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PRIME SUBGROUPS OF ORDERED GROUPS

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In this paper some concepts known for lattice-ordered groups (l-groups) are generalized to ordered groups (henceforth po-groups) and their properties are investigated.

In the first section prime subgroups of G are studied (for l-groups see e.g. [1], [6], [7]). Theorem 1.5 gives equivalent properties of prime subgroups of the 2-isolated Riesz groups. The second section concerns properties of δ-polars in G – see also [5]. (For the basic properties of polars in l-groups see e.g. [4].) A δ-polar is proved to be a directed convex subgroup (hence a dc-subgroup) and the set of δ-polars in the case of 2-isolated subgroups with the property (II) is shown to be a complete Boolean algebra if it is ordered by inclusion (Theorem 2.6). In Theorem 2.9 a relationship between the dual principal δ-polars and the prime subgroups of G is investigated. In the concluding section the notion of an o-filter and that of an o-antifilter in a po-set is introduced and the connection between the prime subgroups of a 2-isolated Riesz group G and the o-filters of G^+ is described, as well as a property of o-antifilters of the po-set of all the dual principal polars in G. By means of this result it has been possible to generalize some results for l-groups contained in [7]. Throughout this paper the terminology of Fuchs's book [2] has been followed.

Note. A. M. W. GLASS has also studied prime subgroups and polars in Riesz groups (in [9]). However, his results do not coincide with those of this paper.

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1. A directed convex subgroup P of a po-group G will be called *prime* if for any two dc-subgroups A, B of G such that $P \supseteq A \cap B$ it holds $P \supseteq A$ or $P \supseteq B$.

Proposition 1.1. *If P is prime, A, B dc-subgroups of G, $A \supset P$, $B \supset P$, then $A \cap B \supset P$.*

Proof. Clearly $A \cap B \supseteq P$. Let $A \cap B = P$. Then $P \supseteq A \cap B$ and thus $P \supseteq A$ or $P \supseteq B$, a contradiction.

If $x_1, \ldots, x_n$ are elements of a *po*-group G, then we denote $U(x_1, \ldots, x_n) = \{y \in G; y \geqq x_i \text{ for all } i = 1, \ldots, n\}$, $L(x_1, \ldots, x_n) = \{z \in G; z \leqq x_i \text{ for all } i = 1, \ldots, n\}$. For any element $x \in G$ we write $|x| = U(x, -x)$.

Lemma. *Let A be a directed subgroup of a po-group G. Then A is convex in G if and only if it satisfies the following condition: if $a \in A$, $x \in G$, $|x| \supseteq |a|$, then $x \in A$.*

Proof. Let A be a *dc*-subgroup of G, $a \in A$, $x \in G$, $|x| \supseteq |a|$. Since A is directed, there exists $y \in |a| \cap A$. Therefore $y \geqq a$, $y \geqq -a$ and by the assumption also $y \geqq x$, $y \geqq -x$, i.e. $-y \leqq x \leqq y$. Since A is convex, $x \in A$. Conversely, let A be a directed subgroup of G satisfying the given condition. Since A is a directed subgroup, the proof of convexity will be given if we show that $a \in A$, $-a \leqq x \leqq a$ implies $x \in A$ (see [2, I.II.4]). Thus, let $a \in A$, $-a \leqq x \leqq a$. However, this implies that $a \geqq x$, $a \geqq -x$. If now $y \geqq a$, $y \geqq -a$, then $y \geqq x$, $y \geqq -x$. Consequently $|a| \subseteq |x|$ and hence by the assumption $x \in A$.

A *po*-group G will be called 2-*isolated* if it holds: If $a \in G$ satisfies $a \geqq -a$, then $a \geqq 0$.

Note. Any *po*-group with an isolated order (and thus any *l*-group) is 2-isolated. (See Proposition 2.1.)

Proposition 1.2. *Let G be a 2-isolated po-group, $\emptyset \neq A \subseteq G^+$, $[A]$ a subsemigroup of G generated by A. Then $C(A) = \{x \in G; |x| \supseteq |p| = U(p) \text{ for some } p \in [A]\}$ is the smallest dc-subgroup of G containing A.*

