $t \in (a, b\rangle$ ($\tilde{T}_+ f(a) = \tilde{T}_- f(a) = 0$ for each $f \in \mathscr{C}_0(\langle a, b\rangle)$). Further, we have

$$\tilde{T}_+ f(t) = \lim_{\substack{[x',t']\to[\varphi(t),t]\\ t'\in\langle a,b\rangle,\, x'>\varphi(t)}} Tf(x', t')\,, \tag{2.5}$$

$$\tilde{T}_- f(t) = \lim_{\substack{[x',t']\to[\varphi(t),t]\\ t'\in\langle a,b\rangle,\, x'<\varphi(t)}} Tf(x', t') \tag{2.6}$$

for each $f \in \mathscr{C}_0(\langle a, b\rangle)$, $t \in \langle a, b\rangle$.

Let F be a given continuous bounded function on B. We define a function F_1 on the interval $(-\infty, \infty)$ by

$$F_1(x) = F(x, a), \qquad x \geqq \varphi(a),$$
$$F_1(x) = F(\varphi(a), a), \quad x < \varphi(a);$$

this function is continuous and bounded on R^1. We can define

$$G_1(x, t) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{(t-a)}} \exp\left(-\frac{(x-\tau)^2}{4(t-a)}\right) F_1(\tau)\, d\tau \tag{2.7}$$

for each $[x, t] \in M_1 = \{[x, t] \in R^2;\ t > a\}$. The function G_1 is the so-called Poisson integral of the function F_1 and is known to be caloric on M_1. Moreover, it is known that

$$\lim_{\substack{[x,t]\to[x_0,a]\\ t>a}} G_1(x, t) = F_1(x_0) \tag{2.8}$$

for each $x_0 \in R^1$. Let us define a function $F_2 \in \mathscr{C}_0(\langle a, b\rangle)$ by

$$F_2(t) = \begin{cases} 0, & t = a \\ F(\varphi(t), t) - G_1(\varphi(t), t), & t \in (a, b\rangle. \end{cases} \tag{2.9}$$

Suppose now that there is a function $f \in \mathscr{C}_0(\langle a, b\rangle)$ such that

$$\tilde{T}_+ f = F_2. \tag{2.10}$$

Putting

$$G(x, t) = Tf(x, t) + G_1(x, t), \quad ([x, t] \in M) \tag{2.11}$$

one can easily verify that for the solution G the condition (2.2) is fulfilled and thus this function solves the boundary value problem stated above for the boundary condition F on B. Now it suffices to answer the question when does such a function f exist. We shall show that if the condition (2.4) is fulfilled then one can find such a function f.

It is well known that by the Riesz-Schauder theory the Fredholm alternative is valid for a continuous linear operator on a Banach space whenever the Fredholm radius of the operator is bigger than 1. But to say that the condition (2.4) is fulfilled is to say that the Fredholm radius of the operator $\overline{T}_0$ is bigger than 1. Thus if this condition is fulfilled then the operator $\tilde{T}_+$ $(=\overline{T}_0 + I)$ either maps the space $\mathscr{C}_0(\langle a, b\rangle)$ onto $\mathscr{C}_0(\langle a, b\rangle)$ or there exists a function $f \in \mathscr{C}_0(\langle a, b\rangle)$ which is not equal to zero function such that $\tilde{T}_+ f = 0$. In the first case, given an arbitrary $g \in \mathscr{C}_0(\langle a, b\rangle)$, one can find exactly one function $f \in \mathscr{C}_0(\langle a, b\rangle)$ such that $\tilde{T}_+ f = 0$. It is evident that it suffices to show the following lemma.

Lemma 2.1. *Let $\omega \tilde{T}_0 < 1$. Then the equations*

$$(2.12) \qquad \tilde{T}_+ f = 0\,,$$

$$(2.13) \qquad \tilde{T}_- f = 0$$

are solved in $\mathscr{C}_0(\langle a, b\rangle)$ only by the zero function.

Proof. Consider, for instance, the equation (2.12).

