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NOTE ON FUNCTIONAL-DIFFERENTIAL EQUATIONS
WITH INITIAL FUNCTIONS OF BOUNDED VARIATION

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In this note we shall deal with the standard functional-differential equation of retarded type

$$(1) \quad \dot{x}(t) = \int_{-r}^0 [d_{\vartheta} P(t, \vartheta)] x(t + \vartheta) + f(t) \quad \text{a.e. on } [a, b],$$

$$(2) \quad x(t) = u(t) \quad \text{on } [a - r, a],$$

where $-\infty < a < b < +\infty$ and the initial functions $u(t)$ are of bounded variation on $[a - r, a]$. We assume that $P(t, \vartheta)$ is a Borel measurable in $(t, \vartheta) \in [a, b] \times \times (-\infty, +\infty)$ $n \times n$ -matrix function such that $p(t) = \text{var}_{-r}^0 P(t, \cdot) < \infty$ for all $t \in [a, b]$ and

$$\int_a^b p(t) dt < \infty,$$

$f(t)$ is an n -vector function Lebesgue integrable on $[a, b]$ ($f(t) \in \mathcal{L}_n(a, b)$). We shall suppose also $P(t, \vartheta) = P(t, -r)$ for $\vartheta \leq -r$ and $P(t, \vartheta) = P(t, 0)$ for $\vartheta \geq 0$. Without any loss of generality we may suppose furthermore that $P(t, \cdot)$ is right continuous on $(-r, 0)$ and $P(t, 0) = 0$ for all $t \in [a, b]$.

Let $\mathcal{BV}_n(a - r, a)$ denote the space of (column) n -vector functions with bounded variation on $[a - r, a]$. $\mathcal{AC}_n(a, b)$ is the space of n -vector functions which are absolutely continuous on $[a, b]$. The introduced spaces are equipped with the usual norms

$$u \in \mathcal{BV}_n(a - r, a) \rightarrow \|u\|_{\mathcal{BV}} = \|u(a)\| + \text{var}_{a-r}^a u,$$

$$x \in \mathcal{AC}_n(a, b) \rightarrow \|x\|_{\mathcal{AC}} = \|x(a)\| + \text{var}_a^b x,$$

$$f \in \mathcal{L}_n(a, b) \rightarrow \|f\|_{\mathcal{L}} = \int_a^b \|f(t)\| dt.$$

Proposition 1. *There exists a unique $n \times n$ -matrix function $Y(t, s)$ defined on $[a, b] \times [a, b]$ and such that*

$$(3) \quad Y(t, s) = \begin{cases} I - \int_s^t Y(t, \sigma) P(\sigma, s - \sigma) d\sigma & \text{for } a \leq t \leq b, \quad a \leq s \leq t, \\ I & \text{for } a \leq t \leq b, \quad t \leq s \leq b, \end{cases}$$

where I is the identity $n \times n$ -matrix. Given $t \in [a, b]$, $Y(t, \cdot)$ is of bounded variation on $[a, b]$ and given $s \in [a, b]$, $Y(\cdot, s)$ is absolutely continuous on $[a, b]$.

(For the proof of a slightly modified assertion see J. K. HALE [2], Theorem 32,2.)

The following representation of solutions of the system (1), (2) is well known (cf. H. T. BANKS [1] or J. K. HALE [2], Theorems 16,1 and 32,2):

Proposition 2. *Given $u \in \mathcal{BV}_n(a - r, a)$, there exists a unique n -vector function $x(t)$ defined on $[a - r, b]$ and absolutely continuous on $[a, b]$ and such that (1) and (2) hold. This function $x(t)$ is on $[a, b]$ given by*

$$(4) \quad x = \Phi u + \Psi f,$$

where

$$\Phi : u \in \mathcal{BV}_n(a - r, a) \rightarrow Y(t, a) u(a) + \int_{a-r}^a \left[d_s \int_a^t Y(t, \sigma) P(\sigma, s - \sigma) d\sigma \right] u(s) \in \mathcal{AC}_n(a, b),$$

$$\Psi : f \in \mathcal{L}_n(a, b) \rightarrow \int_a^t Y(t, s) f(s) ds \in \mathcal{AC}_n(a, b)$$

and $Y(t, s)$ is defined by Proposition 1.