Proof. Let us first prove that $C(A)$ is a subgroup of G. Let $x, y \in C(A)$, i.e. $|x| \supseteq |p|$, $|y| \supseteq |q|$, where $p, q \in [A]$. It holds $|x - y| \supseteq |x| + |y| + |x|$. Indeed, if $z \in |x| + |y| + |x|$, then $z = x_1 + y_1 + x_2$, where $x_1, x_2 \in |x|$, $y_1 \in |y|$. Thus $x_1, x_2 \geqq x, -x$; $y_1 \geqq y, -y$. Since G is 2-isolated, $x_1, x_2, y_1 \geqq 0$ and therefore $x_1 + y_1 + x_2 \geqq x - y + 0 = x - y$, $x_1 + y_1 + x_2 \geqq 0 + y - x = y - x$. Thus $z \in |x - y|$, i.e. $|x - y| \supseteq |x| + |y| + |x|$. Since in any *po*-group G $U(a_1) + U(a_2) + \ldots + U(a_n) = U(a_1 + \ldots + a_n)$ for any $a_1, \ldots, a_n \in G$, it follows in our case that $|p| + |q| + |p| = |p + q + p|$. We can write $|x - y| \supseteq |x| + |y| + |x| \supseteq |p| + |q| + |p| = |p + q + p|$, and since $p + q + p \in [A]$ it is also $x - y \in C(A)$. Let us show that $C(A)$ is directed. Let $x, y \in C(A)$. Then $|x| \supseteq |p|$, $|y| \supseteq |q|$, where $p, q \in [A]$. However, $p \in |p|$ and therefore also $p \in |x|$, i.e. $p \geqq x$. Similarly $q \geqq y$. This implies $p + q \geqq x, y$ and since $p + q \in [A]$, $C(A)$ is directed. Finally, if $|g| \supseteq |c|$, where $g \in G$, $c \in C(A)$, then $|g| \supseteq |c| \supseteq |p|$ for some $p \in [A]$ and consequently $g \in C(A)$. Therefore (by Lemma) $C(A)$ is convex.

Corollary 1. *If G is a 2-isolated po-group, $0 < a \in G$, then the dc-subgroup of G generated by a is $C(a) = \{x \in G; |x| \supseteq |na| = U(na) \text{ for a positive integer } n\}$.*

Corollary 2. *Let G be a 2-isolated po-group, M a dc-subgroup of G, $a > 0$. Then* $C(M, a) = \{x \in G; |x| \supseteq |p| \text{ for some } p \in [M^+, a]\}$.

Proof. By Proposition 1.2, $C(M^+, a) = \{x \in G; |x| \supseteq |p| \text{ for some } p \in [M^+, a]\}$. Obviously M is the smallest *dc*-subgroup containing M^+, hence $M \subseteq C(M^+, a)$, which implies $C(M, a) = C(M^+, a)$.

Let $0 < a \in G$, $0 < b \in G$, where G is a *po*-group. We denote $M_{ab} = C(M, a) \cap \cap C(M, b)$. In [3, Theorem 2.3] the set of all *dc*-subgroups of a *po*-group G was proved to form a complete lattice with respect to its order by inclusion, which in the case of Riesz group G (by [8, Satz 9]) is a distributive sublattice of the lattice of all subgroups of G.

Lemma. *Let G be a 2-isolated Riesz group, M a dc-subgroup of G such that any two dc-subgroups A, B of G for which $A \supset M$, $B \supset M$ satisfy $A \cap B \supset M$. If now $a, b \in G^+ \setminus M$, then there exists an element $0 < x \in M_{ab} \setminus M$.*

Proof. It holds $C(M, a) \supset M$, $C(M, b) \supset M$. By our assumption then $M_{ab} = C(M, a) \cap C(M, b) \supset M$. By [8, Hilfssatz 6] M_{ab} is a *dc*-subgroup of G. Therefore M and M_{ab} are directed and hence $M_{ab}^+ \supset M^+$. It follows that there exists $0 < x \in M_{ab} \setminus M$.

Proposition 1.3. *Let G be a 2-isolated Riesz group, M a dc-subgroup of G such that for any two dc-subgroups A, B of G satisfying $A \supset M$, $B \supset M$, $A \cap B \supset M$ holds. If $a, b \in G^+ \setminus M$, then there exists $0 < x \in (M_{ab} \setminus M) \cap L(a, b)$.*

Proof. Let us consider the subsemigroup M_{ab}^+. It holds $M_{ab}^+ = \{x \in G^+; U(x) \supseteq U(p) \text{ for some } p \in [M^+, a], U(x) \supseteq U(q) \text{ for some } q \in [M^+, b]\}$. $a, b \in G^+$ thus $L(a, b) \cap G^+ \neq \emptyset$ and from $a \in [M^+, a]$, $b \in [M^+, b]$ it follows that $L(a, b) \cap G^+ \subseteq M_{ab}^+$. Let us next suppose that $L(a, b) \cap G^+ \subseteq M^+$. Let x be an arbitrary element of M_{ab}^+, consequently $x \leqq m_1 + a + m_2 + a + \ldots + m_{k-1} + a + m_k$, $x \leqq n_1 + b + n_2 + b + \ldots + n_{l-1} + b + n_l$, where m_i $(i = 1, \ldots, k)$, n_j $(j = 1, \ldots, l)$ are elements of M^+. Therefore $x \in L(m_1 + a + m_2 + \ldots + m_k, n_1 + b + n_2 + \ldots + n_l) \cap G^+$. Let us show that $x \in M^+$. Let first $y \in L(m_1 + a + m_2 + a + \ldots + m_k, b) \cap G^+$, hence $0 \leqq y \leqq m_1 + a + m_2 + a + \ldots + m_k$, $y \leqq b$. Since G is a Riesz group, $y = m'_1 + a_1 + m'_2 + a_2 + \ldots + m'_k$ where $0 \leqq m'_i \leqq m_i$ $(i = 1, \ldots, k)$, $0 \leqq a_i \leqq a$ $(i = 1, \ldots, k - 1)$. (By [2, I.V.13].) Since M is convex, $m'_i \in M^+$ $(i = 1, \ldots, k)$. Further, $0 \leqq a_i \leqq y$ $(i = 1, \ldots, k - 1)$, $y \leqq b$ and therefore $0 \leqq a_i \leqq b$. Hence $a_i \in L(a, b) \cap G^+$ $(i = 1, \ldots, k - 1)$, thus by the assumption $a_i \in M^+$ $(i = 1, \ldots, k - 1)$. Evidently this implies that y is the sum of elements of M^+ and consequently also $y \in M^+$. We have $0 \leqq x \leqq m_1 + a + \ldots + m_k$, $0 \leqq x \leqq n_1 + b + \ldots + n_l$. Hence there exist $0 \leqq n'_j \leqq n_j$ $(j = 1, \ldots, l)$, $0 \leqq b_j \leqq b$ $(j = 1, \ldots, l - 1)$ such that $x = n'_1 + b_1 + n'_2 + b_2 + \ldots + n'_l$. The convexity of M implies that $n'_j \in M^+$ $(j = 1, \ldots, l)$. For b_j $(j =$

