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SPLITTING PROPERTY OF LATTICE ORDERED GROUPS

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Let A be an archimedean linearly ordered group. In [7] it was proved that A is a direct factor of each archimedean lattice ordered group that contains it as an l-ideal. CONRAD [3] defined an archimedean lattice ordered group G to have the splitting property if G is a direct factor of each archimedean l-group H such that G is an l-ideal of H; he proved that if G is an archimedean l-group then the l-group $((G^d)^\wedge)^l$ has the splitting property (for any archimedean lattice ordered group X we denote by X^d, $X^\wedge$ and X^l the divisible hull, the Dedekind completion and the lateral completion of X, respectively). In this note it is shown that a complete lattice ordered group G has the splitting property if and only if it is laterally complete, i.e., if $G = G^l$ (§2). In particular, we obtain the result of Conrad as a corollary. It is proved that every archimedean orthocomplete l-group [1] has the splitting property.

In §3 there is investigated the splitting property of singular lattice ordered groups. We prove that each laterally complete singular l-group is complete and hence that it has the splitting property.

In §4 we show that every laterally complete l-group G contains an l-subgroup which is the greatest l-subgroup of G with respect of being convex in G, complete and having the splitting property. Further it will be proved that if G is archimedean and orthocomplete, then its Dedekind closure $G^\wedge$ is laterally complete (hence $G^\wedge$ has the splitting property). Theorem 8 dealing with subdirect products of l-groups G_i that have the splitting property enables one to construct examples of l-groups which have the splitting property without being laterally complete.

1. PRELIMINARIES

For the terminology, cf. BIRKHOFF [2] and FUCHS [5]. Let us recall some notions we shall use in the sequel. Let A be a lattice ordered group. A subset $M \subset A$ is disjoint if $|m_1| \wedge |m_2| = 0$ for any two distinct elements $m_1, m_2 \in M$ and $m > 0$ for each $m \in M$. The lattice ordered group A is said to be laterally complete (or orthogonally complete) if each disjoint subset $M \subset G^+$ has the least upper bound. For any set $Z \subset A$ we put

$$Z^\delta = \{a \in A : |a| \wedge |z| = 0 \text{ for each } z \in Z\}.$$

Z^δ is called the polar of the set Z in A. Each polar is a convex l-subgroup of A (cf. [12]).

Let B and C be l-subgroups of A such that (i) the group A is the direct product of B and C in the group-theoretical sense, and (ii) for $b \in B$, $c \in C$ we have $b + c \geqq 0$ if and only if $b \geqq 0$ and $c \geqq 0$. Under these assumptions the lattice ordered group A is said to be the direct product of the l-groups B and C and this is denoted by writting $A = B \otimes C$. The l-subgroups B, C are called direct factors of A. The direct factor B is uniquelly determined by A, namely, $B = A^\delta$. For $a \in A$ we denote by aB the component of the element a in the direct factor B. If A is a complete l-group, then by the Theorem of Riesz ([2], Chap. XIII) $A = M^{\delta\delta} \otimes M^\delta$ for each $M \subset A$.

From the definition of the direct product of lattice ordered groups and from the fact that any two direct decompositions of G have a common refinement (cf. [2]) follow the assertions: If $G = A \otimes B$ and C is a direct factor of G, then $C = (A \cap C) \otimes (B \cap C)$. If P, Q are direct factors of G such that $P \cap Q = \{0\}$, then there is a direct factor R of G with $G = P \otimes Q \otimes R$. For each $x \in G$ and any pair of direct factors A, B of G with $A \subset B$ we have $(xB)\,A = xA$.

For each $g \in G$, the principal polar $[g]$ of G generated by the element g is defined to be the set $\{g\}^{\delta\delta}$. The l-group G is called orthocomplete [1] if it is laterally complete and each principal polar of G is a direct factor of G.

An element $0 \leqq s \in A$ is called singular if $t \wedge (s - t) = 0$ for each $t \in A$ with $0 \leqq t \leqq s$ (cf. [4]). The lattice ordered group A is said to be singular if for each $0 < a \in A$ there is a singular element $s \in A$ such that $0 < s \leqq a$.

2. ORTHOCOMPLETE LATTICE ORDERED GROUPS

In this paragraph we assume that $G \neq \{0\}$ is an archimedean lattice ordered group and that G is an l-ideal of an archimedean lattice ordered group H. We denote by $H^\wedge$ the Dedekind completion of H. The lattice ordered group $H^\wedge$ is complete and we can suppose that H is an l-subgroup of $H^\wedge$; for each element $0 < k \in H^\wedge$ there is a subset $M \subset H^+$ such that $\sup M = k$.

For $M \subset H^\wedge$ and $N \subset G$ the symbol M^δ or N^β, respectively, denotes the polar of M in $H^\wedge$ or the polar of N in G. For $y \in H^\wedge$ resp. $x \in G$ we put $\{y\}^{\delta\delta} = [y]$, $\{x\}^{\beta\beta} = [x]'$.