The operators Φ, Ψ in (4) are obviously linear and bounded. The aim of this note is to show that Φ is even completely continuous. By Theorem 3,1 of ŠT. SCHWABIK [5] it suffices to show that the function

$$(5) \quad K(t, s) = \int_a^t Y(t, \sigma) P(\sigma, s - \sigma) d\sigma, \quad (t, s) \in [a, b] \times [a - r, a]$$

is of bounded two-dimensional variation (according to Vitali) on $[a, b] \times [a - r, a]$ ($v(K) < \infty$) and $\text{var}_{a-r}^a K(a, \cdot) + \text{var}_a^b K(\cdot, a) < \infty$. Such functions are said to be of strongly bounded variation on $[a, b] \times [a - r, a]$. (For the definition and basic properties of functions of bounded two-dimensional variation see T. H. HILDEBRANDT [4].)

Lemma 1. *The fundamental matrix solution $Y(t, s)$ defined by Proposition 1 is of strongly bounded variation on $[a, b] \times [a, b]$.*

Proof. Analogously to J. K. HALE in the proof of Theorem 32,2 in [2] we shall introduce the function $W(t, s)$ fulfilling the matrix Volterra integral equation

$$W(t, s) = \begin{cases} -P(t, s - t) - \int_s^t W(t, \sigma) P(\sigma, s - \sigma) d\sigma & \text{for } a \leq t \leq b, \quad a \leq s \leq t, \\ 0 & \text{for } a \leq t \leq b, \quad t \leq s \leq b. \end{cases}$$

The existence of such a function $W(t, s)$ follows from the contraction mapping principle. Moreover, given $t \in [a, b]$, the function $W(t, \cdot)$ is of bounded variation on $[a, b]$. Now, let $s, t \in [a, b]$, $s \leq t$ and let $\{s = s_0 < s_1 < \dots < s_m = t\}$ be an arbitrary subdivision of the interval $[s, t]$. Then

$$\begin{aligned} \sum_{j=1}^m \|W(t, s_j) - W(t, s_{j-1})\| &\leq \sum_{j=1}^m \|P(t, s_j - t) - P(t, s_{j-1} - t)\| + \\ &+ \sum_{j=1}^m \left\{ \int_{s_j}^t \|W(t, \sigma)\| \|P(\sigma, s_j - \sigma) - P(\sigma, s_{j-1} - \sigma)\| d\sigma + \right. \\ &\left. + \int_{s_{j-1}}^{s_j} \|W(t, \sigma) P(\sigma, s_{j-1} - \sigma)\| d\sigma \right\} \leq p(t) + 2 \int_s^t (\text{var}_\sigma^t W(t, \cdot)) p(\sigma) d\sigma, \end{aligned}$$

where $p(t) = \text{var}_t^0 P(t, \cdot)$ for $t \in [a, b]$. Gronwall's inequality yields

$$(6) \quad \|W(t, s)\| \leq \text{var}_s^t W(t, \cdot) \leq p(t) \exp\left(2 \int_s^t p(\sigma) d\sigma\right) < \infty$$

for all $t, s \in [a, b]$, $t \geq s$. It is easy to verify (cf. [2], proof of Theorem 32,2) that for all $t, s \in [a, b]$

$$Y(t, s) = I + \int_s^t W(\tau, s) d\tau.$$

Furthermore, let $v = \{a = t_0 < t_1 < \dots < t_p = b; a = s_0 < s_1 < \dots < s_q = b\}$ be an arbitrary net type subdivision of $[a, b] \times [a, b]$. Then according to (6)