Since by Theorem 1.1

$$\omega \overline{T}_0 = \lim_{r \to 0+} \sup_{t \in (a,b\rangle} \left(\frac{2}{\sqrt{\pi}} V_K(r; \varphi(t), t) + \alpha_K(t) \right),$$

there is $r_0 > 0$ such that

$$(2.14) \qquad \frac{2}{\sqrt{\pi}} V_K(r; \varphi(t), t) + \alpha_K(t) < 1$$

for each $r \in (0, r_0)$, $t \in (a, b\rangle$. Let $f \in \mathscr{C}_0(\langle a, b\rangle)$ and suppose that $\tilde{T}f = 0$ (on $\langle a, b\rangle$). First we show that $f(t) = 0$ for each $t \in \langle a, a + r_0\rangle$ (we may suppose $r_0 < b - a$). There exists a point $t' \in (a, a + r_0\rangle$ such that

$$|f(t')| = \sup_{t \in \langle a, a + r_0\rangle} |f(t)|\,.$$

Then it holds (as the function $\bar{\lambda}^0_{t'}$ is constant on $(t', b\rangle$)

$$(2.15) \qquad |\overline{T}_0 f(t')| = \left| \int_a^b f(\tau)\, \mathrm{d}\bar{\lambda}^0_{t'}(\tau) \right| =$$

$$= \left| \int_a^{t'} f(\tau)\, \mathrm{d}\bar{\lambda}^0_{t'}(\tau) \right| \leqq |f(t')| \operatorname{var}\,[\bar{\lambda}^0_{t'}; \langle a, t'\rangle] =$$

$$= |f(t')| \left(\frac{2}{\sqrt{\pi}} V_K(\varphi(t'), t') + \alpha_K(t') \right) =$$

$$= |f(t')| \left(\frac{2}{\sqrt{\pi}} V_K(r_0; \varphi(t'), t') + \alpha_K(t') \right).$$

Further we have

$$(2.16) \qquad 0 = \tilde{T}_+ f(t') = \overline{T}_0 f(t') + f(t')\,.$$

Comparing (2.15) and (2.16) we obtain from (2.14) that $f(t') = 0$ and thus $f(t) = 0$ for each $t \in \langle a, a + r_0\rangle$.

Suppose further that $2r_0 \leqq b - a$ and prove that $f(t) = 0$ for each $t \in \langle a + r_0, a + 2r_0\rangle$. There is a point $t'' \in \langle a + r_0, a + 2r_0\rangle$ such that

$$|f(t'')| = \sup \{|f(t)|;\ t \in \langle a + r_0,\ a + 2r_0\rangle\}\,.$$

It holds $f(t) = 0$ for each $t \in \langle a, a + r_0 \rangle$ by the first part of this proof so that

$$|\bar{T}_0 f(t'')| = \left| \int_{a+r_0}^{t''} f(\tau) \, \mathrm{d}\bar{\lambda}^0_{t''}(\tau) \right| \leqq$$

$$\leqq |f(t'')| \left(\frac{2}{\sqrt{\pi}} V_K(r_0; \varphi(t''), t'') + \alpha_K(t'') \right).$$

Further, (2.16) is valid also if we write t'' instead of t'. Similarly as above we get from (2.14) that $f(t'') = 0$ and thus $f(t) = 0$ for each $t \in \langle a, a + 2r_0 \rangle$.

Continuing by induction, we conclude that $f(t) = 0$ for each $t \in \langle a, b \rangle$ so that the equation (2.12) is solved only by the zero function.

Analogously for the equation (2.13).

The lemma is proved.

According to Lemma 2.1 and the preceding argument we obtain immediately the following assertion.

Theorem 2.1. *Let φ be a continuous function of bounded variation on the interval $\langle a, b \rangle$ and let K, M, B be defined by (2.1). Suppose that*

$$\lim_{r \to 0+} \sup_{t \in (a,b)} \left(\frac{2}{\sqrt{\pi}} V_K(r; \varphi(t), t) + \alpha_K(t) \right) < 1$$

(this condition requires the validity of (2.3)).

Given a continuous bounded function F on B, let us put

$$F_1(x) = \left\langle \begin{array}{ll} F(x, a) & x \geqq \varphi(a) \\ F(\varphi(a), a) & x < \varphi(a) . \end{array} \right.$$

Then there is one and only one function $f \in \mathscr{C}_0(\langle a, b \rangle)$ such that the function

$$G(x, t) = Tf(x, t) + \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{F_1(\tau)}{\sqrt{(t - a)}} \exp\left(- \frac{(x - \tau)^2}{4(t - a)} \right) \mathrm{d}\tau \quad ([x, t] \in M)$$

is a solution of the boundary value problem for the heat equation on M with the boundary condition F on B.