Lemma 1. *Assume that each principal polar of G is a direct factor of G. Let $0 < e \in G$, $y \in H^\wedge$, $0 \leqq y \leqq e$, $e_1 = e[y]$. Then $e_1 \in G$.*

Proof. Let E be the set of all elements $x \in G$ such that (i) $0 \leqq x \leqq e$, and (ii) $x \wedge y = 0$. At first suppose that $E = \{0\}$. Then $\{y\}^\delta \subset \{e\}^\delta$ hence $e \in [y]$ and so $e[y] = e \in G$. Now assume that $E \neq \{0\}$ and let $E_1 = \{x_i : i \in I\}$ be a maximal disjoint subset of E. According to the assumption, $\{x_i\}^{\beta\beta}$ is a direct factor of G; let $e_i = e[x_i]'$. Then the set $\{e_i\}$ $(i \in I)$ is disjoint and because G is laterally complete there exists the least upper bound e_2 of the set $\{e_i\}$ in G. Obviously $e_i \leqq e$, hence $\bigvee e_i = e_3$ does exist in $H^\wedge$ and $e_3 \leqq e_2$.

We have $0 \leqq e_3 \leqq e$ and $y \wedge e_3 = 0$, hence $e_3 \leqq e[y]^\delta$. Assume that $e_3 < e[y]^\delta$. Consider the interval

$$[0, e] = \{t \in H^\wedge : 0 \leqq t \leqq e\}\,.$$

The lattice $[0, e]$ is distributive and for each $i \in I$, the element $e([x_i]')^\beta$ is a complement of e_i in the lattice $[0, e]$, thus e_i belongs to the center C of $[0, e]$. Moreover, the lattice $[0, e]$ is infinitely distributive and hence C is a closed sublattice of $[0, e]$ (cf. [6]), therefore $e_3 \in C$. Clearly $e[y]^\delta$ also belongs to C. Because C is a Boolean algebra there exists $0 < e_4 \in C$ such that

$$e_3 \wedge e_4 = 0\,, \quad e_3 \vee e_4 = e[y]^\delta\,.$$

Since $0 < e_4 \in H^\wedge$, there is a subset $M \subset H^+$ with $\sup M = e_4$, hence there is $0 < x_0 \in H$ such that $x_0 \leqq e_4$. With respect to $e_4 \leqq e[y]^\delta \leqq e$ we have $0 < x_0 \leqq e$; because $e \in G$ and G is an l-ideal of H, we infer that $x_0 \in G$. Further we have $e_4 \wedge e_i = 0$ and hence $e_4 \wedge x_i = 0$ for each $i \in I$. In fact, from $x_i \leqq e$ it follows

$$x_i = x_i[x_i] \leqq e[x_i] = e_i\,,$$

therefore $0 \leqq e_4 \wedge x_i \leqq e_4 \wedge e_i$. In the same time, $e_4 \leqq e$ and $e_4 \wedge y = 0$ because $e_4 \in [0, e[y]^\delta] \subset [y]^\delta$ thus $x_0 \in E$. This is a contradiction with the maximality of the set $\{x_i : i \in I\}$. Therefore $e_3 = e[y]^\delta$. Because $e_3 \leqq e_2 \leqq e[y]^\delta$ we obtain $e[y]^\delta = e_3 = e_2 \in G$. From this it follows $e_1 = e - e[y]^\delta \in G$.

An element $0 < e$ of an l-group X is called a weak unit of X if $e \wedge x > 0$ for each $0 < x \in X$. If $0 < y \in X$, then y is a weak unit of $[y]$ and for each weak unit z of $[y]$ we have $[z] = [y]$.

Lemma 2. *Let G be orthocomplete, $0 < k \in H$. There exists $0 \leqq g_0 \in G$ such that $g \wedge (k - g_0) = 0$ for each $0 \leqq g \in G$.*

Proof. By the Axiom of choice, there exists a maximal disjoint subset M of G. Because G is laterally complete, $\sup M = e$ exists in G. It is easy to verify that e is a weak unit of G. Denote

$$(e - k) \vee 0 = y_1\,.$$

Then $0 \leqq y_1 \leqq e$. Because $H^\wedge$ is a complete lattice ordered group, $[y_1]$ is a direct summand of $H^\wedge$. Put $e_1 = e[y_1]$. According to Lemma 1, $e_1 \in G$. The element e_1 is a weak unit of $[y_1]$; in fact, if $0 < x \in [y_1]$, then $e \geqq x \wedge y_1 = x_1 > 0$, hence $x_1 \in \in [y_1]$ and therefore

$$e_1 \wedge x \geqq e_1 \wedge x_1 = e[y_1] \wedge x_1[y_1] = (e \wedge x_1)[y_1] = x_1[y_1] = x_1 > 0 .$$

