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ON SOME INVARIANTS OF UNARY ALGEBRAS

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1. PROBLEM

1.0. Notation. If A is a set we denote by $|A|$ the cardinal number of A; similarly, if α is an ordinal then its cardinal number is denoted by $|\alpha|$. We denote by Ord the class of all ordinals. If $\alpha \in \mathrm{Ord}$ then we put $W_\alpha = \{\beta \in \mathrm{Ord};\ \beta < \alpha\}$; further, the least ordinal cofinal with α is denoted by cf α. We denote by N the set of all finite ordinals.

We shall need some simple results concerning ordinals (see [3] and [4]).

(i) If $\alpha, \beta, \gamma \in \mathrm{Ord}$, $\alpha < \beta$ then $\gamma + \alpha < \gamma + \beta$.

(ii) If $\alpha, \beta \in \mathrm{Ord}$, $\alpha \leqq \beta$ then there is precisely one $\xi \in \mathrm{Ord}$ such that $\alpha + \xi = \beta$. We put $\xi = -\alpha + \beta$.

(iii) If $\alpha, \beta \in \mathrm{Ord}$, $\alpha \leqq \beta$, then $\alpha + (-\alpha + \beta) = \beta$, $-\alpha + (\alpha + \beta) = \beta$.

Indeed, the first equation follows directly by definition of $-\alpha + \beta$. If we put $\xi = -\alpha + (\alpha + \beta)$ then $\alpha + \xi = \alpha + \beta$ by definition. Then $\xi = \beta$ follows from the uniqueness of the solution.

(iv) If $\alpha, \beta, \gamma \in \mathrm{Ord}$, $\alpha \leqq \beta < \gamma$, then $-\alpha + \beta < -\alpha + \gamma$.

Indeed, $-\alpha + \beta \geqq -\alpha + \gamma$ would imply $\beta = \alpha + (-\alpha + \beta) \geqq \alpha + (-\alpha + \gamma) = \gamma$ by (iii) and (i).

(v) If $\alpha, \beta \in \mathrm{Ord}$, $\alpha \leqq \beta < \alpha + \omega_0$ then $-\alpha + \beta < \omega_0$.

Indeed, $-\alpha + \beta \geqq \omega_0$ would imply $\beta = \alpha + (-\alpha + \beta) \geqq \alpha + \omega_0$ by (iii) which is a contradiction.

(vi) Suppose $\alpha, \beta, \delta \in \mathrm{Ord}$, $\emptyset \neq \Gamma \subseteq \mathrm{Ord}$, $\beta \leqq \gamma$ for each $\gamma \in \Gamma$, $\delta > \alpha + (-\beta + \gamma)$ for each $\gamma \in \Gamma$. Let ε be the least ordinal greater than all $\gamma \in \Gamma$. Then $\delta \geqq \alpha + (-\beta + \varepsilon)$.

Indeed, suppose, on the contrary, $\delta < \alpha + (-\beta + \varepsilon)$. Since $\delta > \alpha + (-\beta + \gamma)$ for at least one $\gamma \in \Gamma$ we have $\delta \geqq \alpha$ which implies the existence of $-\alpha + \delta$ and $-\alpha + (\alpha + (-\beta + \varepsilon))$ by (ii). Then $-\alpha + \delta < -\alpha + (\alpha + (-\beta + \varepsilon)) = -\beta + \varepsilon$

by (iv) and (iii). It follows $\beta + (-\alpha + \delta) < \beta + (-\beta + \varepsilon) = \varepsilon$ by (i) and (iii). Thus, there is at least one $\gamma_0 \in \Gamma$ such that $\beta + (-\alpha + \delta) \leqq \gamma_0$. It follows $-\alpha + \delta = -\beta + (\beta + (-\alpha + \delta)) \leqq -\beta + \gamma_0$ by (iii) and (iv) which implies $\delta = \alpha + (-\alpha + \delta) \leqq \alpha + (-\beta + \gamma_0)$ by (iii) and (iv) which is a contradiction. Thus, $\delta \geqq \alpha + (-\beta + \varepsilon)$.

Let $\infty \notin \text{Ord}$. If M is an arbitrary set of ordinals then we denote by $<$ the order relation on $M \cup \{\infty\}$ such that its restriction $< \cap (M \times M)$ to M is the natural order relation of ordinals and that $\alpha < \infty$ for each $\alpha \in M$.

If φ is a map of the set A into the set B, $\varphi : A \to B$, and $C \subseteq A$, $D \subseteq B$ then we put $\varphi(C) = \{\varphi(x);\ x \in C\}$; further, we define $\varphi^{-1}(D) = \{x \in A;\ \varphi(x) \in D\}$. If $\varphi : A \to B$ is a map, $C \subseteq A$, then we denote by $\varphi \mid C$ the restriction $\varphi \cap (C \times B)$ of φ; it is a map of C into B.

Let A be a set, f a map of A into A, $f : A \to A$. Then the ordered pair (A, f) is called a unary algebra. For a unary algebra (A, f) we put $f^0 = id_A$, $f^{n+1} = ff^n$ for each $n \in N$. Clearly, $f^{n+m} = f^n f^m$ for all $n, m \in N$. A unary algebra (A, f) is called connected if, for all $x, y \in A$, there are $m, n \in N$ such that $f^m(x) = f^n(y)$. If (A, f) is a unary algebra and $x \in A$ an arbitrary element then we put $[x]_{(A,f)} = \{f^n(x);\ n \in N\}$.

We denote by $\cong$ the relation of isomorphism of algebras.

1.1. Definition. Let (A, f) be a connected unary algebra, $x \in A$. We put $Z(x) = \{y \in A;$ there exists an infinite set $N(y) \subseteq N$ such that $f^n(x) = y$ for each $n \in N(y)\}$.

1.2. Lemma. *Let (A, f) be a connected unary algebra. Then the following assertions hold*:

(a) *If $x \in A$, $y = f(x)$ then $Z(x) = Z(y)$.*

(b) *If $x \in A$, $n \in N$, $y = f^n(x)$ then $Z(x) = Z(y)$.*

(c) *If $x, y \in A$ then $Z(x) = Z(y)$.*

Proof of (a). Suppose $x \in A$, $y = f(x)$, $z \in A$. Then $z \in Z(x)$ iff there is an infinite set $M \subseteq N$ such that $f^n(x) = z$ for each $n \in M$; we can suppose, without loss of generality, that $0 \notin M$. The last condition is equivalent to the condition $f^{n-1}(y) = f^{n-1}(f(x)) = f^n(x) = z$ for each $n \in M$ which is $z \in Z(y)$. Thus, $Z(x) = Z(y)$.

Proof of (b). The assertion (b) follows from (a) by induction.

Proof of (c). If $x, y \in A$ then there exist $m, n \in N$ such that $f^m(x) = f^n(y)$. It follows from (b) that $Z(x) = Z(f^m(x)) = Z(f^n(y)) = Z(y)$.

1.3. Definition. Let (A, f) be a connected unary algebra. We put $Z(A, f) = Z(x)$ where $x \in A$ is an arbitrary element, $R(A, f) = |Z(A, f)|$. Then $Z(A, f)$ is called the *cycle* and $R(A, f)$ the *rang* of (A, f).

1.4. Lemma. *Let (A, f) be a connected unary algebra. Then $(Z(A, f), f \mid Z(A, f))$ is a subalgebra of the algebra (A, f).*

Proof. If $x \in Z(A, f)$ then there exists an infinite set $N(x) \subseteq N$ such that $x = f^n(x)$ for each $n \in N(x)$. It follows $f(x) = f^{n+1}(x)$ for all $n \in N(x)$ which implies $f(x) \in Z(f(x)) = Z(A, f)$.

1.5. Lemma. *Let (A, f) be a connected unary algebra and suppose $x, y \in A$. Then*

(a) *If $n_1, n_2 \in N$, $n_1 \leqq n_2$ are such that $y = f^{n_1}(x) = f^{n_2}(x)$ then $y = f^{n_1 + m(n_2 - n_1)}(x)$ for each $m \in N$.*

(b) *$x \in Z(A, f)$ iff there is $n \in N - \{0\}$ such that $f^n(x) = x$.*

Proof of (a). We put $n_2 - n_1 = d$; thus, $f^{n_1 + 0d}(x) = f^{n_1}(x) = y$. Let $m \in N$ and suppose $f^{n_1 + md}(x) = y$. Then $f^{n_1 + (m+1)d}(x) = f^{d + n_1 + md}(x) = f^d(f^{n_1 + md}(x)) = f^d(y) = f^d(f^{n_1}(x)) = f^{n_1 + d}(x) = y$.

Proof of (b). Suppose, for $x \in A$, the existence of $n \in N - \{0\}$ such that $f^n(x) = x$; then, by (a), we have $x = f^{mn}(x)$ for each $m \in N$. Thus, we have $x = f^p(x)$ for all $p \in \{mn; m \in N\}$ the latter set being infinite. Thus, $x \in Z(x) = Z(A, f)$.

The necessity of the condition for $x \in Z(A, f)$ follows directly from 1.3 and 1.1.

1.6. Lemma. *Let (A, f) be a connected unary algebra. Then the following assertions hold:*

(a) *If $x \in Z(A, f)$ then $|Z(A, f)| = \min \{n \in N - \{0\}; f^n(x) = x\}$.*

(b) $R(A, f) < \aleph_0$.

Proof of (a). We put $d = \min \{n \in N - \{0\}; f^n(x) = x\}$. Since $x \in Z(A, f)$ we have $\{x, f(x), \ldots, f^{d-1}(x)\} \subseteq Z(A, f)$, by 1.4. Let us have $y \in Z(A, f)$. Then $y \in Z(x)$; thus, there exists $m \in N$ such that $f^m(x) = y$. Let $p, q \in N$ be such numbers that $m = pd + q$, $0 \leqq q < d$. Thus, by definition of d and by 1.5 (a), we have $f^{pd}(x) = x$ and $y = f^m(x) = f^q(f^{pd}(x)) = f^q(x)$. Thus, $y \in \{x, f(x), \ldots, f^{d-1}(x)\}$ and we have $\{x, f(x), \ldots, f^{d-1}(x)\} = Z(A, f)$. Therefore, $|Z(A, f)| = d$.

Proof of (b). If $Z(A, f) = \emptyset$ then $R(A, f) = 0 < \aleph_0$. If $Z(A, f) \neq \emptyset$ then there is $x \in Z(A, f)$ and $\{n \in N - \{0\}; f^n(x) = x\} \neq \emptyset$ by 1.5 (b). It follows $R(A, f) = \min \{n \in N - \{0\}; f^n(x) = x\} < \aleph_0$, by (a).

1.7. Definition. Let (A, f) be a connected unary algebra. We put $A^\infty = \{x \in A;$ there is a sequence $(x_i)_{i \in N}$ such that $x_0 = x$ and $f(x_{i+1}) = x_i$ for each $i \in N\}$, $A^0 = \{x \in A; f^{-1}(x) = \emptyset\}$.

Let $\alpha \in \text{Ord}$, $\alpha > 0$ and suppose that the sets $A^\varkappa$ have been defined for all $\varkappa \in W_\alpha$. Then we put $A^\alpha = \{x \in A - \bigcup_{\varkappa \in W_\alpha} A^\varkappa; f^{-1}(x) \subseteq \bigcup_{\varkappa \in W_\alpha} A^\varkappa\}$.

1.8. Lemma. *Let (A, f) be a connected unary algebra, $\alpha, \beta \in \text{Ord}$, $\alpha < \beta$. Then $A^\alpha \cap A^\beta = \emptyset$.*

Proof. Clearly, $A^\beta \subseteq A - \bigcup_{\varkappa \in W_\beta} A^\varkappa$ which implies $A^\beta \cap A^\alpha \subseteq A^\beta \cap \bigcup_{\varkappa \in W_\beta} A^\varkappa = \emptyset$.

1.9. Lemma. *Let (A, f) be a connected unary algebra. Then there is $\vartheta \in \text{Ord}$ such that $A^\vartheta = \emptyset$.*

Proof. Let $\nu \in \text{Ord}$ be such an ordinal number that $|A| \leqq \aleph_\nu$. Suppose $A^\lambda \neq \emptyset$ for each $\lambda \in W_{\omega_{\nu+1}}$. Then $\aleph_{\nu+1} \leqq \sum_{\lambda \in W_{\omega_{\nu+1}}} |A^\lambda| = |\bigcup_{\lambda \in W_{\omega_{\nu+1}}} A^\lambda| \leqq |A| \leqq \aleph_\nu$ by 1.8 which is a contradiction.

Thus, there is $\vartheta \in \text{Ord}$, $\vartheta \in W_{\omega_{\nu+1}}$ such that $A^\vartheta = \emptyset$.

1.10. Lemma. *Let (A, f) be a connected unary algebra.* If *$\vartheta \in \text{Ord}$, $A^\vartheta = \emptyset$ then $A^\lambda = \emptyset$ for each $\lambda \in \text{Ord}$ with the property $\lambda \geqq \vartheta$.*

Proof. We denote by $V(\lambda)$ the following assertion: $A^\lambda = \emptyset$.