Remark 2.1. In exactly the same way as we solved the boundary value problem for the heat equation on M we may solve the boundary value problem for the set

$$M' = \{[x, t]; \; t \in (a, b), \; x < \varphi(t)\}$$

with boundary conditions defined on the set

$$B' = K \cup \{[x, a]; \; x \leqq \varphi(a)\} .$$

We should only use the operator $\bar{T}_- = \bar{T}_0 - I$ instead of $\bar{T}_+$.

Let $\langle a, b\rangle$ be a compact interval in R^1 and consider on this interval two continuous functions φ_1, φ_2 of bounded variation such that

$$\varphi_1(t) < \varphi_2(t) \tag{2.17}$$

for each $t \in \langle a, b\rangle$. Putting

$$\begin{aligned} K_i &= \{[\varphi_i(t), t];\ t \in \langle a, b\rangle\}\,, \quad (i = 1, 2)\,, \\ M &= \{[x, t];\ t \in (a, b),\ \varphi_1(t) < x < \varphi_2(t)\}\,, \\ B &= K_1 \cup K_2 \cup \{[x, a];\ \varphi_1(a) \leqq x \leqq \varphi_2(a)\} \end{aligned} \tag{2.18}$$

we shall solve the Fourier boundary value problem on the region M; i.e., given a continuous function F defined on the set B we look for a function G caloric on M such that

$$\lim_{\substack{[x,t]\to[x_0,t_0]\\ [x,t]\in M}} G(x, t) = F(x_0, t_0) \tag{2.19}$$

holds for each point $[x_0, t_0] \in B$.

By analogy to Def. 1.1 in [17] we define the parabolic variations of the curves φ_1, φ_2 which we shall denote by V_{K_1}, V_{K_2} respectively. Throughout the following we shall suppose that

$$\sup_{t\in\langle a,b\rangle} V_{K_i}(\varphi(t), t) < \infty\,, \quad (i = 1, 2)\,. \tag{2.20}$$

Further, we denote

$$_i\alpha_{\varphi_i(t),t}(t) = \lim_{\tau\to t-} \frac{\varphi_i(t) - \varphi_i(\tau)}{2\sqrt{(t-\tau)}}\,, \quad (i = 1, 2)$$

and

$$\alpha_{K_i}(t) = \left|1 - \frac{2}{\sqrt{\pi}}\, G({}_i\alpha_{\varphi_i(t),t}(t))\right|$$

for each $t \in (a, b\rangle$.

In the same way as we defined the heat potential operator T for the curve φ let us define the heat potential operators T_1, T_2 corresponding to the curves φ_1, φ_2; i.e., we put

$$T_i f(x, t) = \frac{2}{\sqrt{\pi}} \int_a^{\min\{t,b\}} f(\tau) \exp\left(-\frac{(x - \varphi_i(\tau))^2}{4(t-\tau)}\right) \mathrm{d}_\tau \left(\frac{x - \varphi_i(\tau)}{2\sqrt{(t-\tau)}}\right) \quad (i = 1, 2)$$

for each bounded Baire function f defined on $\langle a, b\rangle$ and $x \in R^1$, $t > a$. Put

$$\tilde{T}_1 f(t) = T_1 f(\varphi_1(t), t) + f(t)\left(2 - \frac{2}{\sqrt{\pi}}\, G({}_1\alpha_{\varphi_1(t),t}(t))\right), \tag{2.21}$$

(2.22) $$\tilde{T}_2 f(t) = -T_2 f(\varphi_2(t), t) + f(t) \frac{2}{\sqrt{\pi}} G(_2\alpha_{\varphi_2(t),t}(t))$$

for each $f \in \mathscr{C}_0(\langle a, b\rangle)$, $t \in \langle a, b\rangle$.

In the sequel we shall deal with the space $\mathfrak{C}_0 = \mathfrak{C}_0(\langle a, b\rangle)$ which is defined to be the space of all continuous mappings F from $\langle a, b\rangle$ to R^2 for which $F(a) = [0, 0]$. Thus $\mathfrak{C}_0$ is the set of all pairs $[f_1, f_2]$ with $f_1, f_2 \in \mathscr{C}_0(\langle a, b\rangle)$. We introduce a norm in $\mathfrak{C}_0(\langle a, b\rangle)$ by

$$\|[f_1, f_2]\| = \|[f_1, f_2]\|_{\mathfrak{C}_0} = \|f_1\|_{\mathscr{C}_0} + \|f_2\|_{\mathscr{C}_0}$$

($[f_1, f_2] \in \mathfrak{C}_0$). $\mathfrak{C}_0$ endowed with this norm is a Banach space.