From this we obtain

$$(1) \qquad [y_1] = [e_1] .$$

For any element t of a lattice ordered group we have $(t \vee 0) \wedge ((-t) \vee 0) = 0$, hence

$$(k - e) \vee 0 \in [y_1]^\delta$$

and therefore $((k - e) \vee 0)[y_1] = 0$. Thus according to (1),

$$0 = ((k - e) \vee 0)[e_1] = (k[e_1] - e_1) \vee 0$$

and hence

$$(2) \qquad k[e_1] \leqq e_1 .$$

Further we have $e = e[e_1] + e[e_1]^\delta$, thus

$$(3) \qquad e[e_1]^\delta = e - e_1 .$$

Since $y_1 \in [e_1]$, from $y_1 = (e - k) \vee 0$ we get

$$0 = y_1[e_1]^\delta = (e[e_1]^\delta - k[e_1]^\delta) \vee 0 ,$$

therefore

$$(4) \qquad e[e_1]^\delta < k[e_1]^\delta .$$

Assume that for a positive integer n we have constructed elements $e_1, \ldots, e_n \in G$ such that for each $i \in \{1, \ldots, n\}$

$$(5) \qquad e_i \wedge e_j = 0 \quad \text{for} \quad j \in \{1, \ldots, n\} , \quad i \neq j ,$$

$$(6) \qquad e_i = e[e_i] ,$$

$$(7) \qquad k[e_i] \leqq i e_i ,$$

$$(8) \qquad i[e_i'] \leqq k[e_i^0]^\delta ,$$

where

$$(8a) \qquad e_i^0 = e_1 \vee \ldots \vee e_i , \quad e_i' = e - e_i^0 .$$

For $n = 1$, the relation (5) holds trivially, (6) follows from the definition of e_1, (7) is fulfilled because of (2); the relation (8) is implied by (3) and (4). If $e_n^0 = e$, we put $e_i = 0$ for each positive integer $i > n$. Suppose that $e_n^0 < e$. Define

$$((n + 1)\, e_n' - k) \vee 0 = y_{n+1}\,. \tag{9}$$

We have $0 \leqq y_{n+1} \leqq (n + 1)\, e_n'$. By using Theorem of Riesz we obtain

$$H^\wedge = [e_i] \otimes [e_i]^\delta$$

for $i = 1, \ldots, n$. From (5) it follows $[e_i] \cap [e_j] = \{0\}$ for $i, j \in \{1, \ldots, n\}$, $i \neq j$ and therefore

$$H^\wedge = [e_1] \otimes [e_2] \otimes \ldots \otimes [e_n] \otimes P\,,$$

where $P = [e_1]^\delta \cap [e_2]^\delta \cap \ldots \cap [e_n]^\delta$. The element e_n^0 is a weak unit of $[e_1] \otimes \otimes [e_2] \otimes \ldots \otimes [e_n]$, whence $[e_1] \otimes \ldots \otimes [e_n] = [e_n^0]$ and $P = [e_n^0]^\delta$. Thus

$$e[e_n^0] = e[e_1] + \ldots + e[e_n] = e_1 + \ldots + e_n = e_n^0\,,$$
$$e[e_n^0]^\delta = e - e[e_n^0] = e_n'\,.$$

Therefore $e_n^0 \wedge e_n' = 0$ and from this it follows

$$e_n^0 \wedge y_{n+1} = 0\,. \tag{10}$$

Put $e_{n+1} = e[y_{n+1}]$. By Lemma 1, $e_{n+1} \in G$. Then e_{n+1} is a weak unit of $[y_{n+1}]$, hence $[y_{n+1}] = [e_{n+1}]$ and so

$$e_{n+1} = e[e_{n+1}]\,. \tag{6'}$$

From (10) we obtain

$$e_i \wedge e_{n+1} = 0 \quad \text{for} \quad i = 1, \ldots, n\,. \tag{5'}$$

In view of (9) we have

$$(k - (n + 1)\, e_n') \vee 0 \in [y_{n+1}]^\delta = [e_{n+1}]^\delta\,,$$
$$0 = ((k - (n + 1)\, e_n' \vee 0)\,[e_{n+1}] = (k[e_{n+1}] - (n + 1)\, e_n'[e_{n+1}]) \vee 0\,,$$
$$k[e_{n+1}] \leqq (n + 1)\, e_n'[e_{n+1}]\,. \tag{11}$$