$= 1, \ldots, l - 1)$ the relation $0 \leqq b_j \leqq x$ holds, consequently $0 \leqq b_j \leqq m_1 + a + \ldots \ldots + m_k$. Then $b_j \in L(m_1 + a + \ldots + m_k, b) \cap G^+$ $(j = 1, \ldots, l - 1)$, and following the preceding part we have $b_j \in M^+$ $(j = 1, \ldots, l - 1)$. Then x is a sum of elements of M^+ and thus $x \in M^+$ which evidently leads to $M_{ab}^+ = M^+$. However, by Lemma there exists then $0 < y \in M_{ab} \setminus M$, a contradiction. Therefore there exists $0 < x \in (M_{ab} \setminus M) \cap L(a, b)$.

Proposition 1.4. *Let G be a po-group, M a dc-subgroup of G such that for any two elements $a, b \in G^+ \setminus M$ there exists $0 < x \in (G^+ \setminus M) \cap L(a, b)$. Then M is prime.*

Proof. Let $M \supseteq A \cap B$, where A, B are *dc*-subgroups of G and let $A \nsubseteq M$, $B \nsubseteq M$. Thus there exist $0 < a \in A$, $0 < b \in B$ such that $a, b \in G^+ \setminus M$. (This follows from the fact that the subgroups A, B, M are directed.) By the assumption there exists $0 < x \in (G^+ \setminus M) \cap L(a, b)$. However, then $0 < x \leqq a$, and therefore $x \in A$. Similarly $x \in B$. Hence $x \in A \cap B$. But this means $A \cap B \nsubseteq M$, a contradiction.

The above reasoning implies

Theorem 1.5. *For a dc-subgroup P of a 2-isolated Riesz group G, the following conditions are equivalent:*

(1) *P is prime.*
(2) *If A, B are dc-subgroups of G, $A \supset P$, $B \supset P$, then $A \cap B \supset P$.*
(3) *If $a, b \in G^+ \setminus P$, then there exists $0 < x \in (P_{ab} \setminus P) \cap L(a, b)$.*
(4) *If $a, b \in G^+ \setminus P$, then there exists $0 < x \in (G^+ \setminus P) \cap L(a, b)$.*

Let now G be a *po*-group, S a convex subsemigroup containing 0 of G^+. S will be called *prime in G^+* if it satisfies the following condition: If Q, R are convex subsemigroups containing 0 of G^+, $S \supseteq Q \cap R$, then $S \supseteq Q$ or $S \supseteq R$. Denote the set of all convex subsemigroups with 0 of G^+ by $\bar{\Gamma} = \bar{\Gamma}(G)$ and the set of all *dc*-subgroups of G by $\Gamma = \Gamma(G)$. In [3, Theorems 2.2, 2.3] it is proved that the sets $\Gamma, \bar{\Gamma}$ ordered by inclusion form complete lattices. Hereby the mapping $\varphi : \Gamma \to \bar{\Gamma}$ given by $A\varphi = A^+$ is an isomorphism of the lattice Γ onto the lattice $\bar{\Gamma}$ and the inverse mapping is given by $S\varphi^{-1} = \langle S \rangle$. ($\langle S \rangle$ will always denote the subgroup of G generated by the set S.) Moreover the infimum in $\bar{\Gamma}$ is determined by the intersection. In the case of a Riesz group the infimum of a finite number of elements of Γ is determined by their intersection as well.

