Miroslav Fiedler

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*Czechoslovak Mathematical Journal*, Vol. 24 (1974), No. 3, 392–402

Czechoslovak Mathematical Journal, 24 (99) 1974, Praha

# ADDITIVE COMPOUND MATRICES AND AN INEQUALITY FOR EIGENVALUES OF SYMMETRIC STOCHASTIC MATRICES

MIROSLAV FIEDLER, Praha

(Received January 30, 1973)

**1. Introduction.** In [2], the notion of measure of irreducibility $\mu(A)$ of a doubly stochastic matrix $A$ was introduced and the distance of the "nonstochastic" eigenvalues of $A$ from 1 was estimated from below by a function of $\mu(A)$.

F. SALZMANN [7] observed that if a nonnegative matrix has trace zero and multiple Perron root then a proper non-void subset of its eigenvalues has sum zero. In the present paper, we shall find a quantitative extension of this fact for symmetric stochastic matrices using the mentioned measure $\mu(A)$. In the proof, properties of so called additive compound matrices will play a substantial role. Since these matrices seem to be of interest for themselves, a brief sketch of their theory is included. Related questions have been studied in [4].

**2. Generalized compound matrices.** Let $X$ be an $n$-dimensional vector space over a field $K$. Let $k$ be a fixed integer, $1 \leqq k \leqq n$. Denote by $\Lambda^{(k)}X$, the $k$-th exterior power of $X$, the vector space of all $k$-vectors, i.e. of all linear combinations (over $K$) of exterior products of $k$ vectors in $X$:

$$x_1 \wedge x_2 \wedge \ldots \wedge x_k\,, \quad x_j \in X\,, \quad j = 1, \ldots, k\,.$$

For the exterior products, the following rules are assumed: (cf. [1]):

R 1 If $P = (j_1, \ldots, j_k)$ is any permutation of indices $1, \ldots, k$, $\sigma(P)$ is the sign of $P$ and $x_i \in X$, $i = 1, \ldots, k$, then

$$x_{j_1} \wedge x_{j_2} \wedge \ldots \wedge x_{j_k} = \sigma(P)\, x_1 \wedge x_2 \wedge \ldots \wedge x_k\,;$$

R 2
$$(x_1 + y_1) \wedge x_2 \wedge \ldots \wedge x_k =$$
$$= x_1 \wedge x_2 \wedge \ldots \wedge x_k + y_1 \wedge x_2 \wedge \ldots \wedge x_k\,, \quad x_i \in X\,, \quad y_1 \in X\,;$$

R 3
$$(\lambda x_1) \wedge x_2 \wedge \ldots \wedge x_k = \lambda(x_1 \wedge x_2 \wedge \ldots \wedge x_k)\,, \quad x_i \in X\,, \quad \lambda \in K\,;$$

R 4 There exist linearly independent vectors $x_1, x_2, \ldots, x_n$ in $X$ such that the $\binom{n}{k}$ vectors

$$x_{i_1} \wedge x_{i_2} \wedge \ldots \wedge x_{i_k}, \quad 1 \leqq i_1 < i_2 < \ldots < i_k \leqq n,$$

are linearly independent in $\Lambda^{(k)}X$.

From these rules the following propositions follow easily:

P 1 *The space $\Lambda^{(k)}X$ has dimension $\binom{n}{k}$ and, for any basis $e_1, \ldots, e_n$ of $X$ the $k$-vectors*

$$e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k}, \quad 1 \leqq i_1 < i_2 < \ldots < i_k \leqq n,$$

*form a basis of $\Lambda^{(k)}X$.*

P 2 *If $x_i \in X$, $i = 1, \ldots, k$, then $x_1 \wedge x_2 \wedge \ldots \wedge x_k$ is zero in $\Lambda^{(k)}X$ iff $x_1, \ldots, x_k$ are linearly dependent in $X$.*

P 3 *If $X'$ is a dual space to $X$ with respect to a bilinear form $\langle x, x' \rangle$ then $\Lambda^{(k)}X'$ is dual to $\Lambda^{(k)}X$ with respect to the bilinear form determined by*

$$\langle x_1 \wedge \ldots \wedge x_k, x_1' \wedge \ldots \wedge x_k' \rangle = \det(\langle x_i, x_j' \rangle).$$

*Moreover, if $e_1, \ldots, e_n$ and $e_1', \ldots, e_n'$ are dual bases (i.e. $\langle e_i, e_j' \rangle = \delta_{ij}$) then $e_{i_1} \wedge \ldots \wedge e_{i_k}$, $e_{i_1}' \wedge \ldots \wedge e_{i_k}'$, $1 \leqq i_1 < \ldots < i_k \leqq n$, form also dual bases.*