Since $e_n' = e - (e_1 \vee \ldots \vee e_n) = e - (e_1 + \cdots + e_n)$ and $e_i[e_{n+1}] = 0$ for $i = 1, \ldots, n$ because of (5'), we get $e_n'[e_{n+1}] = e[e_{n+1}] = e_{n+1}$. Thus by (11),

$$k[e_{n+1}] \leqq (n + 1)\, e_{n+1}\,. \tag{7'}$$

From $0 = y_{n+1}[y_{n+1}]^\delta = y_{n+1}[e_{n+1}]^\delta$ and from (9) we obtain

$$0 = ((n + 1)\, e'_n[e_{n+1}]^\delta - k[e_{n+1}]^\delta) \vee 0\,,$$

(12) $$(n + 1)\, e'_n[e_{n+1}]^\delta \leqq k[e_{n+1}]^\delta\,.$$

Denote $e^0_{n+1} = e_1 \vee e_2 \vee \ldots \vee e_{n+1}$, $e'_{n+1} = e - e^0_{n+1}$. Since $0 \leqq e_{n+1} \leqq e^0_{n+1}$, we have

$$[e^0_{n+1}]^\delta \subset [e_{n+1}]^\delta$$

and therefore for each $x \in H^\wedge$,

$$(x[e_{n+1}]^\delta)\,[e^0_{n+1}]^\delta = x[e^0_{n+1}]^\delta\,.$$

Hence it follows from (12)

(13) $$(n + 1)\, e'_n[e^0_{n+1}]^\delta \leqq k[e^0_{n+1}]^\delta\,.$$

The element e^0_{n+1} is a weak unit of $[e_1] \otimes \ldots \otimes [e_{n+1}]$ and hence $[e_1] \otimes \ldots \ldots \otimes [e_{n+1}] = [e^0_{n+1}]$; from this we infer that $e[e^0_{n+1}] = e^0_{n+1}$ and hence $e[e^0_{n+1}]^\delta = = e'_{n+1}$. Thus

$$e'_n[e^0_{n+1}]^\delta = (e - (e_1 + \ldots + e_n))\,[e^0_{n+1}]^\delta = e[e^0_{n+1}]^\delta = e'_{n+1}\,,$$

because $e_i \in [e^0_{n+1}]$ for $i = 1, \ldots, n$. By (13),

(8′) $$(n + 1)\, e'_{n+1} \leqq k[e^0_{n+1}]^\delta\,.$$

Thus we can construct the elements e_i, e^0_i, $e'_i \in G$ $(i = 1, 2, \ldots)$ such that the relations (6)–(8) and (8a) are valid for each positive integer i, and $e_i \wedge e_j = 0$ for any two distinct positive integers i, j. Let $t_i = ie_i \wedge k$. Then $t_i \wedge t_j = 0$ for $i \neq j$ and $t_i \in G$. Because G is laterally complete, the set $\{ie_i \wedge k\}$ $(i = 1, 2, \ldots)$ posesses the least upper bound g_0 in G. Since G is a convex subset of $H^\wedge$, we have

$$g_0 = \bigvee(ie_i \wedge k) \quad (i = 1, 2, \ldots)$$

in $H^\wedge$. Clearly $0 \leqq g_0 \leqq k$. Let j be a positive integer, $g^j = \bigvee(ie_i \wedge k)$ $(i \neq j)$. Since $(ie_i \wedge k) \wedge (je_j \wedge k) = 0$, we have $g^j \wedge (je_j \wedge k) = 0$ and so

$$g_0 = g^j \vee (je_j \wedge k) = g^j + (je_j \wedge k)\,.$$

Further we have $g^j \wedge e_j = 0$ and so $g^j[e_j] = 0$, hence

$$g_0[e_j] = je_j \wedge k\,,$$

for each positive integer j. Put $k - g_0 = k'$ and let $0 \leqq g_1 \in G$; $g_1 \leqq k'$. Assume that $0 < g_1$. If $ie_i \wedge g_1 = g_2 > 0$ for some positive integer i, then

$$k[e_i] = g_0[e_i] + k[e_i] \geqq g_0[e_i] + g_2[e_i] = (ie_i \wedge k) + g_2 > ie_i \wedge k\,.$$

Since $k[e_i] \leqq k$, it follows from (7) that $k[e_i] \leqq ie_i \wedge k$; this is a contradiction. Hence $ie_i \wedge g_1 = 0$ for each $i = 1, 2, \ldots$ and thus $e_i \wedge g_1 = 0$ for $i = 1, 2, \ldots$ Because G is laterally complete there exists the least upper bound e^0 of the set $\{e_i\}$ in G. The elements e_i belong to the center C of the lattice $[0, e] \cap G$. Since this lattice is infinitely distributive, C is a closed sublattice of $[0, e] \cap G$. Further C is a Boolean algebra, the complement of e_i being $e[e_i]^\delta = e - e_i$. If $e^0 = e$, then $g_1 \wedge e = 0$, which is a contradiction, since e is a weak unit of G. Let $e^0 < e$ and let e^1 be the complement of e^0 in C. Then $e^1 > 0$. Because the complement of e_i^0 is e_i' and $e_i^0 \leqq e^0$, we obtain $e_i' \geqq e^1$ for each positive integer i. Hence we have according to (8)

$$ie^1 \leqq k[e_i^0]^\delta \leqq k$$

for each positive integer i, which is impossible since the l-group G is archimedean. The proof is complete.