Lemma. *Let G be a Riesz group. Then a dc-subgroup A of G is prime if and only if A^+ is a prime subsemigroup of G^+.*

Proof. Let M be a prime subgroup of G, $M^+ \supseteq A^+ \cap B^+$, where A, B are *dc*-subgroups of G. Then $M = \langle M^+ \rangle \supseteq \langle A^+ \cap B^+ \rangle$. Since G is a Riesz group, $A^+ \cap$

$\cap B^+ = (A \cap B)^+$, it holds $\langle A^+ \cap B^+ \rangle = \langle (A \cap B)^+ \rangle = A \cap B$. Hence $M \supseteq A \cap B$ and therefore $M \supseteq A$ or $M \supseteq B$. Hence also $M^+ \supseteq A^+$ or $M^+ \supseteq B^+$. Conversely, let M^+ be a prime subsemigroup of G^+, $M \supseteq A \cap B$, where A, B are *dc*-subgroups of G. Then $M^+ \supseteq (A \cap B)^+ = A^+ \cap B^+$ and hence $M^+ \supseteq A^+$ or $M^+ \supseteq B^+$. Consequently also $M \supseteq A$ or $M \supseteq B$.

Theorem 1.6. *Let G be a Riesz group. Then for any prime subgroup M of G there exists a minimal prime subgroup A such that $A \subseteq M$.*

Proof. Let us show that for any prime subsemigroup X of G^+ there exists a minimal prime subsemigroup Y such that $Y \subseteq X$. Let now S_λ $(\lambda \in \Lambda)$ be a decreasing chain of prime subsemigroups of G^+ and let $S = \bigcap_{\lambda \in \Lambda} S_\lambda$. S is a convex subsemigroup with 0 of G^+. Let us show that it is prime. Let $S \supseteq Q \cap R$, where Q, R are convex subsemigroups with 0 of G^+. Then any S_λ $(\lambda \in \Lambda)$ fulfils $S_\lambda \supseteq Q \cap R$, hence every S_λ satisfies $S_\lambda \supseteq Q$ or $S_\lambda \supseteq R$. If every S_λ satisfies both $S_\lambda \supseteq Q$ and $S_\lambda \supseteq R$, then $S \supseteq Q$, $S \supseteq R$. Let λ_0 be such that $S_{\lambda_0} \supseteq Q$, $S_{\lambda_0} \not\supseteq R$. Then, of course, $S_\lambda \supseteq Q$ holds also for $S_\lambda \subseteq S_{\lambda_0}$. Hence $S = \bigcap_{\lambda \in \Lambda} S_\lambda \supseteq Q$ holds. This means that the set of all prime subsemigroups of G is inductive, thus there exists for any prime subsemigroup X a minimal prime one contained in X. Since the lattices Γ and $\bar{\Gamma}$ are isomorphic, there exists also (by Lemma) for any prime subgroup A of G a minimal prime one contained in A.

2. In this section we shall study δ-polars in a *po*-group G. A *δ-polar* (see also [5]) is a generalization of a polar in an l-group. Let us first introduce some concepts and notations. We shall subject G to the following conditions:

(I) *For each $x \in G$, $|x| \neq \emptyset$ holds.*

(II) *For each $x \in G$ there exists $x \vee -x$. ($x \vee -x$ denotes* $\sup(x, -x)$ *in G.)*

(M′) *If $a, b, x \in G$ satisfy $a, b \geqq x, -x, 0$, then there exists $r \in G$ such that* $a, b \geqq r \geqq x, -x, 0$. (See [5].)

G is said to be *regular* if the existence of $\inf(x, y)$ in G^+ implies the existence of $\inf(x, y)$ in G for $x, y \in G^+$. Clearly now $c = \inf_{G^+}(x, y)$ implies $c = \inf_G(x, y)$.

Proposition 2.1. *A po-group with an isolated order is 2-isolated.*

Proof. Let $a \geqq -a$, and consequently $2a \geqq 0$. Since G is isolated, $a \geqq 0$.

Proposition 2.2. *Any Riesz group satisfies the condition* (M′) *and is regular.*

Proof. Let $\inf_{G^+}(x, y) = c$ and $x, y \geqq a$. Then there exists $b \in G$ such that $x, y \geqq b \geqq c, a$. Since $b \geqq 0$, $c = \inf_{G^+}(x, y) \geqq b \geqq a$ and therefore $\inf_{G^+}(x, y) = \inf_G(x, y)$.

Lemma. *If a po-group G is 2-isolated and if there exists $x \vee -x$ for an element $x \in G$, then $x \vee -x \geqq 0$.*

Proof. It holds $x \vee -x \geqq x$, $x \vee -x \geqq -x$, and consequently also $x \geqq$ $\geqq -(x \vee -x)$. Hence $x \vee -x \geqq x \geqq -(x \vee -x)$ and because of G being 2-isolated, we have $x \vee -x \geqq 0$.

Proposition 2.3. *A 2-isolated po-group G satisfying the property* (II) *has the property* (M').

Proof. Let $a, b, x \in G$ such that $a, b \geqq x, -x, 0$. Then $a, b \geqq x \vee -x \geqq x, -x$ and by Lemma $x \vee -x \geqq 0$ holds.