P 4 *If $X$ is unitary (over the field of complex numbers) with the inner product $(x, y)$ then $\Lambda^{(k)}X$ is also unitary with respect to the inner product determined by*

$$(x_1 \wedge \ldots \wedge x_k, y_1 \wedge \ldots \wedge y_k) = \det((x_i, y_j)).$$

Let now $Y$ be an $m$-dimensional vector space over the same field $K$ and assume that the integer $k$ also satisfies $1 \leqq k \leqq m$. Let $\mathscr{A}$ be a linear operator from $X$ into $Y$; we shall write this $\mathscr{A} \in L(X, Y)$. Then the $k$-th compound operator $\mathscr{A}^{(k)}$ is the linear operator in $L(\Lambda^{(k)}X, \Lambda^{(k)}Y)$ defined by

$$\mathscr{A}^{(k)}(x_1 \wedge x_2 \wedge \ldots \wedge x_k) = \mathscr{A}x_1 \wedge \mathscr{A}x_2 \wedge \ldots \wedge \mathscr{A}x_k$$

for any $k$ vectors $x_1, \ldots, x_k$ in $X$.

The following propositions follow then easily:

P 5 *If $k$ does not exceed dimensions of any of the spaces $X, Y, Z$ and if $\mathscr{A} \in L(X, Y)$, $\mathscr{B} \in L(Y, Z)$ then*

$$(\mathscr{B}\mathscr{A})^{(k)} = \mathscr{B}^{(k)}\mathscr{A}^{(k)}.$$

P 6 *If $\mathscr{I} \in L(X, X)$ is the identity operator then $\mathscr{I}^{(k)} \in L(\Lambda^{(k)}X, \Lambda^{(k)}X)$ is the identity operator.*

P 7 *If $\mathscr{A} \in L(X, X)$ is nonsingular then so is $\mathscr{A}^{(k)}$ and*

$$(\mathscr{A}^{(k)})^{-1} = (\mathscr{A}^{-1})^{(k)} .$$

P 8 *If $\mathscr{A}^T \in L(Y', X')$ is the transpose operator to $\mathscr{A} \in L(X, Y)$, i.e. $\langle \mathscr{A}x, y' \rangle = \langle x, \mathscr{A}^T y' \rangle$ for all $x \in X$, $y' \in Y'$, then $(\mathscr{A}^T)^{(k)} \in L(\Lambda^{(k)}Y', \Lambda^{(k)}X')$ is the transpose operator to $\mathscr{A}^k \in L(\Lambda^{(k)}X, \Lambda^{(k)}Y)$.*

P 9 *If $X$ is a unitary space and $\mathscr{A} \in L(X, X)$ is symmetric (unitary, normal) then $\mathscr{A}^{(k)} \in L(\Lambda^{(k)}X, \Lambda^{(k)}X)$ is symmetric (unitary, normal) as well.*

Let now $\mathscr{A} \in L(X, X)$ where $X$ is a general $n$-dimensional vector space. If $1 \leqq k \leqq n$ and $m = 0, 1, 2, \ldots, k$, we define *the generalized k-th compound operators* $\mathscr{A}_m^{(k)} \in L(\Lambda^{(k)}X, \Lambda^{(k)}X)$ as linear operators determined by

$$\mathscr{A}_m^{(k)}(x_1 \wedge x_2 \wedge \ldots \wedge x_k) = \sum_{\substack{(\varepsilon_1, \ldots, \varepsilon_k),\, \varepsilon_i \in \{0, 1\} \\ \sum_1^k \varepsilon_i = m}} \mathscr{A}^{\varepsilon_1}_{x_1} \wedge \mathscr{A}^{\varepsilon_2}_{x_2} \wedge \ldots \wedge \mathscr{A}^{\varepsilon_k}_{x_k}$$

(where $\mathscr{A}^0 = \mathscr{I}$, the identity).

In particular, $\mathscr{A}_k^{(k)}$ is the $k$-th compound operator, $\mathscr{A}_0^{(k)} = \mathscr{I}^{(k)}$. The operator $\mathscr{A}_1^{(k)}$ will be called *additive k-th compound operator* and denoted by $\mathscr{A}^{[k]}$.