Theorem 1. *Each archimedean orthocomplete l-group has the splitting property.*

Proof. Let G be an archimedean orthocomplete l-group and assume that G is an l-ideal of an archimedean l-group H. Under the same denotations as above consider the l-subgroups G and $G^\delta \cap H = G'$ of H. By Lemma 2, each element $k \in H^+$ can be expressed in the form $k = g_0 + k'$ with $0 \leqq g_0 \in G$, $0 \leqq k' \in G'$. Because each element of H is a difference of two positive elements of H, we obtain $H = G + G'$; since $G \cap G' = \{0\}$, H is the direct product of G and G' in the group-theoretical sense. If $k = g_2 + g_3$ with $g_2 \in G$, $g_3 \in G'$, then $g_2 = g_0 \geqq 0$, $g_3 = k' \geqq 0$. Therefore $H = G \otimes G'$.

Corollary [7]. *Every linearly ordered archimedean group has the splitting property.*

Theorem 2. *Let G be a complete lattice ordered group. Then the following conditions are equivalent*: (i) *G has the splitting property*, (ii) *G is laterally complete.*

Proof. Since each complete and laterally complete l-group G is orthocomplete, it follows from Thm. 1 that (i) is implied by (ii). Conversely, let $G \neq \{0\}$ be a complete lattice ordered group having the splitting property. There exists a complete and laterally complete l-group $\bar{G}$ such that G is an l-ideal of $\bar{G}$ and for each $0 < h \in \bar{G}$ there exists a disjoint subset $\{g_j\}$ of G, $0 \leqq g_j$ with $h = \bigvee g_j$ (for the construction of $\bar{G}$, cf., e.g., [8], §2; in fact, $\bar{G} = G^l$ (cf. [3])). Thus $G^\delta = \{0\}$ in $\bar{G}$ (the symbol δ is considered with respect to $\bar{G}$). Since G has the splitting property, $\bar{G} = G \otimes G^\delta = G \otimes \{0\} = G$. Therefore G is laterally complete.

Remark. A particular case of the assertion of Thm. 2 (concerning divisible complete l-groups) was proved by Conrad [3].

Corollary. *Let A be an archimedean lattice ordered group. Then $(A^{\wedge})^l$ has the splitting property.*

In fact, let $G = A$. Then (under the same denotations as in the proof of Thm. 2) the l-group $(A^{\wedge})^l = \bar{G}$ has the splitting property.

If G is an archimedean l-group and G^d is the divisible closure of G, then G^d is archimedean as well; hence, by the Corollary, $((G^d)^{\wedge})^l$ has the splitting property. (Cf. [3].)

Theorem 3. *Let V be an archimedean vector lattice that is laterally complete. Then G has the splitting property.*

Proof. According to [13], V is an orthocomplete lattice ordered group. Therefore according to Thm. 1, V has the splitting property.

From Thm 2 and Thm. 3 we obtain:

Corollary. (Cf. [3]). *Let V be a complete vector lattice. Then the following conditions are equivalent:*

(i) *V is laterally complete;*

(ii) *V has the splitting property.*

Remark. If V is not complete, then (i) is not implied by (ii) (cf. Example 2 below).

3. SINGULAR LATTICE ORDERED GROUPS

Let G be a lattice ordered group. It is easy to verify that an element $0 < s \in G$ is singular if and only if the interval $[0, s]$ of G is a Boolean algebra [9]. Hence if s is singular and $s' \in [0, s]$, then s' is singular as well. G is called conditionally laterally complete if each bounded disjoint subset of G has the least upper bound.

Lemma 3. *Let G be conditionally laterally complete. Let $\{e_i\}$ $(i \in I)$ be a disjoint set of singular elements of G, $e = \bigvee e_i$. Then e is singular.*

Proof. Let $y \in G$, $0 \leqq y \leqq e$, $f_i = e_i \wedge y$, $g_i = e_i - f_i$. Because e_i is singular, $f_i \wedge g_i = 0$. Since G is infinitely distributive, we have $y = \bigvee f_i$. In view of the conditional lateral completeness of G, $\bigvee g_i = z$ does exist in G, since $0 \leqq g_i \leqq e_i \leqq e$ and so the set $\{g_i\}$ is disjoint and bounded. Then

$$y \wedge z = \big(\bigvee_{i\in I} f_i\big) \wedge \big(\bigvee_{j\in I} g_j\big) = \bigvee_{i,j\in I} (f_i \wedge g_j) = 0\,,$$

$$y \vee z = \bigvee_{i\in I} (f_i \vee g_i) = \bigvee_{i\in I} (f_i + g_i) = \bigvee_{i\in I} e_i = e\,,$$

hence $e = y \vee z = y + z$ and $y \wedge (e - y) = 0$. Therefore e is singular.