Let now G be a *po*-group, $x, y \in G$. x, y will be called *disjoint* (notation $x \,\delta\, y$), if there exist $a, b \in G^+$ such that $a \in |x|$, $b \in |y|$, $a \wedge b = 0$. ($a \wedge b$ denotes $\inf(a, b)$ in G.) We denote for $\emptyset \neq A \subseteq G$, $A^\delta = \{x \in G : a \,\delta\, x \text{ for all } a \in A\}$. If $A^\delta \neq \emptyset$, then it will be called a *δ-polar* of the set A. We denote $A^{\delta\delta} = (A^\delta)^\delta$ for $A^\delta \neq \emptyset$. If $A^\delta \neq \emptyset$, then $A \subseteq A^{\delta\delta}$. If $\emptyset \neq A \subseteq G$, $\emptyset \neq B \subseteq G$ such that $A^\delta \neq \emptyset \neq B^\delta$, then $A \subseteq B$ implies $B^\delta \subseteq A^\delta$. Clearly then $A^\delta = A^{\delta\delta\delta}$ for $A^\delta \neq \emptyset$. Further, $\emptyset \neq A \subseteq G$ is a δ-polar in G if and only if $A = A^{\delta\delta}$.

Remark 1. *If a 2-isolated po-group G satisfies the condition* (I), *then $A^\delta \neq \emptyset$ for any $\emptyset \neq A \subseteq G$.*

Proof. Since G satisfies (I), $|a| \neq \emptyset$ for each $a \in G$ and since G is 2-isolated it follows that $|a| \subseteq G^+$. However, then $u \wedge 0 = 0$ for each $u \in |a|$.

Remark 2. *It is obvious that the notion of δ-polars and that of polars in l-groups coincide.*

Proposition 2.4. *If G is a Riesz group or a 2-isolated po-group with the property* (II) *then any δ-polar in G is a convex subgroup of G.*

Proof. Since in both cases G satisfies the condition (M'), the proposition holds by [5, Hilfssatz 12].

Let now G be a *po*-group, $a \in G$. Denote $a^+ = U(a, 0)$, $a^- = -U(-a, 0)$.

Lemma 1. *Let G be a po-group, $a \in G$. Then for each element $x \in a^+$ there exists $y \in a^-$ such that $a = x + y$.*

Proof. Let $x \in a^+$. Then there exists $y \in G$ satisfying $a = x + y$, i.e. $y = -x + a$. Further, $-x \leqq 0$, $-x \leqq -a$ and therefore $-x + a \leqq a$, $-x + a \leqq 0$ which means that $y \in a^-$.

Lemma 2. *Let G be a 2-isolated po-group with the property* (II). *Then a subgroup A is a dc-subgroup if and only if it satisfies the following condition*: *if $a \in A$, $x \in G$, $|x| \supseteq |a|$, then $x \in A$.*

Proof. Let A be a subgroup with the given property. Let $0 \leqq x \leqq a \in A$. Then $|x| \supseteq |a|$ and thus $x \in A$. Consequently A is convex. Since G has the property (II), there exists $a \vee -a$ for each element $a \in A$. Since G is 2-isolated it follows $a \vee -a \in a^+$. Clearly $|a| = |a \vee -a|$, hence by the assumption $a \vee -a \in A$. By Lemma 1 there exists $y \in a^-$ such that $a = (a \vee -a) + y$. A is a subgroup, hence $y \in A$. The element a can be therefore expressed as a difference of two positive elements of A, which implies that A is directed. In any *po*-group the converse inclusion holds in accordance with the Lemma of Proposition 1.2.

Proposition 2.5. *A δ-polar of a 2-isolated po-group G with the property* (II) *is a dc-subgroup of G.*

Proof. Let A^δ be a δ-polar. Let us show that $C(A^{\delta+}) = A^\delta$. Let then $x \in C(A^{\delta+}) = \{y \in G : |y| \supseteq |p| \text{ for some } p \in A^{\delta+}\}$. By Lemma 2, $x \in A^\delta$ holds. Conversely, if $x \in A^\delta$, then for each $a \in A$ there exist $x_1 \in |x|$, $a_1 \in |a|$ satisfying $x_1 \wedge a_1 = 0$. In G there exists $x_0 = x \vee -x = \inf |x|$. Let us show $x_0 \in A^{\delta+}$. Clearly $x_0 \in |x|$. Thus $0 \leqq x_0 \wedge a_1 \leqq x_1 \wedge a_1 = 0$ for $x_0 \in |x_0|$, $a_1 \in |a|$. Therefore $x_0 \in A^{\delta+}$. From this we obtain $x \in C(A^{\delta+})$.