It is easily verified that any linear operator P which maps $\mathfrak{C}_0$ into $\mathfrak{C}_0$ may be expressed (uniquely) in the form

(2.25) $$P(f_1, f_2) = [P_1 f_1 + P_2 f_2,\ P_3 f_1 + P_4 f_2]\,, \quad ([f_1, f_2] \in \mathfrak{C}_0)$$

where P_i $(i = 1, 2, 3, 4)$ are linear operators mapping $\mathscr{C}_0(\langle a, b\rangle)$ into itself. The operator P is bounded iff all operators P_i are and is compact iff all P_i are.

Let us define an operator R acting on $\mathfrak{C}_0$ by

(2.26) $$R(f_1, f_2)(t) = [\tilde{T}_1 f_1(t) - T_2 f_2(\varphi_1(t), t),\ \tilde{T}_2 f_2(t) + T_1 f_1(\varphi_2(t), t)]$$

($[f_1, f_2] \in \mathfrak{C}_0$, $t \in \langle a, b\rangle$) and put

(2.27) $$R_0 = R - I$$

where I is the identity operator on $\mathfrak{C}_0$.

Lemma 2.2. *It holds*

$$\omega R_0 = \max_{i=1,2} \left\{ \lim_{r\to 0+} \sup_{t\in(a,b)} \left(\frac{2}{\sqrt{\pi}} V_{K_i}(r; \varphi_i(t), t) + \alpha_{K_i}(t) \right) \right\}.$$

Proof. We have shown in Corollary 1.1 that the operator defined by the equality (1.16) is compact provided $\psi(t) \neq \varphi(t)$ for each $t \in \langle a, b\rangle$. Applying this assertion to the cases $\psi = \varphi_1$, $\varphi = \varphi_2$ and $\psi = \varphi_2$, $\varphi = \varphi_1$ we conclude that the operator R_2 defined on $\mathfrak{C}_0$ by

(2.28) $$R_2(f_1, f_2)(t) = [-T_2 f_2(\varphi_1(t), t),\ T_1 f_1(\varphi_2(t), t)]$$

($[f_1, f_2] \in \mathfrak{C}_0$, $t \in \langle a, b\rangle$; i.e. $R_2 = [-H_{\varphi_2}^{\varphi_1}, H_{\varphi_1}^{\varphi_2}]$) is a compact operator on $\mathfrak{C}_0$. If P is a continuous linear operator on a Banach space and P_1 is a compact operator acting on the same space then it is seen immediately from the definition of the value ωP that

$$\omega(P + P_1) = \omega P\,.$$

Hence

$$\omega R_0 = \omega R_1 \tag{2.29}$$

where we put $R_1 = R_0 - R_2$.

We have

$$R_1(f_1, f_2) = [\tilde{T}_1 f_1 - f_1,\ \tilde{T}_2 f_2 - f_2]\,, \quad ([f_1, f_2] \in \mathfrak{C}_0)\,. \tag{2.30}$$

In accordance with Theorem 1.1 we get

$$\omega(\tilde{T}_1 - I) = \lim_{r \to 0+} \sup_{t \in \langle a,b \rangle} \left(\frac{2}{\sqrt{\pi}} V_{K_1}(r; \varphi_1(t), t) + \alpha_{K_1}(t) \right) \tag{2.31}$$

and

$$\omega(\tilde{T}_2 - I) = \lim_{r \to 0+} \sup_{t \in \langle a,b \rangle} \left(\frac{2}{\sqrt{\pi}} V_{K_2}(r; \varphi_2(t), t) + \alpha_{K_2}(t) \right). \tag{2.32}$$