Then $V(\vartheta)$ holds.

Let us have $\beta \in \text{Ord}$, $\vartheta < \beta$, suppose that $V(\lambda)$ holds for each λ such that $\vartheta \leqq \lambda < \beta$. Then $\bigcup_{\lambda \in W_\beta} A^\lambda = \bigcup_{\lambda \in W_\vartheta} A^\lambda$ which implies $A^\beta = \{x \in A - \bigcup_{\lambda \in W_\beta} A^\lambda; f^{-1}(x) \subseteq \bigcup_{\lambda \in W_\beta} A^\lambda\} = \{x \in A - \bigcup_{\lambda \in W_\vartheta} A^\lambda; f^{-1}(x) \subseteq \bigcup_{\lambda \in W_\vartheta} A^\lambda\} = A^\vartheta = \emptyset$.

The assertion follows by transfinite induction.

1.11. Definition. Let (A, f) be a connected unary algebra. Then we denote by $\vartheta(A, f)$ the least ordinal ϑ such that $A^\vartheta = \emptyset$.

1.12. Lemma. *Let (A, f) be a connected unary algebra. Then $A^\infty = A - \bigcup_{\varkappa \in W_{\vartheta(A,f)}} A^\varkappa$.*

Proof. (1) If $x \in A - \bigcup_{\varkappa \in W_{\vartheta(A,f)}} A^\varkappa$ then there is an element $x' \in f^{-1}(x)$ such that $x' \in A - \bigcup_{\varkappa \in W_{\vartheta(A,f)}} A^\varkappa$. Indeed, if we had $x' \in \bigcup_{\varkappa \in W_{\vartheta(A,f)}} A^\varkappa$ for each $x' \in f^{-1}(x)$ then we should have $f^{-1}(x) \subseteq \bigcup_{\varkappa \in W_{\vartheta(A,f)}} A^\varkappa$. We denote by ϑ the least ordinal such that $f^{-1}(x) \subseteq \bigcup_{\varkappa \in W_\vartheta} A^\varkappa$. Then $\vartheta \leqq \vartheta(A, f)$ and $x \in A^\vartheta$ by 1.7 which is a contradiction either with $A^{\vartheta(A,f)} = \emptyset$ (in the case $\vartheta = \vartheta(A, f)$) or with $x \in A - \bigcup_{\varkappa \in W_{\vartheta(A,f)}} A^\varkappa$ (in the case $\vartheta < \vartheta(A, f)$).

We put $x_0 = x$ and $x_{n+1} = x'_n$ for $n \in N$. Then $f(x_{n+1}) = x_n$ for $n \in N$ and $x \in A^\infty$. Thus $A - \bigcup_{\varkappa \in W_{\vartheta(A,f)}} A^\varkappa \subseteq A^\infty$.

(2) Let us have $x \in A^\infty \cap (\bigcup_{\varkappa \in W_{\vartheta(A,f)}} A^\varkappa)$. Then there is a sequence $(x_i)_{i \in N}$ such that $x_0 = x$ and $f(x_{i+1}) = x_i$ for each $i \in N$. By 1.8, there exists precisely one $\varkappa_0 \in W_{\vartheta(A,f)}$ such that $x_0 \in A^{\varkappa_0}$.

Suppose that we have constructed ordinals $\varkappa_0 > \varkappa_1 > \ldots > \varkappa_n$ such that $x_i \in A^{\varkappa_i}$ for $i = 0, 1, \ldots, n$ where $n \in N$. Then $x_{n+1} \in f^{-1}(x_n) \subseteq \bigcup_{\varkappa \in W_{\varkappa_n}} A^\varkappa$ which implies the existence of $\varkappa_{n+1} < \varkappa_n$ such that $x_{n+1} \in A^{\varkappa_{n+1}}$. Thus, $(\varkappa_i)_{i \in N}$ is an infinite decreasing sequence of ordinals which is a contradiction.

It follows that $A^\infty \subseteq A - \bigcup_{\varkappa \in W_{\vartheta(A,f)}} A^\varkappa$.

1.13. Theorem. *Let (A, f) be a connected unary algebra. Then $A = \bigcup_{\varkappa \in W_{\vartheta(A,f)} \cup \{\infty\}} A^\varkappa$ with disjoint summands.*

It is a cosequence of 1.12 and 1.8.

1.14. Lemma. *Let (A, f) be a connected unary algebra. Then $(A^\infty, f \mid A^\infty)$ is a subalgebra of (A, f).*

Proof. Let us have $x \in A^\infty$. It follows the existence of a sequence $(x_n)_{n \in N}$ such that $x_n \in A$, $x_0 = x$ and $f(x_{n+1}) = x_n$ for each $n \in N$. We put $f(x) = y = y_0$, $y_n = x_{n-1}$ for each $n \in N - \{0\}$. Then $f(y_{n+1}) = y_n$ for each $n \in N$ which implies $f(x) = y \in A^\infty$.

1.15. Lemma. *Let (A, f) be a connected unary algebra. Then $Z(A, f) \subseteq A^\infty$.*

Proof. $Z(A, f) \subseteq A^\infty$ holds if $Z(A, f) = \emptyset$. Thus, we can suppose $Z(A, f) \neq \emptyset$. Let us have $x \in Z(A, f)$. Then $Z(A, f) = Z(x)$ by 1.3. By 1.1, there exists an infinite set $N(x) \subseteq N$ such that $f^n(x) = x$ for each $n \in N(x)$. We denote by d the least positive element of $N(x)$. Then $f^d(x) = x$ and $f^{md}(x) = x$ for each $m \in N$ by 1.5 (a). We put, for each $n \in N$, $x_n = f^{n(2d-1)}(x)$. Then $f(x_{n+1}) = f(f^{(n+1)(2d-1)}(x)) = f^{n(2d-1)+2d}(x) = f^{n(2d-1)}(f^{2d}(x)) = f^{n(2d-1)}(x) = x_n$ for each $n \in N$ and $x_0 = f^0(x) = x$. Thus, $x \in A^\infty$.

1.16. Lemma. *Let (A, f) be a connected unary algebra, suppose $\lambda, \mu \in W_{\vartheta(A,f)}$, $\lambda < \mu$. Then, for each $x \in A^\mu$, there is an $x' \in A^\lambda$ and an $n \in N - \{0\}$ such that $f^n(x') = x$.*

Proof. Let us have $x \in A^\mu$. Then there is $\nu_1 \in \mathrm{Ord}$, $\lambda \leqq \nu_1 < \mu$ and $x_1 \in A^{\nu_1}$ such that $f(x_1) = x$. Indeed, if no such ν_1 and x_1 exist then $f^{-1}(x) \subseteq \bigcup_{\varkappa \in W_\lambda} A^\varkappa$. Since $x \in A - \bigcup_{\varkappa \in W_\mu} A^\varkappa \subseteq A - \bigcup_{\varkappa \in W_\lambda} A^\varkappa$ we have $x \in A^\lambda$, by 1.7, which contradicts 1.8.

If $\lambda < \nu_1$ we construct similarly $\nu_2 \in \mathrm{Ord}$, $\lambda \leqq \nu_2 < \nu_1$ and $x_2 \in A^{\nu_2}$ such that $f(x_2) = x_1$. As each decreasing sequence of ordinals is finite we construct, after a finite number of such steps, some ordinals $\lambda = \nu_n < \nu_{n-1} \ldots < \nu_1 < \mu$ and some elements $x_i \in A^{\nu_i}$ for $i = 1, 2, \ldots, n$ such that $f(x_{i+1}) = x_i$ for $i = 1, 2, \ldots, n-1$ and $f(x_1) = x$. It follows $f^n(x_n) = x$, $x_n \in A^\lambda$, $n \neq 0$ because $n = 0$ would imply $x = x_n \in A^\lambda \cap A^\mu$ which contradicts 1.8.

1.17. Lemma. *Let (A, f) be a connected unary algebra, $A^\infty \neq \emptyset$. Then the following assertions hold:*

(a) *For each $x \in \bigcup_{\varkappa \in W_{\vartheta(A,f)}} A^\varkappa$ there exists $n(x) \in N$ such that $f^{n(x)}(x) \in A^\infty$.*

(b) *If $A - A^\infty \neq \emptyset$ and $x \in A - A^\infty$ then there is precisely one $i_0 \in N - \{0\}$ such that $f^{i_0 - 1}(x) \in A - A^\infty$, $f^{i_0}(x) \in A^\infty$.*

(c) *If $A - A^\infty \neq \emptyset$ then there is at least one $x \in A - A^\infty$ such that $f(x) \in A^\infty$.*

Proof of (a). We take $y \in A^\infty$. Then there are $m, n \in N$ such that $f^m(x) = f^n(y)$. By 1.14, we have $f^n(y) \in A^\infty$ and we obtain the first assertion.

Proof of (b). By 1.12 and (a), for each $x \in A - A^\infty$, there exists $n(x) \in N$ such that $f^{n(x)}(x) \in A^\infty$. It follows by 1.13 that $n(x) > 0$. Thus, in the set of natural numbers i, $0 < i \leqq n(x)$, there is the least element i_0 such that $f^{i_0}(x) \in A^\infty$. Clearly, $i_0 > 0$ and $f^{i_0 - 1}(x) \in A - A^\infty$.

If $i > i_0$ then $i - 1 \geqq i_0$ and $f^{i-1}(x) = f^{i-1-i_0}(f^{i_0}(x)) \in A^\infty$ as $f^{i_0}(x) \in A^\infty$ and $(A^\infty, f \mid A^\infty)$ is a subalgebra of (A, f) by 1.14. Thus, $f^{i-1}(x) \notin A - A^\infty$.

If $i < i_0$ then $f^i(x) \notin A^\infty$ on the basis of the minimality of i_0.

Thus, i_0 is the only element $i \in N - \{0\}$ such that $f^{i-1}(x) \in A - A^\infty$, $f^i(x) \in A^\infty$.

Proof of (c). We take an arbitrary $z \in A - A^\infty$. By (b), there is precisely one $i_0 \in N - \{0\}$ such that $f^{i_0 - 1}(z) \in A - A^\infty$, $f^{i_0}(z) \in A^\infty$. We put $x = f^{i_0 - 1}(z)$. Then $x \in A - A^\infty$, $f(x) = f^{i_0}(z) \in A^\infty$.

1.18. Definition. Let (A, f) be a connected unary algebra. We define a map $S(A, f) : A \to \mathrm{Ord} \cup \{\infty\}$ by the condition $S(A, f)(x) = \varkappa$ for each $x \in A^\varkappa$, $\varkappa \in W_{\vartheta(A,f)} \cup \{\infty\}$. $S(A, f)(x)$ is called the degree of x.

1.19. Lemma. *Let (A, f) be a connected unary algebra. Then the following assertions hold:*

(a) *If $x \in A$ is such element that $S(A, f)(x) \neq \infty$ then $S(A, f)(f^n(x)) \geqq S(A, f)(x) + n$ for each $n \in N$.*

(b) *If $x \in A$, $\varkappa \in W_{\vartheta(A,f)}$ are arbitrary elements then $|A^\varkappa \cap [x]_{(A,f)}| \leqq 1$.*

Proof of (a). If $n = 0$ then $S(A,f)(f^0(x)) = S(A,f)(x)$. Let $n \in N$ and suppose $S(A,f)(f^n(x)) \geqq S(A,f)(x) + n$. We put $\alpha = S(A,f)(f^{n+1}(x))$. If $\alpha = \infty$ then $\alpha > S(A,f) + n + 1$. If $\alpha < \infty$ then $f^{n+1}(x) \in A^\alpha$ and $f^n(x) \in f^{-1}(f^{n+1}(x)) \subseteq \bigcup_{\beta \in W_\alpha} A^\beta$. Thus, $S(A,f)(f^n(x)) < \alpha$ and $\alpha \geqq S(A,f)(f^n(x)) + 1 \geqq S(A,f)(x) + n + 1$. We have proved the assertion (a).

Proof of (b). Suppose, on the contrary, $|A^\varkappa \cap [x]_{(A,f)}| \geqq 2$; let $y, z \in A^\varkappa \cap [x]_{(A,f)}$, $y \neq z$. Then there is $n \in N - \{0\}$ such that either $f^n(y) = z$ or $f^n(z) = y$. In the first case, we have $\varkappa = S(A,f)(z) \geqq S(A,f)(f^n(y)) \geqq S(A,f)(y) + n > S(A,f)(y) = \varkappa$, by (a), which is a contradiction. Similarly, the second case leads to a contradiction. We have proved the assertion (b).

1.20. Lemma. *Let (A,f), (A_*,f_*) be unary connected algebras, $\varphi : A \to A_*$ an isomorphism of (A,f) onto (A_*,f_*). Then $\vartheta(A,f) = \vartheta(A_*,f_*)$, $\varphi(A^\varkappa) = A_*^\varkappa$ for each $\varkappa \in W_{\vartheta(A,f)} \cup \{\infty\}$ and $\varphi(Z(A,f)) = Z(A_*,f_*)$.*

Proof. For each $\alpha \in \text{Ord}$ we denote by $V(\alpha)$ the following assertion: $\varphi(A^\alpha) = A_*^\alpha$.