The following propositions are immediate:

P 10 *For $\mathscr{A} \in L(X, X)$, $\mathscr{B} \in L(X, X)$, we have*

$$(\mathscr{A} + \mathscr{B})^{[k]} = \mathscr{A}^{[k]} + \mathscr{B}^{[k]} .$$

P 11 *For k fixed, the operators $\mathscr{A}_m^{(k)}$ commute with each other.*

P 12 *If $\mathscr{A} \in L(X, X)$, $\mathscr{B} \in L(X, X)$ are commutative, $\mathscr{A}\mathscr{B} = \mathscr{B}\mathscr{A}$, then all $\mathscr{A}_{m_1}^{(k)}$, $\mathscr{B}_{m_2}^{(k)}$ commute with each other.*

P 13 *If $X$ is unitary and $\mathscr{A} \in L(X, X)$ is symmetric (normal) then $\mathscr{A}_m^{(k)}$ are all symmetric (normal).*

Let us proceed to square matrices. Since any square matrix can be considered as matrix of a linear operator with respect to a basis, assume that $X$ is an $n$-dimensional vector space, $X'$ its dual with respect to a bilinear form $\langle x, x' \rangle$ and let $e_1, \ldots, e_n$, $e'_1, \ldots, e'_n$ be dual bases. If $\mathscr{A} \in L(X, X)$, its matrix $A = (a_{ik})$ where, as usually, $a_{ij} = \langle \mathscr{A}e_j, e'_i \rangle$. We can then define analogously the $k$-th compound matrix $A^{(k)}$

of a matrix $A$ as matrix of the operator $\mathscr{A}^{(k)}$ with respect to the basis $e_{i_1} \wedge e_{i_2} \wedge \ldots \ldots \wedge e_{i_k}$, $1 \leqq i_1 < i_2 < \ldots < i_k \leqq n$. Similarly, the generalized $k$-th compound matrices of $A$, denoted by $A_m^{(k)}$, are matrices of $\mathscr{A}_m^{(k)}$ with respect to the same basis as before.

Let us show that $A^{(k)}$ coincides with the usual $k$-th compound matrix of $A$ (see e.g. [5]). We shall use abbreviations $(i) = \{i_1, \ldots, i_k\}$, $1 \leqq i_1 < i_2 < \ldots < i_k \leqq n$, $(j) = \{j_1, \ldots, j_k\}$, $1 \leqq j_1 < j_2 \ldots < j_k \leqq n$; $A_{(i),(j)}^{(k)}$ will then denote the entry of $A^{(k)}$ with row "index" $(i)$ and column "index" $(j)$.

P 14 *We have*

$$A_{(i),(j)}^{(k)} = \det A((i); (j))$$

*where $A((i); (j))$ is the submatrix of $A$ consisting of rows with indices $i_1, \ldots, i_k$ and columns with indices $j_1, \ldots, j_k$.*

Proof. By P 3,

$$\begin{aligned} A_{(i),(j)}^{(k)} &= \langle \mathscr{A}^{(k)}(e_{j_1} \wedge \ldots \wedge e_{j_k}), e'_{i_1} \wedge \ldots \wedge e'_{i_k} \rangle = \\ &= \langle \mathscr{A} e_{j_1} \wedge \ldots \wedge \mathscr{A} e_{j_k}, e'_{i_1} \wedge \ldots \wedge e'_{i_k} \rangle = \\ &= \det (\langle \mathscr{A} e_{j_p}, e'_{i_q} \rangle) = \det (a_{i_q j_p}) = \det A((i); (j)) \,. \end{aligned}$$

From general properties of matrices of linear operators and from P 5, P 6 and P 7 the well known properties follow:

P 15 *If $A$ and $B$ are $n \times n$ matrices then*

$$(AB)^{(k)} = A^{(k)} B^{(k)} \,,$$

*if $I$ is the $n$-rowed identity matrix then $I^{(k)}$ is the $\binom{n}{k}$-rowed identity matrix and if $A$ is nonsingular then $A^{(k)}$ is nonsingular and*

$$(A^{(k)})^{-1} = (A^{-1})^{(k)} \,.$$

Let us order now the vectors $e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k}$ according to the lexicographical ordering of indices. Then the following proposition holds:

P 16 *If the matrix $A = (a_{ij})$ of $\mathscr{A} \in L(X, X)$ is triangular (i.e. $a_{ij} = 0$ for $i > j$) then all generalized compound matrices $A_m^{(k)}$ are triangular as well. The diagonal entry of $A_m^{(k)}$ corresponding to the index $(i) = (i_1, \ldots, i_k)$, $1 \leqq \leqq i_1 < \ldots i_k \leqq n$, is equal to $E_m(a_{i_1 i_1}, \ldots, a_{i_k i_k})$ where $E_m(\xi_1, \ldots, \xi_k)$ denotes the $m$-th elementary symmetric function of $\xi_1, \ldots, \xi_k$.*