The following theorem generalizes the implication (ii) $\Rightarrow$ (i) of Thm. 2 12, [9].

Theorem 4. *Let G be an archimedean l-group that is conditionally laterally complete and singular. Then G is complete.*

Proof. The case $G = \{0\}$ is trivial; assume that $G \neq \{0\}$. Let $\emptyset \neq A \subset G, 0 < g \in \in G, 0 \leqq a \leqq g$ for each $a \in A$. Let G_1 be the convex l-subgroup of G generated by the element g; hence

$$G_1 = \bigcup[-ng, ng] \quad (n = 1, 2, 3, \ldots).$$

For proving the completeness of G it suffices to verify that sup A exists in G and this is equivalent with the condition that sup A exists in G_1. Since G is singular, the l-group G_1 is singular. From the Axiom of Choice it follows that there exists a disjoint set $\{g_i\}$ of singular elements of G_1 such that, if k is a singular element of G_1 and $k \wedge g_i = 0$ for each g_i, then $k = 0$. Let $0 < h \in G_1$ and assume that $h \wedge g_i = 0$ for each g_i; because G_1 is singular, there is a singular element $0 < g_0 \in G_1$ with $g_0 \leqq h$, and then $0 \leqq g_0 \wedge g_i \leqq h \wedge g_i = 0$ for each g_i, hence $g_0 = 0$ and this is a contradiction. Thus $\{g_i\}$ is a maximal disjoint subset of G_1. Put $e_i = g \wedge g_i$. The elements e_i are singular, $e_i \leqq g$ and $\{e_i\}$ is a maximal disjoint subset of G_1. Moreover, since G is conditionally laterally complete, G_1 is conditionally laterally complete. Thus according to Lemma 3, $e = \bigvee e_i$ is a singular element of G_1. Since $\{e_i\}$ is maximal disjoint in G_1, e_1 is a weak unit of G_1. According to 2.11, [9], the l-group G_1 is complete and hence sup A exists in G_1. Thus sup A exists in G and so G is complete.

Theorem 5. *Let G be a singular l-group that is laterally complete and archimedean. Then G has the splitting property.*

Proof. According to Thm. 4, G is complete and hence by Thm. 2, G has the splitting property.

Let us remark that a singular l-group that has the splitting property need not be laterally complete (cf. Example 1 below).

4. LATERALLY COMPLETE l-GROUPS

Theorem 6. *Let G be a laterally complete l-group. There exists a convex l-subgroup G_0 of G with the following properties:*

(i) *G_0 has the splitting property and is complete;*

(ii) *if G_1 is a convex l-subgroup of G such that G_1 has the splitting property and is complete, then $G_1 \subset G_0$.*

Proof. In [10] it was proved that for every lattice ordered group G there exists a convex l-subgroup G_0 of G such that (a) G_0 is complete, (b) if $0 < g \in G$ and if the interval $[0, g]$ is a complete lattice, then $g \in G_0$. From (b) it follows that $H \subset G_0$ for each convex l-subgroup H of G that is complete. Thus (ii) is valid and it remains to verify that G_0 has the splitting property. Let $\{x_i\}$ be a disjoint subset of G_0. Since G is laterally complete, there exists $x = \bigvee x_i$ in G. Let $y_j \in G$, $0 \leqq y_j \leqq x$ $(j \in J)$. Because the lattice $[0, x_i]$ is complete for each $i \in I$, there is

$$z_i = \bigvee_{j \in J} (x_i \wedge y_j)$$

in $[0, x_i] \subset G_0$. The system $\{z_i\}$ is disjoint, thus there exists $z = \bigvee z_i \leqq x$. For each y_j we have

$$y_j = x \wedge y_j = (\bigvee_{i \in I} x_i) \wedge y_j = \bigvee_{i \in I} (x_i \wedge y_j) \leqq \bigvee_{i \in I} z_i = z\,.$$

Let $v \in G$, $v \leqq x$, $v \geqq y_j$ for each $j \in J$. Then

$$v = v \wedge x = \bigvee_{i \in I} (v \wedge x_i) \geqq \bigvee_{i \in I} z_i = z\,.$$

From this it follows that $z = \bigvee_{j \in J} y_j$. Therefore the lattice $[0, x]$ is complete and hence by (b), the element x belongs to G_0. Thus G_0 is laterally complete and so according to Thm. 2, G_0 has the splitting property.

Lemma 4. *Let G be a conditionally laterally complete l-group such that each principal polar of G is a direct factor of G. Then each polar of G is a direct factor of G.*

The proof differs from the proof of Thm. 3.6, [9] only by the fact that we need not to use Lemma 2.2, [9].