The following conditions are equivalent:

(i) $x \in A^0$

(ii) $f(y) = x$ for no $y \in A$

(iii) $f_*(z) = \varphi(x)$ for no $z \in A_*$

(iv) $\varphi(x) \in A_*^0$.

Indeed, (i) and (ii) are equivalent by 1.7 and (iii) and (iv), too. If $f(y) = x$ for no $y \in A$ and there is $z \in A_*$ such that $f_*(z) = \varphi(x)$ then $f(\varphi^{-1}(z)) = \varphi^{-1}(f_*(z)) = \varphi^{-1}(\varphi(x)) = x$ because φ^{-1} is an isomorphism; we have a contradiction. Thus, (ii) implies (iii) and, similarly, (iii) implies (ii).

It follows that $V(0)$ holds.

Let $\beta > 0$ be an ordinal, suppose that $V(\gamma)$ holds for each $\gamma < \beta$. It follows $\varphi(\bigcup_{\varkappa \in W_\beta} A^\varkappa) = \bigcup_{\varkappa \in W_\beta} A_*^\varkappa$.

The following conditions are equivalent:

(i) $x \in A^\beta$

(ii) $x \in A - \bigcup_{\varkappa \in W_\beta} A^\varkappa$, $f^{-1}(x) \subseteq \bigcup_{\varkappa \in W_\beta} A^\varkappa$

(iii) $\varphi(x) \in A_* - \bigcup_{\varkappa \in W_\beta} A_*^\varkappa$, $f_*^{-1}(\varphi(x)) \subseteq \bigcup_{\varkappa \in W_\beta} A_*^\varkappa$

(iv) $\varphi(x) \in A_*^\beta$.

Indeed, (i) and (ii) are equivalent by 1.7 and (iii) and (iv), too. If $x \in A - \bigcup_{\varkappa \in W_\beta} A^\varkappa$ then $\varphi(x) \in \varphi(A - \bigcup_{\varkappa \in W_\beta} A^\varkappa) = \varphi(A) - \varphi(\bigcup_{\varkappa \in W_\beta} A^\varkappa) = A_* - \bigcup_{\varkappa \in W_\beta} A_*^\varkappa$ by induction hypothesis because φ is a bijection. If $f^{-1}(x) \subseteq \bigcup_{\varkappa \in W_\beta} A^\varkappa$ then each y with the property $f(y) = x$ is in $\bigcup_{\varkappa \in W_\beta} A^\varkappa$. Let us have an arbitrary $z \in f_*^{-1}(\varphi(x))$. Then $f_*(z) = \varphi(x)$ and $f(\varphi^{-1}(z)) = \varphi^{-1}(f_*(z)) = \varphi^{-1}(\varphi(x)) = x$ because φ^{-1} is an isomorphism. It follows $\varphi^{-1}(z) \in \bigcup_{\varkappa \in W_\beta} A^\varkappa$ which implies $z \in \varphi(\bigcup_{\varkappa \in W_\beta} A^\varkappa) = \bigcup_{\varkappa \in W_\beta} A_*^\varkappa$. Thus, $f_*^{-1}(\varphi(x)) \subseteq \subseteq \bigcup_{\varkappa \in W_\beta} A_*^\varkappa$.

We have proved that (ii) implies (iii). Similarly, (iii) implies (ii).

Thus, the validity of $V(\gamma)$ for all $\gamma < \beta$ implies that of $V(\beta)$.

We have $\varphi(A^\alpha) = A_*^\alpha$ for each $\alpha \in \text{Ord}$. Especially, $A^\alpha = \emptyset$ iff $A_*^\alpha = \emptyset$. It follows $\vartheta(A, f) = \vartheta(A_*, f_*)$.

If $x \in A^\infty$ then there is a sequence $(x_i)_{i \in N}$ such that $x_0 = x$ and $f(x_{i+1}) = x_i$ for each $i \in N$. It follows $\varphi(x_0) = \varphi(x)$ and $f_*(\varphi(x_{i+1})) = \varphi(f(x_{i+1})) = \varphi(x_i)$ for each $i \in N$. Thus, $\varphi(x) \in A_*^\infty$. Similarly, $x \in A$, $\varphi(x) \in A_*^\infty$ imply $x \in A^\infty$. We have $\varphi(A^\infty) = = A_*^\infty$.

We have proved $\varphi(A^\varkappa) = A_*^\varkappa$ for each $\varkappa \in W_{\vartheta(A,f)} \cup \{\infty\}$.

If $x \in Z(A, f)$ then there is $n \in N - \{0\}$ such that $f^n(x) = x$ by 1.5 (b). It follows $f_*^n(\varphi(x)) = \varphi(f^n(x)) = \varphi(x)$. Thus, $\varphi(x) \in Z(A_*, f_*)$ by 1.5 (b). Similarly, $x \in A$, $\varphi(x) \in Z(A_*, f_*)$ imply $x \in Z(A, f)$.

We have proved $\varphi(Z(A, f)) = Z(A_*, f_*)$.

1.21. Remark. Let (A, f) be a connected unary algebra. Then the ordinal $\vartheta(A, f)$ and the cardinals $|A^\varkappa|$, $\varkappa \in W_{\vartheta(A,f)} \cup \{\infty\}$ and $R(A, f)$ are preserved under isomorphisms, i.e. they are invariant, by 1.20.

If (A, f), (B, g) are connected unary algebras then the numbers $R(A, f)$, $R(B, g)$ and functions $S(A, f)$, $S(B, g)$ enable to construct all homomorphisms of (A, f) into (B, g). Thus, a very natural problem arises:

1.22. Problem. *Let A be a set, $R \in N$, $S : A \to \text{Ord} \cup \{\infty\}$ a map. Find necessary and sufficient conditions for the existence of a complete unary operation f on A such that (A, f) is connected and $R(A, f) = R$, $S(A, f) = S$.*

2. AUXILIARY CONSTRUCTION

2.1. Definition. Let (A, f) be a connected unary algebra with the property $A^\infty \neq \emptyset$. Then (A, f) is called an *∞-algebra*.

2.2. Definition. Let (A, f) be an ∞-algebra. Then we put $E(A, f) = f^{-1}(A^\infty) - A^\infty$.

2.3. Lemma. *Let (A, f) be an ∞-algebra. Then the following assertions hold:*

(a) *$E(A, f) \neq \emptyset$ iff $A - A^\infty \neq \emptyset$.*

(b) *If $x \in A - A^\infty$ then there is precisely one $n_0 \in N$ such that $f^{n_0}(x) \in E(A, f)$.*

(c) *If $\vartheta(A, f) > 0$ is an isolated ordinal then $\emptyset \neq A^{\vartheta(A,f)-1} \subseteq E(A, f)$.*

Proof of (a). The necessity of the condition is clear.

Let us have $A - A^\infty \neq \emptyset$. Then, by 1.17 (c), there is $x \in A - A^\infty$ such that $f(x) \in A^\infty$. Thus, $x \in E(A, f)$.

Proof of (b). The existence of precisely one $n_0 \in N$ with the property $f^{n_0}(x) \in E(A, f)$ is equivalent to the existence of precisely one $n_0 \in N$ with the properties $f^{n_0}(x) \notin A^\infty$, $f^{n_0+1}(x) \in A^\infty$ which is equivalent to the existence of precisely one $i_0 \in N - \{0\}$ such that $f^{i_0-1}(x) \notin A^\infty$, $f^{i_0}(x) \in A^\infty$. The last assertion holds according to 1.17 (b).

Proof of (c). $A^{\vartheta(A,f)-1} \neq \emptyset$ follows from the definition of $\vartheta(A, f)$. If $x \in A^{\vartheta(A,f)-1}$ then $S(A, f)(f(x)) > S(A, f)(x) = \vartheta(A, f) - 1$ by 1.19 (a). It follows $S(A, f)(f(x)) = \infty$ which implies $f(x) \in A^\infty$. It follows $x \in f^{-1}(A^\infty)$ and we have $A^{\vartheta(A,f)-1} \subseteq f^{-1}(A^\infty)$. Further, $A^{\vartheta(A,f)-1} \cap A^\infty = \emptyset$ by 1.13. It follows $A^{\vartheta(A,f)-1} \subseteq f^{-1}(A^\infty) - A^\infty = E(A, f)$.

2.4. Definition. Let (A, f) be a non empty connected unary algebra. Then it is called a *cone* if $f(A^\varkappa) = A^{\varkappa+1}$ for each $\varkappa \in W_{\vartheta(A,f)}$ such that $\varkappa + 1 \neq \vartheta(A, f)$.

2.5. Examples. 1. A connected unary algebra (A, f) such that $A^\infty = A \neq \emptyset$ is a cone.

2. The unary algebra (N, f) where $f(n) = n + 1$ for each $n \in N$ is a cone.

3. If $(m_n)_{n \in N}$ is a non-increasing sequence of cardinals such that $m_n \neq 0$ for each $n \in N$ and that there is $n_0 \in N$ with the property $m_{n_0} = 1$ then there is a cone (B, g) such that $|B^n| = m_n$ for each $n \in N$.

Indeed, we take mutually disjoint sets B_n such that $|B_n| = m_n$ for each $n \in N$. We put $B = \bigcup_{n \in N} B_n$. For an arbitrary $n \in N$, we take an arbitrary surjection $g_n : B_n \to B_{n+1}$; such a surjection exists because the sequence $(m_n)_{n \in N}$ is non-increasing. We define the map $g : B \to B$ in such a way that $g \mid B_n = g_n$. Then (B, g) is a unary algebra. Clearly, $|B_n| = 1$ for each $n \geqq n_0$. If $x, y \in B$ then there are $m, n \in N$ such that $x \in B_m$, $y \in B_n$. There is $p \in N$, $p \geqq \max\{m, n, n_0\}$. Then $g^{p-m}(x) \in B_p$, $g^{p-n}(y) \in B_p$. Since $|B_p| = 1$ it follows $g^{p-m}(x) = g^{p-n}(y)$. Thus, (B, g) is connected. Clearly, $B^n = B_n$ for each $n \in N$ and $g(B^n) = g(B_n) = g_n(B_n) = B_{n+1} = B^{n+1}$ which implies that (B, g) is a cone such that $|B^n| = m_n$ for each $n \in N$.

2.6. Lemma. *Let (A, f) be a cone. Then $\vartheta(A, f) \leqq \omega_0$.*

Proof. (1) Let $x \in A$, $n \in N$ be such element that $S(A, f)(x) + n \in W_{\vartheta(A,f)}$. We put $S(A, f)(x) = \varkappa$; then $x \in A^\varkappa$. By 2.4, we have $f^n(x) \in A^{\varkappa+n}$ which implies $S(A, f)(f^n(x)) = \varkappa + n = S(A, f)(x) + n$.

(2) Let $\varkappa \in \mathrm{Ord}$, $\varkappa \geqq \omega_0$; we prove that $A^\varkappa = \emptyset$. Indeed, suppose, on the contrary, $y \in A^\varkappa$. Let us have $\lambda \in \mathrm{Ord}$, $\lambda < \omega_0$. Then $\lambda < \varkappa$ and, by 1.16, there exist $z \in A^\lambda$ and $n \in N - \{0\}$ such that $f^n(z) = y$. By (1), we obtain $\varkappa = S(A, f)(y) = S(A, f)(f^n(z)) = S(A, f)(z) + n = \lambda + n < \omega_0$ which is a contradiction.

Thus, $\vartheta(A, f) = \min\{\varkappa \in \mathrm{Ord};\ A^\varkappa = \emptyset\} \leqq \omega_0$.

2.7. Definition. Let $\{(A_i, f_i);\ i \in I\}$ be a non empty system of mutually disjoint ∞-algebras. Let (B, g) be a cone which is disjoint with all ∞-algebras (A_i, f_i), $i \in I$. Let $\varphi : \bigcup_{i\in I} E(A_i, f_i) \to B$ be an arbitrary map. (If $\bigcup_{i\in I} E(A_i, f_i) = \emptyset$ then $\varphi = \emptyset$.)

Then $\bigcup_{i\in I} (A_i, f_i) \oplus_\varphi (B, g)$ denotes a unary algebra (C, h) such that $C = B \cup \bigcup_{i\in I} (A_i - A_i^\infty)$ and that, for each $x \in C$,

$$h(x) = \begin{cases} f_i(x) & \text{if } x \in (A_i - A_i^\infty) - E(A_i, f_i) \text{ for some } i \in I \\ \varphi(x) & \text{if } x \in \bigcup_{i\in I} E(A_i, f_i) \\ g(x) & \text{if } x \in B\,. \end{cases}$$

2.8. Remark. Let $(C, h) = \bigcup_{i\in I} (A_i, f_i) \oplus_\varphi (B, g)$ be a unary algebra defined in 2.7. If $x \in A_i - A_i^\infty$ for some $i \in I$ then $h^{-1}(x) = f_i^{-1}(x)$.