Let us now show that for a linear operator P mapping $\mathfrak{C}_0$ into $\mathfrak{C}_0$ which has the form

$$P(f_1, f_2) = [P_1 f_1, P_2 f_2]\,, \quad ([f_1, f_2] \in \mathfrak{C}_0)$$

it holds

$$\omega P = \max \{\omega P_1, \omega P_2\}\,. \tag{2.33}$$

If $\bar{P}$ is a compact operator acting on $\mathfrak{C}_0$ which is of the form

$$\bar{P}(f_1, f_2) = [\bar{P}_1 f_1 + \bar{P}_2 f_2,\ \bar{P}_3 f_1 + \bar{P}_4 f_2]\,, \quad ([f_1, f_2] \in \mathfrak{C}_0)\,,$$

then the operators $\bar{P}_i$ $(i = 1, 2, 3, 4)$ are compact operators acting on $\mathscr{C}_0(\langle a, b \rangle)$ and we have

$$\begin{aligned} \|(\bar{P} - P)(f_1, f_2)\| &\leqq \|\bar{P}_1 - P_1\|\,\|f_1\| + \|\bar{P}_2\|\,\|f_2\| + \|\bar{P}_3\|\,\|f_1\| + \\ &+ \|\bar{P}_4 - P_2\|\,\|f_2\| \leqq \max\{\|\bar{P}_1 - P_1\|, \|\bar{P}_4 - P_2\|\}\,(\|f_1\| + \|f_2\|) + \\ &+ \|\bar{P}_2\|\,\|f_2\| + \|\bar{P}_3\|\,\|f_1\| \leqq \max\{\|\bar{P}_1 - P_1\|, \|\bar{P}_4 - P_2\|\} + \\ &+ \|\bar{P}_2\|\,\|f_2\| + \|\bar{P}_3\|\,\|f_1\| \end{aligned} \tag{2.34}$$

for each $[f_1, f_2] \in \mathfrak{C}_0$ such that $\|[f_1, f_2]\| \leqq 1$. We may assume $\bar{P}_2, \bar{P}_3$ to be zero operators and conclude (by the definition of ωP, ωP_1, ωP_2 and the definition of the norm of operators) that

$$\omega P \leqq \max \{\omega P_1, \omega P_2\}\,. \tag{2.35}$$

Further it holds for each compact operator $\bar{P}$

$$\|(\bar{P} - P)(f_1, f_2)\| \geqq \|(\bar{P}_1 - P_1) f_1\| + \|(\bar{P}_4 - P_2) f_2\|\,.$$

Suppose, for instance, $\omega P_1 \geqq \omega P_2$ and let $f_2 = 0$. If we let f_1 run over all functions belonging to $\mathscr{C}_0(\langle a, b\rangle)$ such that $\|f_1\| \leqq 1$ we arrive at

$$\|\bar{P} - P\| \geqq \|\bar{P}_1 - P_1\|$$

which implies

$$\omega P \geqq \omega P_1 = \max\{\omega P_1, \omega P_2\}\,. \tag{2.36}$$

(2.33) follows now from (2.35) and (2.36). The assertion follows then from (2.33), (2.30), (2.31), (2.32) and (2.29).

Lemma 2.3. *Suppose $\omega R_0 < 1$. Then the equation*

$$R(f_1, f_2) = 0 \tag{2.37}$$

(*where $0 \in \mathfrak{C}_0$ is the zero element*) *has in $\mathfrak{C}_0$ only the trivial solution.*

Proof. We have shown in the proof of Corollary 1.1 that if $\psi(t) \neq \varphi(t)$ on the whole $\langle a, b\rangle$ then

$$\lim_{r\to 0+} \|H_\varphi^\psi - {}^rH_\varphi^\psi\| = \lim_{r\to 0+} \sup_{t\in\langle a,b\rangle} V_K(r; \psi(t), t) = 0 \tag{2.38}$$

(see (1.24) and (2.25)). Applying (2.38) to the case $\psi = \varphi_1$, $\varphi = \varphi_2 (K = K_2)$ one obtains

$$\lim_{r\to 0+} \sup_{t\in\langle a,b\rangle} V_{K_2}(r; \varphi_1(t), t) = 0\,. \tag{2.39}$$

According to Lemma 2.2 and the relation (1.39) it follows from the assumption $\omega R_0 < 1$ that there is $r_0 > 0$ $(r_0 < b - a)$ such that

$$\frac{2}{\sqrt{\pi}} V_{K_1}(r_0; \varphi_1(t), t) + \alpha_{K_1}(t) + \frac{2}{\sqrt{\pi}} V_{K_2}(r_0; \varphi_1(t), t) < 1 \tag{2.40}$$

for each $t \in (a, b\rangle$.