Proof. Let $(i) = \{i_1, \ldots, i_k\}$, $(j) = \{j_1, \ldots, j_k\}$, $1 \leqq i_1 < \ldots < i_k \leqq n$, $1 \leqq \leqq j_1 < \ldots < j_k \leqq n$. Then the entry of $A_m^{(k)}$ in the position (i), (j) is

$$(A_m^{(k)})_{(i),(j)} = \langle \mathscr{A}_m^{(k)}(e_{j_1} \wedge \ldots \wedge e_{j_k}),\ e'_{i_1} \wedge \ldots \wedge e'_{i_k} \rangle =$$

$$= \sum_{\substack{(\varepsilon_1,\ldots,\varepsilon_k),\ \varepsilon_i \in \{0,1\} \\ \sum_1^k \varepsilon_i = m}} \langle \mathscr{A}^{\varepsilon_1} e_{j_1} \wedge \ldots \wedge \mathscr{A}^{\varepsilon_k} e_{j_k},\ e'_{i_1} \wedge \ldots \wedge e'_{i_k} \rangle =$$

$$= \sum_{\substack{(\varepsilon_1,\ldots,\varepsilon_k),\ \varepsilon_i \in \{0,1\} \\ \sum_1^k \varepsilon_i = m}} \det \left( \langle \mathscr{A}^{\varepsilon_q} e_{j_q},\ e'_{i_p} \rangle \right)$$

where $(\langle \mathscr{A}^{\varepsilon_q} e_{j_q}, e'_{i_p} \rangle)$ are $k \times k$ matrices, $p, q = 1, \ldots, k$.

Let now $(i) > (j)$ in the lexicographical ordering. Then there exists an index $s$, $1 \leqq s < n$, such that $i_1 \geqq j_1, \ldots, i_{s-1} \geqq j_{s-1}$, $i_s > j_s$. Denote, for a moment, by $\alpha_{pq}$, $p, q = 1, \ldots, k$, the entries of one such matrix $(\langle \mathscr{A}^{\varepsilon_q} e_{j_q}, e'_{i_p} \rangle)$. We shall show that $\alpha_{pq} = 0$ whenever $p \geqq s \geqq q$ which will imply, by the Laplace expansion theorem, singularity of this matrix. Thus if $p \geqq s \geqq q$ then

$$i_p \geqq i_s > j_s \geqq j_q\,.$$

If $\varepsilon_q = 0$ then $\alpha_{pq} = \langle e_{j_q}, e'_{i_p} \rangle = 0$. If $\varepsilon_q = 1$ then $\alpha_{pq} = \langle \mathscr{A} e_{j_q}, e'_{i_p} \rangle = a_{i_p j_q} = 0$ since $i_p > j_q$.

Consequently, $(A_m^{(k)})_{(i),(j)} = 0$ and $A_m^{(k)}$ is triangular.

To prove the last assertion, let $(i) = (j)$. Then, by triangularity of $A$ and $A_m^{(k)}$,

$$(A_m^{(k)})_{(i),(i)} = \sum_{\substack{(\varepsilon_1,\ldots,\varepsilon_k),\ \varepsilon_i \in \{0,1\} \\ \sum_1^k \varepsilon_i = m}} \det \left( \langle \mathscr{A}^{\varepsilon_q} e_{i_q},\ e'_{i_p} \rangle \right) =$$

$$= \sum_{\substack{(\varepsilon_1,\ldots,\varepsilon_k),\ \varepsilon_i \in \{0,1\} \\ \sum_1^k \varepsilon_i = m}} \prod_{p=1}^{k} a_{i_p i_p}^{\varepsilon_p} = E_m(a_{i_1 i_1}, \ldots, a_{i_k i_k})$$

(by definition, $a_{kk}^0 = 1$). The proof is complete.