Theorem 2.6. *Any δ-polar of a 2-isolated po-group G with the property* (II) *is a dc-subgroup of G. The set $\Delta = \Delta(G)$ of all δ-polars in G is a complete Boolean algebra with respect to its order by inclusion. An infimum in Δ is formed by intersection.*

Proof. Clearly δ is a symmetric binary relation which is antireflexive. (A relation δ is *antireflexive* if from $x \,\delta\, x$ for an $x \in G$ follows $x \,\delta\, y$ for each $y \in G$.) Define a relation $\prec$ on G as follows: $x \prec y \Leftrightarrow |x| \supseteq |y|$. The relation $\prec$ is a quasiorder, i.e., it is reflexive and transitive. The smallest element in this quasiorder is 0. Indeed, if $a \in |x|$, then $a \geqq 0$, i.e. $a \in |0|$. Consequently $|0| \supseteq |x|$ for all $x \in G$.

To prove that the set Δ ordered by inclusion is a complete Boolean algebra with the infimum in the form of intersection, it suffices by [5, p. 85] to show that the relations δ and $\prec$ satisfy the following conditions $(x, y, z \in G)$:

1. $x \,\delta\, y,\ x \prec y \Rightarrow x \prec 0$;
2. $x \,\delta\, y,\ z \prec y \Rightarrow x \,\delta\, z$;
3. x non δ $y \Rightarrow$ there exists $z \in G$ such that z non $\prec 0$, $z \prec x$, $z \prec y$.

1. Let $x \,\delta\, y$, $x \prec y$. Then there exist $a \in |x|$, $b \in |y|$ such that $a \wedge b = 0$. Since $|x| \supseteq |y|$, $a, b \in |x|$ and therefore $0 = a \wedge b \geqq x, -x$. Consequently $x = 0$, that is $|x| = |0|$. Hence $x \prec 0$.

2. Let $x \,\delta\, y$, $z \prec y$. It means that there exist $a \in |x|$, $b \in |y|$ such that $a \wedge b = 0$ and at the same time $|z| \supseteq |y|$. Thus $b \in |z|$ and $a \wedge b = 0$ for $a \in |x|$, $b \in |z|$, hence $x \,\delta\, z$.

3. Since G satisfies (II), there exists $x \vee -x$ for each $x \in G$. If $x \,\delta\, y$, then there exist $a \in |x|$, $b \in |y|$ satisfying $a \wedge b = 0$. Then for $x_0 = x \vee -x$, $y_0 = y \vee -y$ we have $0 \leqq x_0 \leqq a$, $0 \leqq y_0 \leqq b$ and therefore $x_0 \wedge y_0 = 0$. Consequently, for a 2-isolated *po*-group with the property (II) $x \,\delta\, y$ holds if and only if $(x \vee -x) \wedge \wedge (y \vee -y) = 0$. If x non δ y, then there exists $0 < c \in G$ such that $x \vee -x \geqq \geqq c$, $y \vee -y \geqq c$. Hence $|c| \supseteq |x|$, $|c| \supseteq |y|$, $|c| \supsetneqq |0|$, i.e. $c \prec x$, $c \prec y$, c non $\prec 0$.

Proposition 2.7. *Let G be a 2-isolated po-group with the property* (II), A_λ^δ $(\lambda \in \Lambda)$ *δ-polars in G. Then*

$$\bigwedge_{\lambda\in\Lambda} A_\lambda^\delta = \bigcap_{\lambda\in\Lambda} A_\lambda^\delta = (\bigcup_{\lambda\in\Lambda} A_\lambda)^\delta, \quad \bigvee_{\lambda\in\Lambda} A_\lambda^\delta = (\bigcup_{\lambda\in\Lambda} A_\lambda^\delta)^{\delta\delta}.$$

Proof. 1. The proposition concerning the supremum follows from the fact that for any δ-polar A^δ in G, $A^{\delta\delta}$ is the intersection of all δ-polars in G containing A.

2. Let $y \in (\bigcup_{\lambda\in\Lambda} A_\lambda)^\delta$. Thus for each $a \in \bigcup_{\lambda\in\Lambda} A_\lambda$ there exist elements $a_1 \in |a|$, $y_1 \in |y|$ such that $a_1 \wedge y_1 = 0$, therefore $y \in \bigcap_{\lambda\in\Lambda} A_\lambda^\delta$. The converse inclusion follows from the properties of complements in a Boolean algebra and from the proposition on the supremum:

$$\bigwedge_{\lambda\in\Lambda} A_\lambda^\delta = (\bigvee_{\lambda\in\Lambda} A_\lambda^{\delta\delta})^\delta = (\bigcup_{\lambda\in\Lambda} A_\lambda^{\delta\delta})^{\delta\delta\delta} = (\bigcup_{\lambda\in\Lambda} A_\lambda^{\delta\delta})^\delta \subseteq (\bigcup_{\lambda\in\Lambda} A_\lambda)^\delta.$$

Let G be a *po*-group. Any δ-polar $a^\delta = \{a\}^\delta$ in G where $a \in G$, will be called a *dual principal δ-polar*; similarly a δ-polar $a^{\delta\delta} = \{a\}^{\delta\delta}$ will be called a *principal δ-polar.* The set of all dual principal δ-polars in G will be denoted by $\Pi^\delta(G)$.