Theorem 7. *Let G be an orthocomplete l-group. Every polar of G is a direct factor of G. If G is archimedean, then $G^\wedge$ is laterally complete and so it has the splitting property.*

Proof. The first assertion is a consequence of Lemma 4. Let G be archimedean; then $G^\wedge$ exists. Since $G^\wedge$ is complete, we have to show that $G^\wedge$ is laterally complete. Let $\{y_i\}$ $(i \in I)$ be a disjoint subset of $G^\wedge$. For each $i \in I$ there is an element $z_i \in G$ and a subset X_i of G such that $0 < x_i$ for each $x_i \in X_i$ and the relations

$$y_i = \sup X_i\,, \quad y_i \leqq z_i$$

hold in $G^\wedge$. Let $A_i = X_i^{\delta\delta}$ where the symbol δ is taken with respect to G. Every A_i is a direct factor of G. Because the set $\{y_i\}$ is disjoint, for any two distinct elements $i, j \in I$ and any $x_i \in X_i$, $x_j \in X_j$ we have $x_i \wedge x_j = 0$. Therefore $A_i \cap A_j = \{0\}$. Put $t_i = z_i A_i$. Since

$$x_i \leqq z_i \quad \text{for each} \quad x_i \in X_i\,,$$

we infer that

$$x_i = x_i A_i \leqq z_i A_i = t_i$$

holds in G and therefore we have in $G^\wedge$

$$y_i = \sup X_i \leqq t_i .$$

From $A_i \cap A_j = \{0\}$ for $i \neq j$ it follows $t_i \wedge t_j = 0$ and so, because G is laterally complete, the element $t = \sup \{t_i\}$ exists in G. Hence the set $\{y_i\}$ $(i \in I)$ is bounded in $G^\wedge$ and thus $y = \bigvee y_i$ exists in $G^\wedge$. We have proved that $G^\wedge$ is laterally complete and so by Thm. 2, $G^\wedge$ has the splitting property.

Corollary. *Let G be a σ-complete lattice ordered group that is laterally complete. Then each polar of G is a direct factor of G and $G^\wedge$ has the splitting property.*

Proof. From the σ-completeness of G if follows that G is archimedean and that each principal polar of G is a direct factor of G (cf. [11]), hence G is orthocomplete. Now it suffices to apply Thm. 7.

The complete direct product of lattice ordered groups G_i $(i \in I)$ will be denoted by πG_i $(i \in I)$. Let i be a fixed element of I. We denote by $\bar{G}_i$ the set of all elements $g \in \pi G_i$ $(i \in I)$ such that $g(j) = 0$ for each $j \in I, j \neq i$. Let G be an l-subgroup of πG_i $(i \in I)$ such that $\bigcup_{i \in I} \bar{G}_i \subset G$. Then G is called a completely subdirect product of l-groups G_i.

Theorem 8. *Let G_i $(i \in I)$ be lattice ordered groups having the splitting property and let G be a completely subdirect product of l-groups G_i such that for each $0 < f \in \pi G_i$ $(i \in I)$ there is $g \in G$ with $f \leqq g$. Then G has the splitting property.*

Proof. Assume that G is an l-ideal of an archimedean l-group H, $0 \leqq k \in H$. Let $i \in I$. Then the l-group $\bar{G}_i$ is an l-ideal of H and because $\bar{G}_i$ is isomorphic to G_i, we obtain that $\bar{G}_i$ has the splitting property; therefore $\bar{G}_i$ is a direct factor of H. There is $f \in \pi G_i$ $(i \in I)$ such that

$$f\bar{G}_i = k\bar{G}_i \quad \text{for each} \quad i \in I .$$

According to the assumption, there exists $g \in G$ with $f \leqq g$. Put $g \wedge k = g_0$. Then

$$g_0 \bar{G}_i = g\bar{G}_i \wedge k\bar{G}_i = g\bar{G}_i \wedge f\bar{G}_i = (g \wedge f)\,\bar{G}_i = f\bar{G}_i$$

for each $i \in I$, hence $f = g_0 \in G$. Let $0 \leqq g' \in G$, $g' \leqq k$. For each $i \in I$ we have

$$g'\bar{G}_i \leqq k\bar{G}_i = f\bar{G}_i ,$$

thus $g' \leqq g_0$. Let $g_1 = k - g_0$. If there is $0 < x \in G$, $x \leqq g_1$, then for $g' = g_0 + x$ we have $g_0 < g_0 + x \leqq k$ and $g_0 + x \in G$, which is a contradiction. Hence $g_1 \in G^\delta$

where the symbol δ is taken with respect to the l-group H. Now we can use a similar method as in the proof of Thm. 1 and we obtain $H = G \otimes G^\delta$.