We are able now to prove

**Theorem 2,1.** *Let $A$ be an $n \times n$ matrix over a field $K$ with eigenvalues $\alpha_1, \ldots, \alpha_n$. Let $k, m$ be integers, $1 \leqq k \leqq n$, $0 \leqq m \leqq k$. Then the generalized $k$-th compound matrix $A_m^{(k)}$ has eigenvalues $E_m(\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_k})$, $1 \leqq i_1 < i_2 < \ldots < i_k \leqq n$.*

Proof. In a suitable extension field $K'$, $A$ can be expressed in the form

$$A = STS^{-1}$$

where $S$ is nonsingular and $T$ triangular with diagonal entries $\alpha_1, \ldots, \alpha_n$. Hence in an $n$-dimensional vector space $X$ over $K'$, there exist two bases $e_1, \ldots, e_n$ and $\tilde{e}_1, \ldots, \tilde{e}_n$, and in $L(X, X)$ a linear operator $\mathscr{A}$ such that $A$ is the matrix of $\mathscr{A}$ with respect to the basis $e_1, \ldots, e_n$ and $T$ is the matrix of $\mathscr{A}$ with respect to $\tilde{e}_1, \ldots, \tilde{e}_n$. The generalized $k$-th compound operator $\mathscr{A}_m^{(k)} \in L(\Lambda^{(k)}X, \Lambda^{(k)}X)$ has then matrices $A_m^{(k)}$ with respect to the basis $e_{i_1} \wedge \ldots \wedge e_{i_k}$ and $T_m^{(k)}$ with respect to the basis $\tilde{e}_{i_1} \wedge \ldots \wedge \tilde{e}_{i_k}$. By P 16, $T_m^{(k)}$ is triangular and its eigenvalues, equal to diagonal entries, are $E_m(\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_k})$, $1 \leqq i_1 < i_2 < \ldots < i_k \leqq n$. Thus $\mathscr{A}_m^{(k)}$ as well as $A_m^{(k)}$ has these eigenvalues. The proof is complete.

This Theorem can be generalized as follows:

**Theorem 2,2.** *Let $A$ be an $n \times n$ matrix with eigenvalues $\alpha_1, \ldots, \alpha_n$, $1 \leqq k \leqq n$. Let $S(\xi_1, \ldots, \xi_k)$ be a polynomial symmetric function of $\xi_1, \ldots, \xi_k$. If $S$ is expressed (which is, according to a well known theorem, always possible) as a polynomial in the elementary symmetric functions $E_1, \ldots, E_k$:*

$$S = \Phi(E_1, E_2, \ldots, E_k),$$

*then the matrix polynomial*

$$\Phi(A_1^{(k)}, A_2^{(k)}, \ldots, A_k^{(k)})$$

*is a matrix the eigenvalues of which are exactly all numbers $S(\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_k})$, $1 \leqq i_1 < i_2 < \ldots < i_k \leqq n$.*

Proof. If $A$ is triangular, the assertion follows immediately from P 16 and from the fact that the diagonal entries of $\Phi(A_1^{(k)}, \ldots, A_k^{(k)})$ are $\Phi(E_1(\alpha_{i_1}, \ldots, \alpha_{i_k}), \ldots, \ldots, E_k(\alpha_{i_1}, \ldots, \alpha_{i_k})) = S(\alpha_{i_1}, \ldots, \alpha_{i_k})$. The general case follows from the decomposition (in a suitable extension field) $A = STS^{-1}$ where $S$ is nonsingular and $T$ triangular. The proof is complete.

**Corollary 2,3.** *Let $A$ be an $n \times n$ matrix with eigenvalues $\alpha_1, \ldots, \alpha_n$. Then the matrix $(A^{[2]})^2 - 4A^{(2)}$ (or $(A_1^{(2)})^2 - 4A_2^{(2)}$) has eigenvalues $(\alpha_i - \alpha_j)^2$, $1 \leqq i < < j \leqq n$.*

Remark. This matrix can be called *discriminant of $A$* since its nonsingularity is equivalent to simplicity of all eigenvalues of $A$.

Proof is immediate.

In the sequel, we shall be interested in the $k$-th additive compound matrix $A^{[k]}$ only. The following Theorem presents the complete description of $A^{[k]}$ if $A$ is given.

**Theorem 2,4.** *Let $A = (a_{ij})$ be an $n \times n$ matrix. Let $1 \leqq k \leqq n$. Then $A^{[k]}$ has the following entry $A_{(i),(j)}^{[k]}$ in the row corresponding to the set $(i) = \{i_1, \ldots, i_k\}$ and column corresponding to $(j) = \{j_1, \ldots, j_k\}$, $1 \leqq i_1 < \ldots < i_k \leqq n$, $1 \leqq j_1 < \ldots \ldots < j_k \leqq n$:*

$$A^{[k]}_{(i),(j)} = \sum_{s=1}^{k} a_{i_s i_s} \qquad \text{if } \operatorname{card}((i) \cap (j)) = k \ (\text{i.e. if } (i) = (j));$$

$$A^{[k]}_{(i),(j)} = (-1)^{\sigma} a_{pq} \qquad \text{if } \operatorname{card}((i) \cap (j)) = k - 1, \ \text{where } p = (i) \setminus ((i) \cap (j)), \ q = (j) \setminus ((i) \cap (j)) \text{ and } \sigma \text{ is the number of elements in } (i) \cap (j) \text{ between } p \text{ and } q;$$

$$A^{[k]}_{(i),(j)} = 0 \qquad \text{if } \operatorname{card}((i) \cap (j)) \leqq k - 2.$$