Lemma. *If in a 2-isolated po-group G there exists $x \vee -x$ for an element $x \in G$, then $x^\delta = (x \vee -x)^\delta$.*

Proof. Let $y \in x^\delta$, then there exist $y_1 \in |y|$, $x_1 \in |x|$ such that $x_1 \wedge y_1 = 0$. From $x \vee -x \geqq 0$ it follows that $(x \vee -x) \wedge y_1 = 0$ and hence $y \in (x \vee -x)^\delta$. Conversely, let $z \in (x \vee -x)^\delta$, then there exist $z_1 \in |z|$, $x_2 \in |x \vee -x|$ such that $z_1 \wedge x_2 = 0$. Since $x_2 \geqq x \vee -x$, it is $x_2 \geqq x, -x$. Consequently, $z \in x^\delta$.

Proposition 2.8. *If G is a 2-isolated po-group with the property* (II), *then for each two elements $a, b \in G$*

$$a^\delta \cap b^\delta = [(a \vee -a) + (b \vee -b)]^\delta.$$

Thus the set $\Pi^\delta(G)$ ordered by inclusion is a $\wedge$-subsemilattice of the lattice $\Delta(G)$.

Proof. Because of the Lemma it suffices to consider positive elements. Let $a, b \in \in G^+$. If $x \in (a + b)^\delta$, there exist $x_1 \in |x|$, $c_1 \in |a + b|$ such that $x_1 \wedge c_1 = 0$. Since $c_1 \geqq a + b$ it is also $c_1 \geqq a, b$. Therefore $x \in a^\delta \cap b^\delta$.

Conversely, let $y \in a^\delta \cap b^\delta$, i.e., let $y_1, y_2 \in |y|$, $a_1 \in |a|$, $b_1 \in |b|$ exist such that $y_1 \wedge a_1 = y_2 \wedge b_1 = 0$. Since $y_0 = y \vee -y \geqq 0$, $y_0 \wedge a_1 = y_0 \wedge b_1 = 0$. $a \wedge \wedge c = b \wedge c = 0$ implies $(a + b) \wedge c = 0$ for any *po*-group G and $a, b, c \in G$. (See [5, Hilfssatz 2].) Thus in our case $y_0 \wedge (a_1 + b_1) = 0$ and since $a_1 + b_1 \geqq \geqq a + b$, $y \in (a + b)^\delta$.

We shall now point to the connection between the dual principal δ-polars and the prime subgroups.

Theorem 2.9. *Let G be a 2-isolated Riesz group, P a prime subgroup of G. Then it holds: If $a \in G \setminus P$, then $a^\delta \subseteq P$.*

Proof. Let P be prime, $a \notin P$. If $u \in a^\delta$, then there exist $u_1 \in |u|$, $a_1 \in |a|$ such that $u_1 \wedge a_1 = 0$, hence $L(u_1, a_1) \cap G^+ = 0$. If $u \neq 0$, then $u_1 > 0$, $a_1 > 0$. Further $-a_1 \leqq a \leqq a_1$ and since $a \notin P$, $a_1 \notin P$. Let us show that $u_1 \in P$. Indeed, if $u_1 \notin P$, then by Theorem 1.5 $L(u_1, a_1) \cap G^+ \neq 0$, hence $u_1 \wedge a_1 = 0$ could not be valid. Since $-u_1 \leqq u \leqq u_1$, $u \in P$.

Note. For *l*-groups the converse implication is satisfied as well. (See [6, Theorem 2.3].) It remains an open problem whether the converse theorem for 2-isolated Riesz groups is valid or not.

3. In this section we introduce the notion of an *o-filter* and that of an *o-antifilter* of *po*-sets and we shall point out their connection with prime subgroups of *po*-groups. We shall thus generalize some results valid for *l*-groups treated by F. Šik in [6] and [7].

Let M be a *po*-set. A subset $\emptyset \neq F \subseteq M$ will be called an *o-filter* of M, if

1. $L(a, b) \cap F \neq \emptyset$ for each $a, b \in F$, i.e., F is *l*-directed.
2. If $a \in F$, $x \in M$ such that $a \leqq x$, then $x \in F$.
3. The smallest element of M (if it exists) does not belong to F.

An *o-antifilter* of M is defined dually.

Evidently, in lattices we obtain in this way filters and antifilters in the ordinary sense.

o-filters and *o*-antifilters exist in any ordered set M that contains elements different from the smallest one. If, for example, $a \in M$ is different from the smallest element, then the set $F_a = \{x \in M;\ a \leqq x\}$ $(A_a = \{y \in M;\ y \leqq a\})$ is an *o*-filter (an *o*-antifilter) of M. The maximal elements in a by inclusion ordered set of all *o*-filters (*o*-antifilters) of a *po*-set M will be called *o-ultrafilters* (*o-ultraantifilters*). Clearly, a set-union of an increasing chain of *o*-filters (*o*-antifilters) of M is again an *o*-filter (an *o*-antifilter). Therefore any *o*-filter (*o*-antifilter) of M is contained in an *o*-ultrafilter (*o*-ultraantifilter) of this set.