Corollary. *If G is the complete direct product of lattice ordered groups G_i such that each G_i has the splitting property, then G has the splitting property.*

Theorem 9. *Let G_i $(i \in I)$ be lattice ordered groups having the splitting property and let G be a completely subdirect product of l-groups G_i such that G is convex in πG_i $(i \in I)$. Then G has the splitting property if and only if $G = \pi G_i$ $(i \in I)$.*

Proof. If $G = \pi G_i$ $(i \in I)$, then G has the splitting property according to the Corollary. Assume that G has the splitting property. Then πG_i $(i \in I) = G \otimes G^\delta$. For each $i \in I$ there is $g \in G$ with $g(i) > 0$ and hence $G^\delta = \{0\}$, thus $G = \pi G_i$ $(i \in I)$.

We conclude by two examples showing that there exist lattice ordered groups that have the splitting property without being laterally complete.

Example 1. In this example we construct an archimedean lattice ordered group G such that (i) G has the splitting property, (ii) G is singular, (iii) G is not conditionally orthogonally complete, (iv) each polar of G is a direct factor of G.

Let I be an infinite set and for each $i \in I$ let G_i be the additive group of all integers with the natural linear order. Let E be the set of all even integers and let G be the set of all elements $g \in \pi G_i$ $(i \in I)$ with the property that the set

$$S(g) = \{i \in I : g(i) \notin E\}$$

is finite. G is a completely subdirect product of l-groups G_i. Let $0 \leqq f \in \pi G_i$ $(i \in I)$. There exists $g \in \pi G_i$ $(i \in I)$ such that $g(i)$ is even for each $i \in I$ and $f \leqq g$. Because $g \in G$, according to Thm. 8, G has the splitting property. Let $e_i \in G$ such that $e_i(i) = 1$ and $e_i(j) = 0$ for each $j \in I$, $j \neq i$. The element e_i is singular and for each $0 < g \in G$ there is $i \in I$ with $e_i \leqq g$; therefore G is singular. The set $\{e_i\}$ $(i \in I)$ is bounded in G, disjoint and it has no least upper bound. Hence (iii) is valid. For $A \subset G$ we denote

$$s(A) = \{i \in I\colon \text{there is } a \in A \text{ with } a(i) \neq 0\}\,,$$
$$s'(A) = I - s(A)\,.$$

Then we have

$$A^\delta = \{g \in G : g(i) = 0 \text{ for each } i \in s(A)\}\,,$$
$$A^{\delta\delta} = \{g \in G : g(i) = 0 \text{ for each } i \in s'(A)\}\,.$$

Obviously, $G = A^\delta \otimes A^{\delta\delta}$.

Example 2. Let G be as in Example 1 and for each $i \in I$ let H_i be the additive group of all reals with the natural order, $H_0 = \pi H_i$ $(i \in I)$. If $f \in H_0$ and α is a real, then

$\alpha f \in H_0$ (where αf has the usual meaning). Let H be the set of all $h \in H_0$ with the property that there exist a real α and an element $g \in G$ such that $h = \alpha g$. Then H is a vector lattice fulfilling the conditions (i), (iii) and (iv) from Example 1 (with G replaced by H).

References

[1] *S. J. Bernau:* Orthocompletion in lattice groups. Proc. London Math. Soc. *16* (1966), 444—480.

[2] *G. Birkhoff:* Lattice theory, third edition, Providence 1967.

[3] *P. Conrad:* The essential closure of an archimedean lattice-ordered group. Duke Math. J. *38* (1971), 151—160.

[4] *P. Conrad, D. McAllister:* The completion of a lattice ordered group. J. Austral. Math. Soc. *9* (1969), 182—208.

[5] *Л. Фукс:* Частично упорядоченные алгебраические системы, Москва 1965.

[6] *J. Jakubík:* Centrum nekonečne distributívnych zväzov. Matem. fyz. čas. *7* (1957), 116—120.

[7] *J. Jakubík:* Konvexe Ketten in *l*-Gruppen. Časop. pěst. mat. *84* (1959), 53—63.

[8] *Я. Якубик:* Представления и расширения *l*-групп. Чех. мат. ж. *13* (1963), 267—283.

[9] *J. Jakubík:* On σ-complete lattice ordered groups. Czech. Math. J. *23* (1973), 164—174.

[10] *J. Jakubík:* Conditionally α-complete sublattices of a distributive lattice. Alg. univ. *2* (1972), 9—15.

[11] *G. Jameson:* Ordered linear spaces. Lecture Notes 141, Springer Verlag, Berlin 1970.

[12] *Ф. Шик:* К теории структурно упорядоченных групп. Чех. мат. ж. *6* (1956), 1—25.

[13] *А. Й. Векслер, В. А. Гейлер:* О порядковой и дизьюнктной полноте линейных полуупорядоченных пространств. Сибир. мат. ж. *13* (1972), 43—51.

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