Proof. Let $X$ be an $n$-dimensional vector space over the field containing all elements of $A$, $e_1, \ldots, e_n$ a basis of $X$, $\mathscr{A}$ the operator which has $A$ as its matrix with respect to this basis. Then $A^{[k]}$ is the matrix, with respect to the basis $e_{i_1} \wedge \ldots \wedge e_{i_k}$, $1 \leqq i_1 < \ldots < i_k \leqq n$, of the operator $\mathscr{A}_1^{(k)}$ defined above. Then

$$\begin{aligned} \mathscr{A}_1^{(k)}(e_{j_1} \wedge \ldots \wedge e_{j_k}) = {} & \mathscr{A}e_{j_1} \wedge e_{j_2} \wedge \ldots \wedge e_{j_k} + \\ & + e_{j_1} \wedge \mathscr{A}e_{j_2} \wedge \ldots \wedge e_{j_k} + \\ & + \ldots\ldots\ldots\ldots\ldots\ldots + \\ & + e_{j_1} \wedge e_{j_2} \wedge \ldots \wedge \mathscr{A}e_{j_k} \end{aligned}$$

and we obtain the entry $A^{[k]}_{(i),(j)}$ as the coefficient at the term $e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k}$ in the expansion of the right hand side as a linear combination of the vectors $e_{p_1} \wedge \ldots \wedge e_{p_k}$, $1 \leqq p_1 < \ldots < p_k \leqq n$. Since $\mathscr{A}e_{j_s} = \sum_{i=1}^{n} a_{ij_s} e_i$, $s = 1, \ldots, k$, the cases $(i) = (j)$ and $\operatorname{card}((i) \cap (j)) \leqq k - 2$ are clear. Let now $\operatorname{card}((i) \cap (j)) = k - 1$, $(i) \cap (j) = \{s_1, \ldots, s_{k-1}\}$, $1 \leqq s_1 < \ldots < s_{k-1} \leqq n$, $(i) = \{s_1, \ldots, s_v, p, s_{v+1}, \ldots, s_{k-1}\}$, $(j) = \{s_1, \ldots, s_w, q, s_{w+1}, \ldots, s_{k-1}\}$. We obtain the coefficient $A^{[k]}_{(i),(j)}$ from the term

$$e_{s_1} \wedge \ldots \wedge e_{s_w} \wedge \mathscr{A}e_q \wedge e_{s_{w+1}} \wedge \ldots \wedge e_{s_{k-1}}$$

only, in particular from the term $a_{pq}e_p$ of $\mathscr{A}e_q$. We obtain thus

$$\begin{aligned} & a_{pq}(e_{s_1} \wedge \ldots \wedge e_{s_w} \wedge e_p \wedge e_{s_{w+1}} \wedge \ldots \wedge e_{s_{k-1}}) = \\ & = (-1)^{\sigma} a_{pq}(e_{s_1} \wedge \ldots \wedge e_{s_v} \wedge e_p \wedge e_{s_{v+1}} \wedge \ldots \wedge e_{s_{k-1}}) \end{aligned}$$

where $\sigma$ is the number of indices $s_t$ between $p$ and $q$. The proof is complete.

**Corollary 2,5.** *Let $A = (a_{ik})$ be a real $n \times n$ matrix, $1 \leqq k \leqq n$. Let $M = \{i_1, \ldots, i_k\}$, $1 \leqq i_1 < \ldots < i_k \leqq n$. Then the sum of squares of all the off-diagonal entries of the matrix $A^{[k]}$ in the row corresponding to indices in $M$ is equal to*

$$\sum_{i \in M, j \notin M} a_{ij}^2 .$$

Proof. It follows from Theorem 2,4 that the non-zero off-diagonal entries of the row of $A^{[k]}$ corresponding to $M$ are of the form $\pm a_{pq}$ where $p \in M$ and $q \notin M$, each pair $(p, q)$ occurring exactly once.