Let $[f_1, f_2] \in \mathfrak{C}_0$ and let $R(f_1, f_2) = 0$. Then there are $t' \in (a, a + r_0\rangle$, $i \in \{1, 2\}$ such that

$$|f_i(t')| = \max_{j=1,2}\{\sup_{t\in\langle a,a+r_0\rangle} |f_j(t)|\}\,.$$

Suppose, for instance, $i = 1$. Then

$$0 = T_1 f_1(\varphi_1(t'), t') + f_1(t')\left(1 - \frac{2}{\sqrt{\pi}} G({}_1\alpha_{\varphi_1(t'),t'}(t'))\right) + \tag{2.41}$$
$$+ f_1(t') - T_2 f_2(\varphi_1(t'), t')$$

and, on the other hand,

$$(2.42)\quad \left|T_1 f_1(\varphi_1(t'), t') + f_1(t')\left(1 - \frac{2}{\sqrt{\pi}}\, G({}_1\alpha_{\varphi_1(t'),t'}(t'))\right) - T_2 f_2(\varphi_1(t'), t')\right| \leqq$$

$$\leqq |f_1(t')| \left(\frac{2}{\sqrt{\pi}} V_{K_1}(r_0; \varphi_1(t'), t') + \alpha_{K_1}(t') + \frac{2}{\sqrt{\pi}} V_{K_2}(r_0; \varphi_1(t'), t')\right).$$

If we compare (2.41), (2.42) and (2.40) we get $f_1(t') = 0$ so that the functions f_1, f_2 vanish on the interval $\langle a, a + r_0 \rangle$.

The proof may be completed by induction similarly to the proof of Lemma 2.1.

Theorem 2.2. *Let φ_1, φ_2 be continuous functions of bounded variation on the interval $\langle a, b \rangle$ and suppose that $\varphi_1(t) < \varphi_2(t)$ for each $t \in \langle a, b \rangle$. Let K_1, K_2, M, B be defined by (2.18) and let*

$$(2.43)\qquad \max_{i=1,2} \left\{ \lim_{r \to 0+} \sup_{t \in \langle a,b \rangle} \left(\frac{2}{\sqrt{\pi}} V_{K_i}(r; \varphi_i(t), t) + \alpha_{K_i}(t) \right) \right\} < 1$$

(the condition (2.20) is included in (2.43)). Let F be a continuous function on B and put

$$(2.44)\qquad F_1(x) = \begin{cases} F(\varphi_1(a), a) & x \leqq \varphi_1(a) \\ F(x, a) & x \in (\varphi_1(a), \varphi_2(a)) \\ F(\varphi_2(a), a) & x \geqq \varphi_2(a)\,. \end{cases}$$

Then there is a unique pair of functions $g_1, g_2 \in \mathscr{C}_0(\langle a, b \rangle)$ such that the function G defined on M by

$$(2.45)\qquad G(x, t) = \frac{2}{\sqrt{\pi}} \int_a^t g_1(\tau) \exp\left(-\frac{(x - \varphi_1(\tau))^2}{4(t - \tau)}\right) \mathrm{d}_\tau \left(\frac{x - \varphi_1(\tau)}{2\sqrt{(t - \tau)}}\right) -$$

$$- \frac{2}{\sqrt{\pi}} \int_a^t g_2(\tau) \exp\left(-\frac{(x - \varphi_2(\tau))^2}{4(t - \tau)}\right) \mathrm{d}_\tau \left(\frac{x - \varphi_2(\tau)}{2\sqrt{(t - \tau)}}\right) +$$

$$+ \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{F_1(\tau)}{\sqrt{(t - a)}} \exp\left(-\frac{(x - \tau)^2}{4(t - a)}\right) \mathrm{d}\tau \quad ([x, t] \in M)$$

is a solution of the boundary value problem for the heat equation on M with the boundary condition F on B.

Proof of this theorem is based on the Riesz-Schauder theory similarly to the proof of Theorem 2.1. If we put

$$G_1(x, t) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{F_1(\tau)}{\sqrt{(t - a)}} \exp\left(-\frac{(x - \tau)^2}{4(t - a)}\right) \mathrm{d}\tau$$

for $t > a$, $x \in R^1$ and define functions f_1, f_2 by

$$f_i(t) = F(\varphi_i(t), t) - G_1(\varphi_i(t), t) \quad t \in (a, b\rangle\,, \quad i = 1, 2\,,$$
$$f_1(a) = f_2(a) = 0$$

then we can find (using the Riesz-Schauder theory) a unique pair $[g_1, g_2] \in \mathfrak{C}_0$ for which

$$R(g_1, g_2) = [f_1, f_2]\,.$$

It is easily verified that this pair $[g_1, g_2]$ possesses the required properties.

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