$$\begin{aligned} \sum_{j=1}^p \sum_{k=1}^q \Delta \Delta_{j,k} Y &= \sum_{j=1}^p \sum_{k=1}^q \|Y(t_j, s_k) - Y(t_{j-1}, s_k) - Y(t_j, s_{k-1}) + Y(t_{j-1}, s_{k-1})\| \leq \\ &\leq \sum_{j=1}^p \sum_{k=1}^q \left\| \int_{t_{j-1}}^{t_j} (W(\tau, s_k) - W(\tau, s_{k-1})) d\tau \right\| \leq \int_a^b \sum_{k=1}^q \|W(\tau, s_k) - W(\tau, s_{k-1})\| d\tau \leq \\ &\leq \int_a^b \text{var}_a^\tau W(\tau, \cdot) d\tau = \int_a^b p(\tau) \exp\left(2 \int_a^\tau p(\sigma) d\sigma\right) d\tau = M < \infty. \end{aligned}$$

Thus

$$v(Y) = \sup \sum_{j=1}^p \sum_{k=1}^q \Delta \Delta_{j,k} Y \leq M < \infty$$

which completes the proof.

Corollary 1. *There exists $M < \infty$ such that for all $t, s \in [a, b]$*

$$\|Y(t, s)\| + \text{var}_a^b Y(t, \cdot) + \text{var}_a^b Y(\cdot, s) + v(Y) \leq M.$$

Lemma 2. *The function $K(t, s)$ defined by (5) is of strongly bounded variation on $[a, b] \times [a - r, a]$.*

Proof. a) $K(a, \cdot) = 0$ on $[a - r, a]$.

b) Let $\{a = t_0 < t_1 < \dots < t_m = b\}$ be an arbitrary subdivision of $[a, b]$. Then by Corollary 1

$$\begin{aligned} \sum_{j=1}^m \|K(t_j, a) - K(t_{j-1}, a)\| &= \sum_{j=1}^m \left\| \int_{t_{j-1}}^{t_j} Y(t_j, \sigma) P(\sigma, a - \sigma) d\sigma + \right. \\ &\left. + \int_a^{t_{j-1}} (Y(t_j, \sigma) - Y(t_{j-1}, \sigma)) P(\sigma, a - \sigma) d\sigma \right\| \leq M \int_a^b p(\sigma) d\sigma < \infty. \end{aligned}$$

Hence $\text{var}_a^b K(\cdot, a) < \infty$.

c) Given a net type subdivision $\{a = t_0 < t_1 < \dots < t_p = b; a - r = s_0 < s_1 < \dots < s_q = a\}$ of $[a, b] \times [a - r, a]$, we have by Corollary 1

$$\begin{aligned} &\sum_{j=1}^p \sum_{k=1}^q \|K(t_j, s_k) - K(t_{j-1}, s_k) - K(t_j, s_{k-1}) + K(t_{j-1}, s_{k-1})\| = \\ &= \sum_{j=1}^p \sum_{k=1}^q \left\| \int_a^{t_{j-1}} (Y(t_j, \sigma) - Y(t_{j-1}, \sigma)) (P(\sigma, s_k - \sigma) - P(\sigma, s_{k-1} - \sigma)) d\sigma + \right. \\ &\quad \left. + \int_{t_{j-1}}^{t_j} Y(t_j, \sigma) (P(\sigma, s_k - \sigma) - P(\sigma, s_{k-1} - \sigma)) d\sigma \right\| \leq \\ &\leq \int_a^b (\text{var}_a^b Y(\cdot, \sigma) + \sup_{\tau \in [a, b]} \|Y(\tau, \sigma)\|) \text{var}_{-r}^0 P(\sigma, \cdot) d\sigma \leq M \int_a^b p(\sigma) d\sigma < \infty. \end{aligned}$$

Consequently, $v(K) < \infty$ and this completes the proof of Lemma 2.

The following theorem is a direct consequence of Theorem 3,1 from [5] and of Lemma 2.

Theorem. *The Cauchy operator Φ in the variation — of — constants formula (4) is completely continuous.*

References

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