**3. An application to symmetric stochastic matrices.** In this section, all vectors and matrices will be real. We shall be using vector and matrix norms (see e.g. [3]). By the norm $\|x\|$ of a column vector $x = (x_1, \ldots, x_n)^T$ we shall mean the euclidean norm, i.e. $\|x\| = (\sum_1^n x_i^2)^{1/2}$. If $C$ is an $n \times n$ matrix then its norm $\|C\|$ is, as usual, defined as $\sup_{\|x\| \leqq 1} \|Cx\|$ which is well known to be equal to the nonnegative square root of the maximum eigenvalue of $CC^T$.

Let us also recall that a stochastic matrix is a nonnegative square matrix with all row-sums equal to one, a doubly stochastic matrix is a stochastic matrix with all column-sums equal to one. For such a doubly stochastic $n \times n$ matrix $A = (a_{ij})$, the measure of irreducibility $\mu(A)$ was defined in [2] as

$$\mu(A) = \min_{\emptyset \neq M \subset N, M \neq N} \sum_{i \in M, j \notin M} a_{ij}$$

where $N = \{1, 2, \ldots, n\}$. It was proved in [2] that every nonstochastic eingevalue $\lambda$ of $A$ satisfies the inequality

$$(1) \qquad |1 - \lambda| \geqq 2\left(1 - \cos\frac{\pi}{n}\right)\mu(A) .$$

We shall need the following particular case of a theorem by L. Mirsky [6]:

*Let $A, B$ be symmetric $n \times n$ matrices with eigenvalues $\alpha_1 \geqq \ldots \geqq \alpha_n$, $\beta_1 \geqq \ldots \ldots \geqq \beta_n$ respectively. Then,*

$$\max_{i=1,\ldots,n} |\alpha_i - \beta_i| \leqq \|A - B\| .$$

From this we shall derive easily

**Corollary 3,1.** *Let* $A = \begin{pmatrix} a_{11} & a_1^T \\ a_1 & A_{22} \end{pmatrix}$ *be a symmetric $n \times n$ matrix in a partitioned form, $A_{22}$ being $(n-1) \times (n-1)$. Then at least one eigenvalue of $A$ is contained in the interval*

$$|a_{11} - x| \leqq \|a_1\| .$$

Proof. We put

$$B = \begin{pmatrix} a_{11} & 0 \\ 0 & A_{22} \end{pmatrix}.$$

Then $A - B = C$ where

$$C = \begin{pmatrix} 0 & a_1^T \\ a_1 & 0 \end{pmatrix}.$$

Since

$$CC^T = \begin{pmatrix} a_1^T a_1 & 0 \\ 0 & a_1 a_1^T \end{pmatrix}$$

has eigenvalues $\|a_1\|^2, \|a_1\|^2, 0, \dots, 0, \|A - B\| = \|a_1\|$. The matrix $B$ has an eigenvalue $\beta = a_{11}$. It follows from the preceding theorem that for some eigenvalue $\alpha$ of $A$

$$|\alpha - \beta| \leqq \|A - B\| ,$$

i.e.

$$|a_{11} - \alpha| \leqq \|a_1\| .$$

The proof is complete.

We shall prove now the following generalization of this Corollary:

**Theorem 3,2.** *Let $A = (a_{ik})$ be a symmetric $n \times n$ matrix. Let $\emptyset \neq M \subsetneqq N = \{1, 2, \dots, n\}$, card $M = m$. Then there exist $m$ eigenvalues $\lambda_1, \dots, \lambda_m$ of $A$ such that*

$$\left| \sum_{i \in M} a_{ii} - \sum_{i=1}^{m} \lambda_i \right| \leqq \Big( \sum_{i \in M, j \notin M} a_{ij}^2 \Big)^{1/2} .$$

Proof. It suffices to prove this for $M = \{1, \dots, m\}$, $1 \leqq m \leqq n - 1$. Then the additive $m$-th compound matrix $A^{[m]}$ has, according to Theorem 2,4 and Corollary 2,5, $\sum_{i \in M} a_{ii}$ as its upper left corner entry and $\sum_{i \in M, j \notin M} a_{ij}^2$ as the sum of squares of all the off-diagonal entries in the first row. Since $A^{[m]}$ is symmetric by P 13, it follows from Corollary 3,1 that there exists an eigenvalue $\omega$ of $A^{[m]}$ in the interval

$$\left| \sum_{i \in M} a_{ii} - x \right| \leqq \Big( \sum_{i \in M, j \notin M} a_{ij}^2 \Big)^{1/2} .$$

However, all eigenvalues of $A^{[m]}$ are sums of $m$ eigenvalues of $A$ by Theorem 2,1. The proof is complete.