2.9. Lemma. *Let $(C, h) = \bigcup_{i\in I} (A_i, f_i) \oplus_\varphi (B, g)$ be a unary algebra defined in 2.7. Then (C, h) is a connected unary algebra.*

Proof. (1) Let $x \in C$ be arbitrary. Then there is $m \in N$ such that $h^m(x) \in B$. Indeed, if $x \in B$ then we have nothing to prove.

If $x \in A_i - A_i^\infty$ for some $i \in I$ then, by 2.3 (b), there is precisely one $m \in N$ such that $f_i^m(x) \in E(A_i, f_i)$. It follows, for $n \in N$, $n < m$, that $f_i^n(x) \notin A_i^\infty$, since $f_i^n(x) \in A_i^\infty$ would imply $f_i^m(x) = f_i^{m-n}(f_i^n(x)) \in A_i^\infty$ by 1.14 which is a contradiction as $A_i^\infty \cap E(A_i, f_i) = \emptyset$. Thus, $0 \leqq n < m$ implies $f_i^n(x) \in A_i - A_i^\infty - E(A_i, f_i)$. It follows $h^n(x) = f_i^n(x)$ for each n, $0 \leqq n < m$ and especially $h^{m-1}(x) = f_i^{m-1}(x) \in A_i - A_i^\infty - E(A_i, f_i)$ which implies $h^m(x) = f_i^m(x) \in E(A_i, f_i)$ and $h^{m+1}(x) = h(h^m(x)) = h(f_i^m(x)) = \varphi(f_i^m(x)) \in B$.

(2) Let us have $x, y \in C$. Then there are $n, m \in N$ such that $h^n(x) \in B$, $h^m(y) \in B$ by (1). Since (B, g) is connected there are $p, q \in N$ such that $g^p(h^n(x)) = g^q(h^m(y))$ which implies $h^{p+n}(x) = g^p(h^n(x)) = g^q(h^m(y)) = h^{q+m}(y)$. Thus, (C, h) in connected.

2.10. Lemma. *Let* $(C, h) = \bigcup_{i \in I} (A_i, f_i) \oplus_{\varphi} (B, g)$ *be a unary algebra defined in 2.7. Then the following assertions hold:*

(a) *If* $i \in I$ *and* $\varkappa \in \mathrm{Ord}$ *then* $A_i^\varkappa = C^\varkappa \cap A_i$.

(b) $C^\infty = B^\infty$.

(c) $Z(C, h) = Z(B, g)$.

(d) *Putting* $I(\varkappa) = \{i \in I;\ \varkappa < \vartheta(A_i, f_i)\}$ *for each* $\varkappa \in W_{\vartheta(C,h)}$ *we have* $C^\varkappa \subseteq (B - B^\infty) \cup \bigcup_{i \in I(\varkappa)} (A_i - A_i^\infty)$.

(e) *We put* $\vartheta_I = \sup_{i \in I} \vartheta(A_i, f_i)$. *If* $\vartheta_I \leqq \lambda < \vartheta(C, h)$ *then* $C^\lambda \subseteq B - B^\infty$.

(f) *If* $i \in I$ *then* $\vartheta(A_i, f_i)$ *is the least ordinal greater than* $S(C, h)(x)$ *for all* $x \in E(A_i, f_i)$.

(g) *If* $x \in C - C^\infty$ *and there exists* $i \in I$ *such that* $\emptyset \neq h^{-1}(x) = E(A_i, f_i)$ *then* $S(C, h)(x) = \vartheta(A_i, f_i)$.

(h) *If* $i \in I$, $\varkappa \in \mathrm{Ord}$ *and* $C^\varkappa \subseteq A_i - A_i^\infty$ *then* $C^\varkappa = A_i^\varkappa$.

Proof of (a). Let $i \in I$ be an arbitrary element. If $A_i - A_i^\infty = \emptyset$ then $W_{\vartheta(A_i, f_i)} = \emptyset$. It follows $A_i^\varkappa = \emptyset$ and $C^\varkappa \cap A_i \subseteq C \cap A_i \subseteq A_i - A_i^\infty = \emptyset$.

Thus, we can suppose $A_i - A_i^\infty \neq \emptyset$. We have $A_i^0 = C^0 \cap A_i$ because $x \in A_i^0$ iff $x \in A_i$ and $f_i^{-1}(x) = \emptyset$; by 2.8, it is equivalent to $x \in A_i$ and $h^{-1}(x) = \emptyset$ which means $x \in C^0 \cap A_i$.

Let us have $\lambda \in \mathrm{Ord}$, $\lambda > 0$ and suppose $A_i^\varkappa = C^\varkappa \cap A_i$ for each $\varkappa \in W_\lambda$. Then

$$(*) \qquad \bigcup_{\varkappa \in W_\lambda} A_i^\varkappa = \bigcup_{\varkappa \in W_\lambda} (C^\varkappa \cap A_i) = A_i \cap \Big(\bigcup_{\varkappa \in W_\lambda} C^\varkappa\Big)$$

and

$$(**) \qquad A_i - \bigcup_{\varkappa \in W_\lambda} A_i^\varkappa = (A_i \cap C) - \Big(A_i \cap \Big(\bigcup_{\varkappa \in W_\lambda} C^\varkappa\Big)\Big) = A_i \cap \Big(C - \bigcup_{\varkappa \in W_\lambda} C^\varkappa\Big).$$

It follows that, for $x \in C$, the following assertions are mutually equivalent:

(i) $x \in A_i^\lambda$

(ii) $x \in A_i - \bigcup_{\varkappa \in W_\lambda} A_i^\varkappa$, $f_i^{-1}(x) \subseteq \bigcup_{\varkappa \in W_\lambda} A_i^\varkappa$

(iii) $x \in A_i - \bigcup_{\varkappa \in W_\lambda} A_i^\varkappa$, $h^{-1}(x) \subseteq \bigcup_{\varkappa \in W_\lambda} A_i^\varkappa$

(iv) $x \in A_i$, $x \in C - \bigcup_{\varkappa \in W_\lambda} C^\varkappa$, $h^{-1}(x) \subseteq A_i \cap \Big(\bigcup_{\varkappa \in W_\lambda} C^\varkappa\Big)$

(v) $x \in A_i$, $x \in C - \bigcup_{\varkappa \in W_\lambda} C^\varkappa$, $h^{-1}(x) \subseteq \bigcup_{\varkappa \in W_\lambda} C^\varkappa$

(vi) $x \in A_i \cap C^\lambda$.

Indeed, (i) and (ii) are equivalent by 1.7, (v) and (vi), too. Clearly, $x \in C$, $x \in A_i^\lambda$ implies $x \in A_i - A_i^\infty$ which implies $h^{-1}(x) = f_i^{-1}(x)$ by 2.8. Thus, (ii) and (iii) are equivalent. Since $h^{-1}(x) = f_i^{-1}(x) \subseteq A_i$ (iv) and (v) are equivalent. The equivalence of (iii) and (iv) follows by (*) and (**).

We have proved $A_i^\lambda = C^\lambda \cap A_i$. The assertion (a) follows by transfinite induction.

Proof of (b). Let us have $x \in C^\infty$. Then there is a sequence $(x_k)_{k \in N}$ such that $x_0 = x$ and $h(x_{k+1}) = x_k$ for each $k \in N$. If $x_k \in B$ for all $k \in N$ then $g(x_{k+1}) = h(x_{k+1}) = x_k$ for all $k \in N$ which implies $x \in B^\infty$. If there is $k \in N$ such that $x_k \notin B$ then $x_k \in A_i - A_i^\infty$ for some $i \in I$. Clearly, for each $l \geqq k$, we have $x_l \in A_i$. Thus, for all $l \in N$, $l \geqq k$, we obtain $f_i(x_{l+1}) = h(x_{l+1}) = x_l$. It follows $x_k \in A_i^\infty$ which is a contradiction. Thus, $x \in B^\infty$ and $C^\infty \subseteq B^\infty$.

If $x \in B^\infty$ then there is a sequence $(x_k)_{k \in N}$, $x_k \in B$ for each $k \in N$ such that $x_0 = x$ and $g(x_{k+1}) = x_k$ for each $k \in N$. It follows $h(x_{k+1}) = x_k$ for each $k \in N$. Thus $x \in C^\infty$.

We have proved $C^\infty = B^\infty$.

Proof of (c). Let us have $x \in Z(C, h)$. Then there is $n \in N - \{0\}$ such that $h^n(x) = x$, by 1.5 (b). Then $Z(C, h) \subseteq C^\infty = B^\infty \subseteq B$ by (b) and 1.15 which implies $x \in B$ and $[x]_{(C,h)} \subseteq B$. Thus, $h^n(x) = g^n(x)$ which implies $x \in Z(B, g)$, by 1.5 (b).

Suppose $x \in Z(B, g)$. Then there is $n \in N - \{0\}$ such that $g^n(x) = x$, by 1.5 (b). We have $h^n(x) = g^n(x) = x$ which implies $x \in Z(C, h)$.

Thus, $Z(C, h) = Z(B, g)$.

Proof of (d). Let us have $i \in I - I(\varkappa)$. By (a), it follows $C^\varkappa \cap A_i = A_i^\varkappa = \emptyset$ because $\varkappa \geqq \vartheta(A_i, f_i)$. By (b), we have $C^\varkappa \subseteq C - C^\infty = (B - B^\infty) \cup \bigcup_{i \in I(\varkappa)} (A_i - A_i^\infty)$ which implies (d).

Proof of (e). We have $C^\lambda \subseteq (B - B^\infty) \cup \bigcup_{i \in I(\lambda)} (A_i - A_i^\infty)$ by (d) where $I(\lambda) = \{i \in I;\ \lambda < \vartheta(A_i, f_i)\}$. Since $\vartheta(A_i, f_i) \leqq \vartheta_I \leqq \lambda$ for each $i \in I$ we have $I(\lambda) = \emptyset$ and $C^\lambda \subseteq B - B^\infty$.

Proof of (f). Since $E(A_i, f_i) \subseteq \bigcup_{\lambda \in W_{\vartheta(A_i, f_i)}} A_i^\lambda$, then, for each $x \in E(A_i, f_i)$, there is $\lambda \in W_{\vartheta(A_i, f_i)}$ such that $x \in A_i^\lambda \subseteq C^\lambda$ by (a). It follows $S(C, h)(x) = \lambda < \vartheta(A_i, f_i)$.

Suppose the existence of $\beta \in \mathrm{Ord}$, $\beta < \vartheta(A_i, f_i)$ such that $S(C, h)(x) < \beta$ for each $x \in E(A_i, f_i)$. Then there is $y \in A_i^\beta = A_i \cap C^\beta$ by (a). Then $y \in A_i - A_i^\infty$. By 2.3 (b), there is precisely one $n \in N$ such that $f_i^n(y) \in E(A_i, f_i)$. Clearly, $f_i^j(y) \in A_i - A_i^\infty$ for $j = 0, 1, \ldots, n$. It follows $h^n(y) = f_i^n(y)$ and $\beta = S(C, h)(y) \leqq S(C, h)(y) + n \leqq S(C, h)(h^n(y)) = S(C, h)(f_i^n(y)) < \beta$ by 1.19 (a), which is a contradiction.

Thus, $\vartheta(A_i, f_i)$ is the least ordinal greater than $S(C, h)(x)$ for all $x \in E(A_i, f_i)$.

Proof of (g). Let $y \in E(A_i, f_i)$ be arbitrary. Then $x = h(y)$ which implies $S(C, h)(x) = S(C, h)(h(y)) > S(C, h)(y)$ by 1.19 (a). It follows $S(C, h)(x) \geqq \vartheta(A_i, f_i)$ by (f).

Suppose $S(C, h)(x) > \vartheta(A_i, f_i)$. Then there are $z \in C$, $n \in N - \{0\}$ such that $S(C, h)(z) = \vartheta(A_i, f_i)$ and $h^n(z) = x$, by 1.16. We put $t = h^{n-1}(z)$. Then $h(t) = h^n(z) = x$ which implies $t \in E(A_i, f_i)$. It follows $\vartheta(A_i, f_i) = S(C, h)(z) \leqq S(C, h)(z) + n - 1 \leqq S(C, h)(h^{n-1}(z)) = S(C, h)(t) < \vartheta(A_i, f_i)$ by 1.19 (a) and (f) which is a contradiction.

Thus, $S(C, h)(x) = \vartheta(A_i, f_i)$.

Proof of (h). We have $C^\varkappa = C^\varkappa \cap (A_i - A_i^\infty) \subseteq C^\varkappa \cap A_i = A_i^\varkappa \subseteq C^\varkappa$ by (a). It follows $C^\varkappa = A_i^\varkappa$.

2.11. Definition. Let $\emptyset \neq M \subseteq \mathrm{Ord}$, $\alpha \in \mathrm{Ord}$. Then we put $M \leqq \alpha$ if $\beta \leqq \alpha$ for each $\beta \in M$.