The following theorem is a consequence of Theorem 1.5.

Theorem 3.1. *Let G be a 2-isolated Riesz group, P a dc-subgroup of G. Then P is prime if and only if $G^+ \setminus P$ is an o-filter of the po-set G^+.*

Proof. Let P be prime, $a, b \in G^+ \setminus P$. Then there exists $0 < x \in L(a, b) \cap \cap (G^+ \setminus P)$, hence $G^+ \setminus P$ is l-directed. Let $a \in G^+ \setminus P$, $x \in G^+$, $a \leqq x$. If $x \in P^+$, then $0 \leqq a \leqq x$, hence $a \in P^+$, a contradiction. Finally, 0 is the smallest element in G^+, $0 \in P$, hence $0 \notin G^+ \setminus P$. This means that $G^+ \setminus P$ is an o-filter of G^+. Conversely, let $G^+ \setminus P$ be an o-filter of G^+, $a, b \in G^+ \setminus P$. Then $(G^+ \setminus P) \cap L(a, b) \neq \emptyset$ and consequently there exists $0 < x \in (G^+ \setminus P) \cap L(a, b)$. This means that P is prime.

Theorem 3.2. *Let G be a 2-isolated Riesz group and P a dc-subgroup of G such that $G^+ \setminus P$ is an o-ultrafilter of G^+. Then P is minimal prime.*

Proof. If there exists Q prime, $Q \subseteq P$, then $G^+ \setminus P \subseteq G^+ \setminus Q$. $G^+ \setminus P$, $G^+ \setminus Q$ are by Theorem 3.1 o-filters of G^+ and since $G^+ \setminus P$ is an o-ultrafilter, $G^+ \setminus P = G^+ \setminus Q$ must hold. Hence $P^+ = Q^+$ and since P, Q are directed, $P = Q$. Consequently, P is minimal prime.

Note. For l-groups $\neq 0$ the converse implication is satisfied, too. (See [7, Theorem 7.5].) It remains an open question whether the converse theorem for the 2-isolated Riesz groups is valid or not.

Theorem 3.3. *Let G be a 2-isolated Riesz group, $\Pi^\delta(G)$ the set of all dual principal polars in G ordered by inclusion. Let $\mathbf{x}$ be an o-antifilter of $\Pi^\delta(G)$. Then $\bigcup \mathbf{x}$ is a dc-subgroup of G.*

Note. $\bigcup \mathbf{x}$ is a set-union of elements of $\mathbf{x}$ considered as subsets of G.

Proof of Theorem 3.3. Let $x, y \in \bigcup \mathbf{x}$. Then there exist $a^\delta, b^\delta \in \mathbf{x}$ such that $x \in a^\delta$, $y \in b^\delta$. Since δ-polars are dc-subgroups of G it holds: if $c^\delta \supseteq a^\delta$, $c^\delta \supseteq b^\delta$, then $c^\delta \supseteq \langle a^\delta, b^\delta \rangle$. Since $\mathbf{x}$ is an o-antifilter of $\Pi^\delta(G)$, there exists at least one δ-polar $d^\delta \in \mathbf{x}$ such that $d^\delta \supseteq \langle a^\delta, b^\delta \rangle$. (Since $U(a^\delta, b^\delta) \cap \mathbf{x} \neq \emptyset$.) Then $\langle a^\delta, b^\delta \rangle \subseteq \bigcup \mathbf{x}$ and hence $\bigcup \mathbf{x}$ is a subgroup of G. Further, let again $x, y \in \bigcup \mathbf{x}$, $x \in a^\delta$, $y \in b^\delta$, $a^\delta, b^\delta \in \mathbf{x}$. If $z \in G$, $x \leqq z \leqq y$, $d^\delta \supseteq \langle a^\delta, b^\delta \rangle$, $d^\delta \in \mathbf{x}$, then $x, y \in d^\delta$, hence $z \in \bigcup \mathbf{x}$. It means that $\bigcup \mathbf{x}$ is convex. Finally, since d^δ is directed, there exists $z \in d^\delta$, $z \geqq x$, $z \geqq y$ and therefore $\bigcup \mathbf{x}$ is directed as well.

Note. Some results of this paper concern the class $\mathfrak{G}$ of all 2-isolated po-groups with the property (II). Let us show that the class $\mathfrak{L}$ of all l-groups is a proper subclass of $\mathfrak{G}$.

Let Z be the linearly ordered additive group of all integers and let G be the subgroup of the direct sum $Z \oplus Z$ that is formed by exactly all elements $(a, b) \in Z \oplus Z$, where $a + b$ is an even number. In [5, Beispiel III] F. Šik has proved that G is not a Riesz group (thus G is not an l-group). Hence for $(a, b) \geqq (-a, -b)$ it holds $(a, b) \geqq (0, 0)$ and for every $x = (a, b)$ there exists $x \vee -x = (|a|, |b|)$.

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