In the following two theorems, the matrix $A$ will be a stochastic symmetric matrix (and therefore doubly stochastic).

**Theorem 3,3.** *Let $A = (a_{ij})$ be a symmetric stochastic $n \times n$ matrix with eigenvalues $\lambda_1, \dots, \lambda_n$. Then there exist two non-void proper subsets $M_1, M_2$ of $N = \{1, \dots, n\}$ with the same number of elements such that*

$$0 \leqq \sum_{i \in M_1} a_{ii} - \sum_{i \in M_2} \lambda_i \leqq \mu(A) .$$

Proof. According to the definition of $\mu(A)$ there exists a non-void proper subset $M_0$ of $N$ such that

$$\mu(A) = \sum_{i \in M_0, j \notin M_0} a_{ij} .$$

By Theorem 3,2, there exists a subset $M$ of $N$, with the same number of elements as $M_0$, such that

$$\left|\sum_{i\in M_0} a_{ii} - \sum_{i\in M} \lambda_i\right| \leqq \left(\sum_{i\in M_0, j\notin M_0} a_{ij}^2\right)^{1/2} \leqq \sum_{i\in M_0, j\notin M_0} a_{ij} = \mu(A)\,.$$

If $\sum_{i\in M_0} a_{ii} - \sum_{i\in M} \lambda_i \geqq 0$, we put $M_1 = M_0$, $M_2 = M$; in the other case, we put $M_1 = N \setminus M_0$, $M_2 = N \setminus M$. The proof is complete.

**Theorem 3,4.** *Let* $1 = \lambda_1 \geqq \lambda_2 \geqq \ldots \geqq \lambda_n$ *be the eigenvalues of a symmetric* $n \times n$ *stochastic matrix. Then*

$$\lambda_1 - \lambda_2 + 2\left(1 - \cos\frac{\pi}{n}\right)\left(\sum_{i\in N} \lambda_i - \min_{M\neq N, \emptyset\neq M\subset N, \sum_{i\in M}\lambda_i \geqq 0} \sum_{i\in M} \lambda_i\right) \geqq 0\,.$$

Proof. Let $A = (a_{ik})$ be that symmetric stochastic matrix. By Theorem 3.3, there exist two non-void proper subsets $M_1$, $M_2$ of $N$ with the same number of elements such that

$$0 \leqq \sum_{i\in M_1} a_{ii} - \sum_{i\in M_2} \lambda_i \leqq \mu(A)\,.$$

This implies

$$\sum_{i\in M_2} \lambda_i \leqq \sum_{i\in M_1} a_{ii} \leqq \sum_{i\in N} a_{ii} = \sum_{i\in N} \lambda_i$$

and

$$\sum_{i\in M_2} \lambda_i + \mu(A) \geqq \sum_{i\in M_1} a_{ii} \geqq 0\,.$$

Thus, there exists a non-void proper subset of $N$, namely $M_2$, such that

$$\sum_{i\in N} \lambda_i \geqq \sum_{i\in M_2} \lambda_i \geqq -\mu(A)\,,$$

which is equivalent to

$$0 \leqq \sum_{i\in N\setminus M_2} \lambda_i \leqq \sum_{i\in N} \lambda_i + \mu(A)\,.$$

Hence also

$$\min_{M\neq N, \emptyset\neq M\subset N, \sum_{i\in M}\lambda_i \geqq 0} \sum_{i\in M} \lambda_i \leqq \sum_{i\in N} \lambda_i + \mu(A)\,.$$

By (1),

$$2\left(1 - \cos\frac{\pi}{n}\right)\mu(A) \leqq \lambda_1 - \lambda_2\,.$$

Combining these two inequalities, we obtain the result of the Theorem.

Remark. The modified Salzman's example [7] stating that no stochastic $5 \times 5$ matrix can have eigenvalues $1, 1, -\frac{2}{3}, -\frac{2}{3}, -\frac{2}{3}$ can be extended by Theorem 3,4 for symmetric stochastic matrices as follows:

None of the 5-tuples 1, $1 - \alpha_1$, $-\frac{2}{3} - \alpha_2$, $-\frac{2}{3} - \alpha_3$, $-\frac{2}{3} - \alpha_4$ with $|\alpha_1| + |\alpha_2| + |\alpha_3| + |\alpha_4| < \frac{2}{15}(3 - 2\cos\frac{1}{5}\pi) \doteq 0.184$ consists of eigenvalues of a symmetric stochastic $5 \times 5$ matrix.

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*Author's address:* 115 67 Praha 1, Žitná 25, ČSSR (Matematický ústav ČSAV).