2.12. Lemma. *Let* $(C, h) = \bigcup_{i \in I} (A_i, f_i) \oplus_{\varphi} (B, g)$ *be a unary algebra defined in* 2.7. *We put* $\vartheta_I = \sup_{i \in I} \vartheta(A_i, f_i)$, *then* $C^{\vartheta_I} \subseteq B - B^\infty$ *and we put*

$$n^* = \begin{cases} \min \{n \in W_{\vartheta(B,g)};\ B^n \cap C^{\vartheta_I} \neq \emptyset\} & \text{if} \quad C^{\vartheta_I} \neq \emptyset \\ \vartheta(B, g) \quad \text{if} \quad C^{\vartheta_I} = \emptyset\,. \end{cases}$$

If $m \in W_{\vartheta(B,g)}$, $m \geqq n^*$ *then* $S(C, h)(B^m) \leqq \vartheta_I + (m - n^*)$.

Proof. $C^{\vartheta_I} \subseteq B - B^\infty$ by 2.10 (e).

Let us have $m \in W_{\vartheta(B,g)}$, $m \geqq n^*$. Then $n^* \leqq m < \vartheta(B, g)$. We denote by $V(m)$ the following assertion: $S(C, h)(B^m) \leqq \vartheta_I + (m - n^*)$.

Then $V(n^*)$ holds: Suppose, on the contrary, the existence of $y_0 \in B^{n^*}$ such that $S(C, h)(y_0) > \vartheta_I$. By 2.10 (b) $S(C, h)(y_0) \neq \infty$. By 1.16, there is $z \in C^{\vartheta_I}$ and $n_0 \in N - \{0\}$ such that $h^{n_0}(z) = g^{n_0}(z) = y_0$ which implies $n^* \leqq S(B, g)(z) < S(B, g)(z) + n_0 \leqq S(B, g)(g^{n_0}(z)) = S(B, g)(y_0) = n^*$ by 1.19 (a) which is a contradiction. Thus, $S(C, h)(B^{n^*}) \leqq \vartheta_I$.

Let us have $k \in W_{\vartheta(B,g)}$, $k \geqq n^*$. Suppose that $V(k)$ holds and that $k + 1 \in W_{\vartheta(B,g)}$.

Let us have $y \in B^{k+1}$. Then $h^{-1}(y) \subseteq B^k \cup \bigcup_{i \in I} E(A_i, f_i)$ because (B, g) is a cone. By 2.10 (f), we have $S(C, h)(E(A_i, f_i)) < \vartheta(A_i, f_i) \leqq \vartheta_I \leqq \vartheta_I + (k - n^*)$ for each $i \in I$. The validity of $V(k)$ means $S(C, h)(B^k) \leqq \vartheta_I + (k - n^*)$. It follows $S(C, h)(h^{-1}(y)) \leqq \vartheta_I + (k - n^*)$. According to the definition of $S(C, h)$ we obtain $S(C, h)(y) \leqq \vartheta_I + (k + 1 - n^*)$ which is $V(k + 1)$.

It follows by induction that $V(m)$ holds for each $m \in W_{\vartheta(B,g)}$ with the property $m \geqq n^*$.

2.13. Theorem. Let $(C, h) = \bigcup_{i\in I} (A_i, f_i) \oplus_{\varphi} (B, g)$ *be a unary algebra defined in* 2.7, *let* ϑ_I *and* n^* *be defined by* 2.12. *Then*

$$\vartheta(C, h) = \vartheta_I + (-n^* + \vartheta(B, g)) .$$

Proof. (1) Suppose $n^* = \vartheta(B, g)$.

If $i \in I$, $\varkappa \in W_{\vartheta(A_i,f_i)}$ then $\emptyset \neq A_i^{\varkappa} = C^{\varkappa} \cap A_i$ by 2.10 (a) which implies $C^{\varkappa} \neq \emptyset$. It follows $\vartheta(C, h) > \varkappa$ for each $\varkappa \in W_{\vartheta(A_i,f_i)}$. It follows $\vartheta(C, h) \geqq \vartheta(A_i, f_i)$ which implies $\vartheta(C, h) \geqq \vartheta_I$.

Suppose $\vartheta(C, h) > \vartheta_I$. By 1.10, there is $x \in C$ such that $S(C, h)(x) = \vartheta_I$ which implies $C^{\vartheta_I} \neq \emptyset$. It is a contradiction with the fact $n^* = \vartheta(B, g)$.

Thus, $\vartheta(C, h) = \vartheta_I$.

(2) Suppose $n^* < \vartheta(B,g)$.

Then $C^{\vartheta_I} \neq \emptyset$ and there exists at least one $x \in B^{n^*}$ such that $S(C, h)(x) = \vartheta_I$. Let us have $n \in W_{\vartheta(B,g)}$, $n \geqq n^*$. By 1.19 (a), we have $S(C, h)(h^{n-n^*}(x)) \geqq \vartheta_I + (n - n^*)$. Since (B, g) is a cone and $x \in B^{n^*} \subseteq B$ we have $h^{n-n^*}(x) = g^{n-n^*}(x) \in g^{n-n^*}(B^{n^*}) = B^n$. Thus, by 2.12 we have $S(C, h)(B^n) \leqq \vartheta_I + (n - n^*)$ which implies $S(C, h)(h^{n-n^*}(x)) \leqq \vartheta_I + (n - n^*)$. It follows $S(C, h)(h^{n-n^*}(x)) = \vartheta_I + (n - n^*)$.

Thus, for each $n \in W_{\vartheta(B,g)}$, $n \geqq n^*$, we have $\vartheta(C, h) > S(C, h)(h^{n-n^*}(x)) = \vartheta_I + (n - n^*) = \vartheta_I + (-n^* + n)$ which implies $\vartheta(C, h) \geqq \vartheta_I + (-n^* + \vartheta(B, g))$ by 1.0 (vi).

Suppose $\vartheta(C, h) > \vartheta_I + (-n^* + \vartheta(B, g))$. We put $\varkappa = \vartheta_I + (-n^* + \vartheta(B, g))$.

By 1.10, there exists $y \in C^{\varkappa}$. Since $\varkappa > \vartheta_I$, there is $z \in C^{\vartheta_I}$ and $n \in N - \{0\}$ such that $h^n(z) = y$, by 1.16. It follows $C^{\varkappa} \subseteq B - B^{\infty}$, $C^{\vartheta_I} \subseteq B - B^{\infty}$ by 2.10 (e). It follows the existence of $m \in W_{\vartheta(B,g)}$ such that $y \in B^m$. Since $z \in B$ we have $g^n(z) = y$ which implies $S(B, g)(y) = S(B, g)(g^n(z)) \geqq S(B, g)(z) + n$ by 1.19 (a). Clearly, $z \in C^{\vartheta_I}$ implies $n^* \leqq S(B, g)(z) < S(B, g)(y) = m$. By 2.12, we have $\vartheta_I + (-n^* + \vartheta(B, g)) = \varkappa = S(C, h)(y) \leqq \vartheta_I + (m - n^*) = \vartheta_I + (-n^* + m)$. It follows $-n^* + \vartheta(B, g) = -\vartheta_I + (\vartheta_I + (-n^* + \vartheta(B, g))) \leqq -\vartheta_I + (\vartheta_I + (-n^* + m)) = -n^* + m$ by 1.0 (iii) and (iv) which implies $\vartheta(B, g) = n^* + (-n^* + \vartheta(B, g)) \leqq n^* + (-n^* + m) = m$ by 1.0 (iii) and (i). Thus, $\vartheta(B, g) \leqq m$ which is a contradiction.

It follows $\vartheta(C, h) = \vartheta_I + (-n^* + \vartheta(B, g))$.

3. NECESSARY CONDITIONS

3.1. Lemma. *Let* (A, f) *be a connected unary algebra. If* $|A^{\infty}| < \aleph_0$ *then* $Z(A, f) = A^{\infty}$ *and* $R(A, f) = |A^{\infty}|$.

Proof. By 1.15 we have $Z(A, f) \subseteq A^{\infty}$.

Let us suppose $|A^\infty| < \aleph_0$. We prove $A^\infty \subseteq Z(A, f)$. It holds if $A^\infty = \emptyset$. Thus, we can suppose $A^\infty \neq \emptyset$. Let us have $x \in A^\infty$. Then there is a sequence $(x_i)_{i \in N}$ such that $f(x_{i+1}) = x_i$ for each $i \in N$ and $x_0 = x$. Clearly, $x_i \in A^\infty$ for each $i \in N$. From the finiteness of A^∞, it follows the existence of $i, j \in N$, $i < j$, such that $x_i = x_j$. We prove by an easy induction that $f^n(x_n) = x$ for each $n \in N$. It follows $f^i(x_i) = x = f^j(x_j) = f^j(x_i)$. We put $d = j - i > 0$. By 1.5 (b), we have $x \in Z(A, f)$ because $f^d(x) = f^d(f^i(x_i)) = f^j(x_i) = x$.

We have proved $Z(A, f) = A^\infty$ which implies $R(A, f) = |A^\infty|$.

3.2. Lemma. *Let (A, f) be a connected unary algebra, suppose $\lambda, \mu \in W_{\vartheta(A,f)}$, $\lambda < \mu$. Then the following assertions hold:*

(a) *If $x, y \in A^\mu$, $x' \in A^\lambda$, $m, n \in N - \{0\}$, $f^m(x') = x$, $f^n(x') = y$ then $x = y$.*

(b) *If $\varphi : A^\mu \to A^\lambda$ is a map such that, for each $x \in A^\mu$, there exists $n(x) \in N - \{0\}$ with the property $f^{n(x)}(\varphi(x)) = x$ then φ is injective.*

Proof of (a). Let us have $x, y \in A^\mu$, $x' \in A^\lambda$, $m, n \in N - \{0\}$, $f^m(x') = x$, $f^n(x') = y$. Suppose $m \geqq n$. Then $x = f^m(x') = f^{m-n}(f^n(x')) = f^{m-n}(y)$. Thus, $f^{m-n}(y) = x \in A^\mu$, $f^0(y) = y \in A^\mu$ which implies $x = y$ by 1.19 (b).

Proof of (b). Suppose that $\varphi : A^\mu \to A^\lambda$ is such a map that, for each $x \in A^\mu$, there exists $n(x) \in N - \{0\}$ with the property $f^{n(x)}(\varphi(x)) = x$. Let $s, t \in A^\mu$ be such elements that $\varphi(s) = \varphi(t)$. Then there exist $n(s)$, $n(t) \in N - \{0\}$ such that $s = f^{n(s)}(\varphi(s))$, $t = f^{n(t)}(\varphi(t)) = f^{n(t)}(\varphi(s))$. Then, by (a), we have $s = t$ and (b) holds.

3.3. Lemma. *Let (A, f) be a connected unary algebra, suppose $\lambda, \mu \in W_{\vartheta(A,f)}$, $\lambda \leqq \mu$. Then $|A^\mu| \leqq |A^\lambda|$.*

Proof. By 1.16, there exists a map $\varphi : A^\mu \to A^\lambda$ such that, for each $x \in A^\mu$, there is $n(x) \in N - \{0\}$ such that $f^{n(x)}(\varphi(x)) = x$. By 3.2 (b), this map is injective. Thus $|A^\mu| \leqq |A^\lambda|$.

3.4. Lemma. *Let (A, f) be a connected unary algebra and α a limit ordinal with the property $\alpha \leqq \vartheta(A, f)$. If (A, f) is no ∞-algebra suppose $\alpha < \vartheta(A, f)$. Then $|A^\varkappa| \geqq |\mathrm{cf}\, \alpha|$ for each $\varkappa \in W_\alpha$.*

Proof. If $\alpha = 0$ then we have nothing to prove as $W_\alpha = \emptyset$.

Suppose $\alpha > 0$.

(1) Suppose first $\alpha \neq \vartheta(A, f)$. Then $x \in A^\alpha$ implies $f^{-1}(x) \subseteq \bigcup_{\varkappa \in W_\alpha} A^\varkappa$, $x \in A - \bigcup_{\varkappa \in W_\alpha} A^\varkappa$.

Let $\varkappa \in W_\alpha$ be an arbitrary ordinal. Then there is an ordinal $\lambda \in W_\alpha$, $\lambda > \varkappa$ and an element $y \in A^\lambda$ such that $f(y) = x$. Indeed, if such λ, y do not exist then there is an ordinal $\mu \in W_\alpha$ such that $f^{-1}(x) \subseteq \bigcup_{\nu \in W_\mu} A^\nu$. Further, $x \in A - \bigcup_{\nu \in W_\alpha} A^\nu \subseteq A - \bigcup_{\nu \in W_\mu} A^\nu$ which implies $x \in A^\mu$ in contradiction to 1.8.

(2) Let $(\varkappa_\nu)_{\nu \in W_{\mathrm{cf}\,\alpha}}$ be an arbitrary increasing sequence of ordinals such that $\sup_{\nu \in W_{\mathrm{cf}\,\alpha}} \varkappa_\nu = \alpha$. By (1), there is an ordinal $\mu_0 \in W_\alpha$, $\mu_0 > \varkappa_0$ and an element $x_{\mu_0} \in A^{\mu_0}$ such that $f(x_{\mu_0}) = x$.

Let $\varrho \in W_{\mathrm{cf}\,\alpha}$ be an arbitrary ordinal and suppose that we have constructed, for each ordinal $\nu < \varrho$, an ordinal μ_ν such that $\varkappa_\nu < \mu_\nu < \alpha$ and an element $x_{\mu_\nu} \in A^{\mu_\nu}$ in such a way that $(\mu_\nu)_{\nu \in W_\varrho}$ is an increasing sequence. Then $\sup_{\nu \in W_\varrho} \mu_\nu < \alpha$ because $\varrho < \mathrm{cf}\,\alpha$ and $\mathrm{cf}\,\alpha$ is the least ordinal cofinal with α. Thus, we can take an ordinal $\mu_\varrho \in W_\alpha$ such that $\mu_\varrho > \varkappa_\varrho$, $\mu_\varrho > \sup_{\nu \in W_\varrho} \mu_\nu$ and an element $x_{\mu_\varrho} \in A^{\mu_\varrho}$ such that $f(x_{\mu_\varrho}) = x$, by (1).

By transfinite induction, we obtain an increasing sequence of ordinals $(\mu_\nu)_{\nu \in W_{\mathrm{cf}\,\alpha}}$ and a sequence of elements $(x_{\mu_\nu})_{\nu \in W_{\mathrm{cf}\,\alpha}}$ such that $f(x_{\mu_\nu}) = x$ for each $\nu \in W_{\mathrm{cf}\,\alpha}$. Further, we have $\alpha = \sup_{\nu \in W_{\mathrm{cf}\,\alpha}} \varkappa_\nu \leqq \sup_{\nu \in W_{\mathrm{cf}\,\alpha}} \mu_\nu \leqq \alpha$; thus, $\sup_{\nu \in W_{\mathrm{cf}\,\alpha}} \mu_\nu = \alpha$.

(3) Let $\varkappa \in W_\alpha$ be arbitrary. By 1.16, for each $\nu \in W_{\mathrm{cf}\,\alpha}$ such that $\mu_\nu > \varkappa$ there exists $y_\nu \in A^\varkappa$ and $n_\nu \in N - \{0\}$ such that $f^{n_\nu}(y_\nu) = x_{\mu_\nu}$. We put $X = \{x_{\mu_\nu};\ \nu \in W_{\mathrm{cf}\,\alpha}, \varkappa < \mu_\nu\}$. Clearly, $\nu, \nu' \in W_{\mathrm{cf}\,\alpha}$, $\nu \neq \nu'$, $\varkappa < \mu_\nu, \mu_{\nu'}$, imply $\mu_\nu \neq \mu_{\nu'}$ because $(\mu_\lambda)_{\lambda \in W_{\mathrm{cf}\,\alpha}}$ is an increasing sequence. It implies $x_{\mu_\nu} \neq x_{\mu_{\nu'}}$ by 1.8. Suppose $y_\nu = y_{\nu'}$, $n_\nu \leqq n_{\nu'}$. We put $d = n_{\nu'} - n_\nu$ and we have $x_{\mu_{\nu'}} = f^{n_{\nu'}}(y_{\nu'}) = f^{n_\nu + d}(y_\nu) = f^d(f^{n_\nu}(y_\nu)) = f^d(x_{\mu_\nu})$. Then, for $d = 0$, we have $x_{\mu_{\nu'}} = x_{\mu_\nu}$ which is a contradiction. Thus, $d > 0$ and $x_{\mu_{\nu'}} = f^d(x_{\mu_\nu}) = f^{d-1}(f(x_{\mu_\nu})) = f^{d-1}(x)$ which implies $x = f(x_{\mu_{\nu'}}) = f(f^{d-1}(x)) = f^d(x)$. It follows from $d > 0$ that $x \in Z(A, f)$ by 1.5 (b); thus $x \in A^\infty$ by 1.3 and 1.15 which contradicts 1.13. Thus, $y_\nu \neq y_{\nu'}$.

We have proved that there exists an injection of X into $A^\varkappa$. Clearly, $|X| = |\mathrm{cf}\,\alpha|$. Thus $|A^\varkappa| \geqq |\mathrm{cf}\,\alpha|$.

(4) Suppose now $\alpha = \vartheta(A, f)$. By our hypothesis, (A, f) is an ∞-algebra. Let (B, g) be a cone such that $B = B^0 \cup B^\infty$ where $|B^0| = 1$, $|B^\infty| = 1$. Let $\varphi : E(A, f) \to B^0$ is the only map of $E(A, f)$ onto B^0. We put $I = \{1\}$, $A_1 = A$, $f_1 = f$, $(C, h) = \bigcup_{i \in I} (A_i, f_i) \oplus_\varphi (B, g)$.

By 2.9, (C, h) is a connected unary algebra. We define ϑ_I, n^* by 2.12. Clearly, $\vartheta_I = \vartheta(A, f)$. If $x \in B^0$ then $S(C, h)(x) = \vartheta(A, f)$ by 2.10 (g) and we have $x \in C^{\vartheta_I}$ which implies $n^* = 0$. Clearly, $\vartheta(B, g) = 1$. It follows $\vartheta(C, h) = \vartheta(A, f) + 1$ by 2.13. Thus, $\alpha = \vartheta(A, f) < \vartheta(C, h)$. For each $\varkappa \in W_\alpha$, we have $|C^\varkappa| \geqq |\mathrm{cf}\,\alpha|$ by (1), (2), (3). By 2.10 (d), we have $C^\varkappa \subseteq (B - B^\infty) \cup (A - A^\infty) = B^0 \cup (A - A^\infty)$. As we have seen, $S(C, h)(x) = \vartheta(A, f)$ for $x \in B^0$. It follows $C^\varkappa \subseteq A - A^\infty$. By 2.10 (h), we have $C^\varkappa = A^\varkappa$. Thus $|A^\varkappa| \geqq |\mathrm{cf}\,\alpha|$.

3.5. Lemma. *Let (A, f) be a non empty connected unary algebra which is not an ∞-algebra. Then the following assertions hold:*

(a) *$\vartheta(A, f)$ is a limit ordinal cofinal with ω_0.*

(b) *If $\lambda \in W_{\vartheta(A,f)}$ is such an ordinal that $|A^\lambda| < \aleph_0$ then there is such an ordinal $\mu \in W_{\vartheta(A,f)}$ that $|A^\mu| = 1$.*

Proof of (a). Suppose that $\vartheta(A, f)$ is an isolated ordinal. Then there is $x \in A$ such that $S(A, f)(x) = \vartheta(A, f) - 1$. By 1.19 (a), we have $S(A, f)(x) < S(A, f)(f(x))$. If $S(A, f)(f(x)) \in \mathrm{Ord}$ then $S(A, f)(f(x)) \geqq \vartheta(A, f)$ which is impossible. Thus, $S(A, f)(f(x)) = \infty$ which contradicts the hypothesis $A^\infty = \emptyset$. Thus, $\vartheta(A, f)$ is a limit ordinal.

Let $x \in A$ be such an element that $S(A, f)(x) = 0$; such an element exists because $\vartheta(A, f) > 0$. For each $\varkappa \in W_{\vartheta(A,f)}$ there is an element $y_\varkappa \in A$ such that $S(A, f)(y_\varkappa) = \varkappa$. Since (A, f) is connected there are $m_\varkappa, n_\varkappa \in N$ such that $f^{n_\varkappa}(x) = f^{m_\varkappa}(y_\varkappa)$. By 1.19 (a), we have $S(A, f)(f^{n_\varkappa}(x)) = S(A, f)(f^{m_\varkappa}(y_\varkappa)) \geqq \varkappa$. Thus $(S(A, f)(f^n(x)))_{n \in N}$ is a sequence of the type ω_0 such that $W_{\vartheta(A,f)}$ is cofinal with this sequence.

Proof of (b). Let us have $\lambda \in W_{\vartheta(A,f)}$, $|A^\lambda| < \aleph_0$. If $|A^\lambda| = 1$ then we have nothing to prove. Suppose $|A^\lambda| \geqq 2$, let $x, y \in A^\lambda$ be such elements that $x \neq y$. As (A, f) is connected there are $n, m \in N - \{0\}$ such that $f^n(x) = f^m(y) = z$. Since $A^\infty = \emptyset$ there is $\lambda_1 \in W_{\vartheta(A,f)}$, $\lambda_1 > \lambda$ such that $z \in A^{\lambda_1}$. By 1.16, there is a map $\varphi : A^{\lambda_1} \to A^\lambda$ such that $\varphi(z) = x$ and that, for each $t \in A^{\lambda_1}$, there is $k \in N - \{0\}$ such that $f^k(\varphi(t)) = t$. By 3.2 (b), this map is injective.

We prove that $y \notin \varphi(A^{\lambda_1})$. Suppose, on the contrary, the existence of $z' \in A^{\lambda_1}$ with the property $\varphi(z') = y$. Then there is $p \in N - \{0\}$ such that $f^p(y) = f^p(\varphi(z')) = z' \in A^{\lambda_1}$. We have $f^m(y) = z \in A^{\lambda_1}$. It follows $z = z'$ by 3.2 (a) which implies $x = \varphi(z) = \varphi(z') = y$ which is a contradiction. Thus $\varphi : A^{\lambda_1} \to A^\lambda$ is not a surjection. Since A^λ is a finite set we have $|A^\lambda| > |A^{\lambda_1}|$.

We proceed similarly with the set A^{λ_1}, $\lambda_1 \in W_{\vartheta(A,f)}$ as A^{λ_1} is a finite set. Since $\vartheta(A, f)$ is a limit ordinal, we obtain, after a finite number of steps, an ordinal $\mu \in W_{\vartheta(A,f)}$ such that $|A^\mu| = 1$.

3.6. Definition. Let $\alpha \in \mathrm{Ord}$ and suppose that $(m_\varkappa)_{\varkappa \in W_\alpha \cup \{\infty\}}$ is a sequence of cardinals. We put

$$\operatorname{crit}(m_\varkappa)_{\varkappa \in W_\alpha \cup \{\infty\}} = \begin{cases} W_\alpha \cup \{\alpha\} & \text{if } m_\infty \neq 0 \\ W_\alpha & \text{if } m_\infty = 0\,. \end{cases}$$

3.7. Definition. Let $\Gamma \subseteq \mathrm{Ord} \cup \{\infty\}$ and suppose that $(m_\varkappa)_{\varkappa \in \Gamma}$ is a sequence of cardinals. This sequence is called *suitable* if the following conditions are satisfied:

(1) $\Gamma = W_\alpha \cup \{\infty\}$ for some $\alpha \in \mathrm{Ord}$, the sequence $(m_\varkappa)_{\varkappa \in W_\alpha}$ is non-increasing and $m_\alpha \neq 0$ for each $\varkappa \in W_\alpha$.

(2) If $m_\infty = 0$ then (a) α is a limit ordinal cofinal with ω_0, (b) the existence of $\lambda \in W_\alpha$ with the property $m_\lambda < \aleph_0$ implies the existence of $\mu \in W_\alpha$ with the property $m_\mu = 1$.

(3) For an arbitrary limit ordinal $\mu \in \operatorname{crit}(m_\varkappa)_{\varkappa \in W_\alpha \cup \{\infty\}}$ and for an arbitrary $\lambda \in W_\mu$ we have $m_\lambda \geqq |\operatorname{cf} \mu|$.

3.8. Theorem. *Let (A, f) be a non empty connected unary algebra. Then the following assertions hold:*

(a) *If $|A^\infty| < \aleph_0$ then $R(A, f) = |A^\infty|$.*

(b) *The sequence $(|A^\varkappa|)_{\varkappa \in W_{\vartheta(A,f)} \cup \{\infty\}}$ is suitable.*

Proof. (a) follows by 3.1. The property (1) of 3.7 follows by definition of $\vartheta(A, f)$ and by 3.3, the property (2) of 3.7 follows by 3.5 and the property (3) of 3.7 follows by 3.4.

3.9. Lemma. *Let $\alpha \in \mathrm{Ord}$, let $(m_\varkappa)_{\varkappa \in W_\alpha \cup \{\infty\}}$ be a suitable sequence of cardinals with the property $m_\infty = 1$. If $\beta \in W_\alpha$ then $(m_\varkappa)_{\varkappa \in W_\beta \cup \{\infty\}}$ is a suitable sequence with the property $m_\infty = 1$.*

Proof. The sequence $(m_\varkappa)_{\varkappa \in W_\beta \cup \{\infty\}}$ satisfies the condition (1) of 3.7. The condition (2) is satisfied trivialy as $m_\infty = 1$. If $\mu \in \operatorname{crit}(m_\varkappa)_{\varkappa \in W_\beta \cup \{\infty\}}$ then $\mu \leqq \beta$ which implies $\mu \in \operatorname{crit}(m_\varkappa)_{\varkappa \in W_\alpha \cup \{\infty\}}$. Thus, for each limit ordinal $\mu \in \operatorname{crit}(m_\varkappa)_{\varkappa \in W_\beta \cup \{\infty\}}$ and each $\lambda \in W_\mu$ we have $m_\lambda \geqq |\operatorname{cf} \mu|$ which is (3) of 3.7.

3.10. Lemma. *Let $\alpha \in \mathrm{Ord}$, let $(m_\varkappa)_{\varkappa \in W_\alpha \cup \{\infty\}}$ be a suitable sequence of cardinals such that $m_\infty = 0$. We put $m'_\varkappa = m_\varkappa$ for each $\varkappa \in W_\alpha$, $m'_\infty = 1$. Then $(m'_\varkappa)_{\varkappa \in W_\beta \cup \{\infty\}}$ is a suitable sequence for each $\beta \in W_\alpha$.*

Proof. The condition (1) of 3.7 is satisfied by the sequence $(m'_\varkappa)_{\varkappa \in W_\beta \cup \{\infty\}}$, the condition (2) of 3.7 is satisfied trivially as $m'_\infty = 1$. Clearly, $\beta \in W_\alpha$ implies $\beta \in \operatorname{crit}(m_\varkappa)_{\varkappa \in W_\alpha \cup \{\infty\}}$ which implies $\operatorname{crit}(m'_\varkappa)_{\varkappa \in W_\beta \cup \{\infty\}} = W_\beta \cup \{\beta\} \subseteq W_\alpha = \operatorname{crit}(m_\varkappa)_{\varkappa \in W_\alpha \cup \{\infty\}}$. If $\mu \in \operatorname{crit}(m'_\varkappa)_{\varkappa \in W_\beta \cup \{\infty\}}$ is a limit ordinal and $\lambda \in W_\mu$ then $\mu \in \operatorname{crit}(m_\varkappa)_{\varkappa \in W_\alpha \cup \{\infty\}}$ which implies $|m'_\lambda| = |m_\lambda| \geqq |\operatorname{cf} \mu|$. Thus, the condition (3) of 3.7 is satisfied by the sequence $(m'_\varkappa)_{\varkappa \in W_\beta \cup \{\infty\}}$.

3.11. Lemma. *Let $\alpha \in \mathrm{Ord}$, let $(m_\varkappa)_{\varkappa \in W_\alpha \cup \{\infty\}}$ be a suitable sequence of cardinals such that $m_\infty \neq 0$. We put $m'_\varkappa = m_\varkappa$ for each $\varkappa \in W_\alpha$, $m'_\infty = 1$. Then $(m'_\varkappa)_{\varkappa \in W_\alpha \cup \{\infty\}}$ is a suitable sequence.*

Proof. $(m'_\varkappa)_{\varkappa \in W_\alpha \cup \{\infty\}}$ satisfies obviously the condition (1) and (2) of 3.7. Clearly, $\operatorname{crit}(m'_\varkappa)_{\varkappa \in W_\alpha \cup \{\infty\}} = W_\alpha \cup \{\alpha\} = \operatorname{crit}(m_\varkappa)_{\varkappa \in W_\alpha \cup \{\infty\}}$. Thus, if $\mu \in \operatorname{crit}(m'_\varkappa)_{\varkappa \in W_\alpha \cup \{\infty\}}$ is a limit ordinal and $\lambda \in W_\mu$ then $\mu \in \operatorname{crit}(m_\varkappa)_{\varkappa \in W_\alpha \cup \{\infty\}}$ and $m'_\lambda = m_\lambda \geqq |\operatorname{cf} \mu|$. Thus, $(m'_\varkappa)_{\varkappa \in W_\alpha \cup \{\infty\}}$ satisfies the condition (3) of 3.7.

4.1. Lemma. *Let* $(C, h) = \bigcup_{i \in I} (A_i, f_i) \oplus_{\varphi} (B, g)$ *be a unary algebra defined in* 2.7. *We put* $\vartheta_I = \sup_{i \in I} \vartheta(A_i, f_i)$. *We suppose that* $\emptyset \neq B^0 \subseteq C^{\vartheta_I}$. *Then the following conditions hold*:

(a) $n^* = 0$ *where* n^* *is defined according to* 2.12, $\vartheta(C, h) = \vartheta_I + \vartheta(B, g)$ *and, if we put* $n(\varkappa) = -\vartheta_I + \varkappa$ *for each* $\varkappa$ *with the property* $\vartheta_I \leqq \varkappa < \vartheta(C, h)$ *then* $\{n(\varkappa); \vartheta_I \leqq \varkappa < \vartheta(C, h)\} = W_{\vartheta(B,g)}$.

(b) $C^\varkappa = B^{n(\varkappa)}$ *for each* $\varkappa$, $\vartheta_I \leqq \varkappa < \vartheta(C, h)$.

Proof of (a). If $\emptyset \neq B^0 \subseteq C^{\vartheta_I}$ then $n^* = 0$ by 2.12. It follows $\vartheta(C, h) = \vartheta_I + (-n^* + \vartheta(B, g)) = \vartheta_I + \vartheta(B, g)$ by 2.13.

Further, if $\vartheta_I \leqq \varkappa < \vartheta(C, h)$ then $n(\varkappa) = -\vartheta_I + \varkappa < -\vartheta_I + \vartheta(C, h) = \vartheta(B, g)$. On the other hand, if $n < \vartheta(B, g)$ then $n = -\vartheta_I + (\vartheta_I + n)$ where $\vartheta_I \leqq \vartheta_I + n < \vartheta_I + \vartheta(B, g) < \vartheta(C, h)$.

Proof of (b). (1) For each $m < \vartheta(B, g)$, we have $S(C, h)(B^m) \leqq \vartheta_I + (-n^* + m) = \vartheta_I + m$ by 2.12 and (a). Further, $x \in B^m$ implies the existence of $y \in B^0 \subseteq C^{\vartheta_I}$ such that $h^m(y) = g^m(y) = x$ because (B, g) is a cone. It follows $S(C, h)(x) = S(C, h)(h^m(y)) \geqq S(C, h)(y) + m = \vartheta_I + m$ by 1.19 (a). Thus, $S(C, h)(B^m) = \vartheta_I + m$ for each $m < \vartheta(B, g)$. It implies, for each $\varkappa$, $\vartheta_I \leqq \varkappa < \vartheta(C, h)$, that $B^{n(\varkappa)} \subseteq C^{\vartheta_I + n(\varkappa)} = C^\varkappa$ by (a).

(2) $\vartheta_I \leqq \varkappa < \vartheta(C, h)$ implies $C^\varkappa \subseteq B - B^\infty = \bigcup_{n \in W_{\vartheta(B,g)}} B^n$ by 2.10 (e). It implies $C^\varkappa = C^\varkappa \cap \bigcup_{n \in W_{\vartheta(B,g)}} B^n = \bigcup_{n \in W_{\vartheta(B,g)}} (C^{\vartheta_I + n(\varkappa)} \cap B^n) = C^{\vartheta_I + n(\varkappa)} \cap B^{n(\varkappa)}$ by (a) and (1). It follows $C^\varkappa \subseteq B^{n(\varkappa)}$.

Thus, we have $C^\varkappa = B^{n(\varkappa)}$ for each $\varkappa$, $\vartheta_I \leqq \varkappa < \vartheta(C, h)$ by (1) and (2).

4.2. Lemma. *Let* (A, f) *be an* ∞*-algebra such that* $\vartheta(A, f) > 0$ *is an isolated ordinal,* (B, g) *a cone disjoint with* (A, f) *such that* $B^0 \neq \emptyset$, $|B^0| \leqq |A^{\vartheta(A,f)-1}|$. *Then the following assertions hold*:

(a) *There exists a surjection* $\psi : A^{\vartheta(A,f)-1} \to B^0$ *which is a restriction of a surjection* $\varphi : E(A, f) \to B^0$.

We put $I = \{1\}$, $A_1 = A$, $f_1 = f$ and let $(C, h) = \bigcup_{i \in I} (A_i, f_i) \oplus_{\varphi} (B, g)$ *be a unary algebra defined in* 2.7.

(b) (C, h) *is a connected unary algebra.*

(c) $\vartheta(C, h) = \vartheta(A, f) + \vartheta(B, g)$.

(d) $C^\varkappa = A^\varkappa$ *for each* $\varkappa < \vartheta(A, f)$, $C^\varkappa = B^{n(\varkappa)}$ *for each* $\varkappa$, $\vartheta(A, f) \leqq \varkappa < \vartheta(C, h)$ *where* $n(\varkappa)$ *is defined according to* 4.1 (a), $C^\infty = B^\infty$.

Proof of (a). Since $\vartheta(A, f) > 0$ is an isolated ordinal then $\emptyset \neq A^{\vartheta(A,f)-1} \subseteq E(A, f)$ by 2.3 (c) and since $|A^{\vartheta(A,f)-1}| \geqq |B^0|$ then there is a surjection $\psi : A^{\vartheta(A,f)-1} \to B^0$ which is a restriction of a surjection $\varphi : E(A, f) \to B^0$.

Proof of (b). (C, h) is a connected unary algebra by 2.9.

Proof of (c). If $x \in B^0$ then there exists $z \in A^{\vartheta(A,f)-1}$ such that $h(z) = \psi(z) = \varphi(z) = x$. Since $A^{\vartheta(A,f)-1} \subseteq C^{\vartheta(A,f)-1}$ by 2.10 (a) we have $S(C, h)(z) = \vartheta(A, f) - 1$. It follows $S(C, h)(x) = S(C, h)(h(z)) > S(C, h)(z) = \vartheta(A, f) - 1$ by 1.19 (a) because $x \notin B^\infty = C^\infty$ with regard to 2.10 (b). Since $g^{-1}(x) = \emptyset$ we have $h^{-1}(x) = \varphi^{-1}(x) \subseteq A - A^\infty = \bigcup_{\varkappa \in W_{\vartheta(A,f)}} A^\varkappa \subseteq \bigcup_{\varkappa \in W_{\vartheta(A,f)}} C^\varkappa$ by 2.10 (a). Further, $x \in C - \bigcup_{\varkappa \in W_{\vartheta(A,f)}} C^\varkappa$ because $S(C, h)(x) \geqq \vartheta(A, f)$. It follows $x \in C^{\vartheta(A,f)}$ which is $x \in C^{\vartheta_I}$ because $\vartheta_I = \sup_{i \in I} \vartheta(A_i, f_i) = \vartheta(A, f)$. We have proved $B^0 \subseteq C^{\vartheta_I}$.

It implies $\vartheta(C, h) = \vartheta_I + \vartheta(B, g) = \vartheta(A, f) + \vartheta(B, g)$ by 4.1 (a).

Proof of (d). We have proved $B^0 \subseteq C^{\vartheta_I}$. It follows $B = B^\infty \cup \bigcup_{m \in W_{\vartheta(B,g)}} B^m = C^\infty \cup \bigcup_{m \in W_{\vartheta(B,g)}} C^{\vartheta(A,f)+m}$ by 2.10 (b), 4.1 (b) and 4.1 (a). Thus, $C^\varkappa \subseteq A - A^\infty$ for each $\varkappa < \vartheta(A, f)$ which implies $C^\varkappa = A^\varkappa$ for each $\varkappa < \vartheta(A, f)$ by 2.10 (h).

Further, $C^\varkappa = B^{n(\varkappa)}$ for each $\varkappa$, $\vartheta(A, f) \leqq \varkappa < \vartheta(C, h)$ by 4.1 (b).

Finally, $C^\infty = B^\infty$ follows by 2.10 (b).

4.3. Lemma. *Let $\{(A_i, f_i);\ i \in I\}$ be a set of mutually disjoint ∞-algebras such that $\vartheta(A_i, f_i) > 0$ for each $i \in I$. We put $\vartheta_I = \sup_{i \in I} \vartheta(A_i, f_i)$ and $I(\varkappa) = \{i \in I;\ \varkappa \in W_{\vartheta(A_i,f_i)}$ for each $\varkappa < \vartheta_I\}$. We suppose that, for each $\varkappa < \vartheta_I$, there is a cardinal $m_\varkappa \geqq \max\{|I|, \aleph_0\}$ such that $|A_i^\varkappa| = m_\varkappa$ for each $i \in I(\varkappa)$. Let (B, g) be a cone disjoint with all ∞-algebras (A_i, f_i) such that $|B - B^\infty| = |I|$. Then the following assertions hold:*

(a) *There exists a surjection $\varphi : \bigcup_{i \in I} E(A_i, f_i) \to B - B^\infty$ such that, for each $x \in B - B^\infty$, there is (precisely one) $i \in I$ such that $\varphi^{-1}(x) = E(A_i, f_i)$.*

Let $(C, h) = \bigcup_{i \in I} (A_i, f_i) \oplus_\varphi (B, g)$ be a unary algebra defined in 2.7.

(b) *(C, h) is a connected unary algebra.*

(c) *Let $I = W_\alpha$ for some limit ordinal α and suppose that $B - B^\infty \neq B^0$ implies $\alpha = \omega_0$ and $\varphi(E(A_i, f_i)) = B^i$ for each $i \in I$. If $(\vartheta(A_i, f_i))_{i \in I}$ is an increasing sequence then $\vartheta(C, h) = \vartheta_I$.*

(c') *If there is an ∞-algebra (A, f) such that $(A_i, f_i) \cong (A, f)$ for each $i \in I$ then $\vartheta(C, h) = \vartheta(A, f) + \vartheta(B, g)$.*

(d) *$|C^\varkappa| = m_\varkappa$ for each $\varkappa < \vartheta_I$, $C^\infty = B^\infty$.*

(d') *If there is an ∞-algebra (A,f) such that $(A_i, f_i) \cong (A,f)$ for each $i \in I$ then $\vartheta_I < \vartheta(C, h)$ and, for each $\varkappa$, $\vartheta_I \leqq \varkappa < \vartheta(C, h)$, $C^\varkappa = B^{n(\varkappa)}$ where $n(\varkappa)$ is defined according to* 4.1 (a).

Proof of (a). If $i \in I$ then $\vartheta(A_i, f_i) > 0$ which implies $A_i - A_i^\infty \neq \emptyset$; it follows $E(A_i, f_i) \neq \emptyset$ by 2.3 (a). Let $\psi : I \to B - B^\infty$ be a bijection; we put $\varphi(t) = \psi(i)$ for each $i \in I$ and $t \in E(A_i, f_i)$. Then $\varphi : \bigcup_{i \in I} E(A_i, f_i) \to B - B^\infty$ is a surjection with the property: for each $x \in B - B^\infty$, there is (precisely one) $i \in I$ such that $\varphi^{-1}(x) = E(A_i, f_i)$.

Proof of (b). (C, h) is connected unary algebra by 2.9.

Proof of (c). (1) We put $\{e_i\} = \varphi(E(A_i, f_i))$ for each $i \in I$. If $x \in B - B^\infty$ is arbitrary then, by (a), there is (precisely one) $i \in I$ such that $e_i = x$. Thus, $B - B^\infty = \{e_i;\ i \in I\}$.

(2) We prove that $S(A, f)(e_i) = \vartheta(A_i, f_i)$ for each $i \in I = W_\alpha$. Indeed, if $B - B^\infty = B^0$ then, for each $i \in I$, $e_i \in B^0$ and $h^{-1}(e_i) = \varphi^{-1}(e_i) = E(A_i, f_i)$ by 2.7. Thus, $S(C, h)(e_i) = \vartheta(A_i, f_i)$ by 2.10 (g).

We suppose that $B - B^\infty \neq B^0$. Then $I = W_{\omega_0}$ and $\{e_i\} = B^i$ for each $i \in I$. The assertion $S(C, h)(e_i) = \vartheta(A_i, f_i)$ $(i \in I)$ will be proved by induction.

If $i = 0$ then $e_0 \in B^0$ and, by 2.7 and 2.10 (g), $S(C, h)(e_0) = \vartheta(A_0, f_0)$.

Let $i \in I - \{0\}$ and suppose $S(C, h)(e_{i-1}) = \vartheta(A_{i-1}, f_{i-1})$. By 2.7, we have $h^{-1}(e_i) = E(A_i, f_i) \cup B^{i-1} = E(A_i, f_i) \cup \{e_{i-1}\}$. By 2.10 (f), it follows $\vartheta(A_i, f_i) = \min\{\alpha \in \mathrm{Ord};\ \alpha > S(C, h)(E(A_i, f_i))\}$ (see 2.11). Further, $(\vartheta(A_i, f_i))_{i \in I}$ is increasing and it implies $\vartheta(A_i, f_i) > \vartheta(A_{i-1}, f_{i-1}) = S(C, h)(e_{i-1})$. Thus, $\vartheta(A_i, f_i) = \min\{\alpha \in \mathrm{Ord};\ \alpha > S(C, h)(E(A_i, f_i) \cup \{e_{i-1}\}) = \min\{\alpha \in \mathrm{Ord};\ \alpha > S(C, h)(h^{-1}(e_i))\} = S(C, h)(e_i)$.

(3) By (1) and (2), there exists, for each $x \in B - B^\infty$, $i \in I$ such that $S(C, h)(x) = \vartheta(A_i, f_i)$. Further, we have $\vartheta(A_i, f_i) \neq \vartheta_I$ for each $i \in I$ because $(\vartheta(A_i, f_i))_{i \in I}$ is increasing and $|I| \geqq \aleph_0$. Thus, $S(C, h)(x) \neq \vartheta_I$ for each $x \in B - B^\infty$. It follows $C^{\vartheta_I} = \emptyset$ and $n^* = \vartheta(B, g)$ by 2.12. We obtain $\vartheta(C, h) = \vartheta_I + (-n^* + \vartheta(B, g)) = \vartheta_I$ by 2.13.

Proof of (c'). $\vartheta_I = \sup_{i \in I} \vartheta(A_i, f_i) = \vartheta(A, f)$ by 1.20. Further, $B^0 \neq \emptyset$ because $B - B^\infty \neq \emptyset$ and we have $S(C, h)(x) = \vartheta(A, f)$ for each $x \in B^0$ by (a) and 2.10 (g). It implies $B^0 \subseteq C^{\vartheta_I}$. We obtain $n^* = 0$ by 4.1 (a) and $\vartheta(C, h) = \vartheta_I + (-n^* + \vartheta(B, g)) = \vartheta(A, f) + \vartheta(B, g)$ by 2.13.

Proof of (d). We put $\varphi(E(A_i, f_i)) = \{e_i\}$ for each $i \in I$; then $B - B^\infty \subseteq \bigcup_{i \in I} [e_i]_{(C,h)}$ by (a). Let us have $\varkappa < \vartheta_I$. We put $m^* = |C^\varkappa \cap (B - B^\infty)|$. Then $C^\varkappa \cap (B - B^\infty) \subseteq \bigcup_{i \in I} (C^\varkappa \cap [e_i]_{(C,h)})$ which implies $m^* \leqq \sum_{i \in I} |C^\varkappa \cap [e_i]_{(C,h)}| \leqq |I|$ because $|C^\varkappa \cap$

$\cap [e_i]_{(C,h)}| \leqq 1$ by 1.19 (b). We have $C^\varkappa \subseteq (B - B^\infty) \cup \bigcup_{i \in I(\varkappa)} (A_i - A_i^\infty)$ by 2.10 (d) which implies $C^\varkappa = C^\varkappa \cap ((B - B^\infty) \cup \bigcup_{i \in I(\varkappa)} (A_i - A_i^\infty) = (C^\varkappa \cap (B - B^\infty)) \cup \bigcup_{i \in I(\varkappa)} A_i$ with disjoint summans by 2.10 (a) and 1.13. It follows $|C^\varkappa| = m^* + \sum_{i \in I(\varkappa)} |A_i^\varkappa| = m^* + |I(\varkappa)| \, m_\varkappa = m_\varkappa$ because $m_\varkappa \geqq \aleph_0$, $m_\varkappa \geqq |I| \geqq |I(\varkappa)|$, $m_\varkappa \geqq |I| \geqq m^*$ and $I(\varkappa) \neq \emptyset$.

$C^\infty = B^\infty$ follows by 2.10 (b).

Proof of (d'). $\vartheta(C, h) = \vartheta(A, f) + \vartheta(B, g) = \vartheta_I + \vartheta(B, g)$ by (c') and $\vartheta(B, g) > 0$ because $B - B^\infty \neq \emptyset$. It follows $\vartheta_I < \vartheta(C, h)$. Further, we have proved $\emptyset \neq B_0 \subseteq C^{\vartheta_I}$ in the proof of (c'). It implies $C^\varkappa = B^{n(\varkappa)}$ for each $\varkappa$, $\vartheta_I \leqq \varkappa < \vartheta(C, h)$, by 4.1 (b).

4.4. Definition. Let us have $\alpha \in \mathrm{Ord}$, let $(m_\varkappa)_{\varkappa \in W_\alpha \cup \{\infty\}}$ be a suitable sequence of cardinals, (A, f) an ∞-algebra. Then (A, f) is said *to have the property* (β) with respect to the given sequence if $\beta \in W_\alpha$, $\vartheta(A, f) = \beta$ and $|A^\varkappa| = m_\varkappa$ for each $\varkappa \in W_\beta$.

4.5. Lemma. *Let us have $\alpha \in \mathrm{Ord}$, $\alpha \geqq 2$, let $(m_\varkappa)_{\varkappa \in W_\alpha \cup \{\infty\}}$ be a suitable sequence of cardinals with the property $m_\infty \leqq 1$. If, for each $\beta \in W_\alpha$, there is an ∞-algebra having the property (β) with respect to the given sequence then there exists a connected unary algebra (A, f) such that $\vartheta(A, f) = \alpha$ and $|A^\varkappa| = m_\varkappa$ for each $\varkappa \in W_\alpha \cup \{\infty\}$.*

Proof. (I) If α is an isolated ordinal then $m_\infty = 1$ because $m_\infty \neq 0$ by 3.7. Thus, there exists $\alpha - 1 \in \mathrm{Ord}$ because $\alpha \geqq 2$ and an ∞-algebra (A, f) having the property $(\alpha - 1)$ with respect to the given sequence.

Two cases can occur:

(1) Suppose $m_\varkappa \geqq \aleph_0$ for each $\varkappa \in W_{\alpha-1}$.

Let $\{(A_i, f_i);\ i \in I\}$ be a set of mutually disjoint ∞-algebras such that $(A_i, f_i) \cong (A, f)$ for each $i \in I$ and $|I| = m_{\alpha-1}$. Let (B, g) be a cone disjoint with all agebras (A_i, f_i) such that $B = B^0 \cup B^\infty$, $|B^0| = m_{\alpha-1}$, $|B^\infty| = 1$.

Then $\vartheta(A_i, f_i) = \vartheta(A, f) = \alpha - 1 > 0$ for each $i \in I$ by 1.20. Further, we have $\vartheta_I = \sup_{i \in I} \vartheta(A_i, f_i) = \vartheta(A, f) = \alpha - 1$ and, for each $\varkappa < \alpha - 1$, $m_\varkappa \geqq m_{\alpha-1} = |I|$ which implies $m_\varkappa \geqq \max_{i \in I} \{|I|, \aleph_0\}$ and $|A_i^\varkappa| = m_\varkappa$ for each $i \in I = I(\varkappa)$ (see 4.3). Finally, we have $|B - B^\infty| = |B^0| = m_{\alpha-1} = |I|$. Then there exists a surjection $\varphi : \bigcup_{i \in I} E(A_i, f_i) \to B - B^\infty$ such that, for each $x \in B - B^\infty$, there is (precisely one) $i \in I$ such that $\varphi^{-1}(x) = E(A_i, f_i)$ by 4.3 (a). We put $(C, h) = \bigcup_{i \in I} (A_i, f_i) \oplus_\varphi (B, g)$.

By 4.3 (b), (C, h) is a connected unary algebra and $\vartheta(C, h) = \vartheta(A, f) + \vartheta(B, g) = (\alpha - 1) + 1 = \alpha$ by 4.3 (c') because $\vartheta(B, g) = 1$.

By 4.3 (d), we have $|C^{\varkappa}| = m_{\varkappa}$ for each $\varkappa < \alpha - 1$. By 4.3 (d'), we obtain $C^{\alpha-1} = B^0$ because $\vartheta(C, h) = \alpha$. It implies $|C^{\alpha-1}| = |B^0| = m_{\alpha-1}$. By 4.3 (d), we obtain $|C^{\infty}| = |B^{\infty}| = 1 = m_{\infty}$.

We have constructed a connected unary algebra (C, h) with the following properties: $\vartheta(C, h) = \alpha$, $|C^{\varkappa}| = m_{\varkappa}$ for each $\varkappa \in W_{\alpha} \cup \{\infty\}$.

(2) Suppose the existence of $\varkappa_0 \in W_{\alpha-1}$ such that $m_{\varkappa_0} < \aleph_0$.

Then $\alpha \geqq 2$ implies $\alpha - 1 \geqq 1$. Clearly, $\alpha - 1 \in \operatorname{crit}(m_{\varkappa})_{\varkappa \in W_{\alpha} \cup \{\infty\}}$. If $\alpha - 1$ were a limit ordinal then we should have $m_{\varkappa_0} \geqq |\operatorname{cf}(\alpha - 1)|$ by 3.7 (3) which is a contradiction to the finiteness of $m_{\varkappa_0}$. Thus, $\alpha - 1$ is an isolated ordinal.

Let (B, g) be a cone disjoint with (A, f) such that $B = B^0 \cup B^{\infty}$ and $|B^0| = m_{\alpha-1}$, $|B^{\infty}| = 1$.

$\vartheta(A, f) = \alpha - 1 > 0$ is an isolated ordinal and we have $B^0 \neq \emptyset$, $|B^0| = m_{\alpha-1} \leqq m_{\alpha-2} = |A^{\alpha-2}| = |A^{\vartheta(A,f)-1}|$. Then there exists a surjection $\psi : A^{\vartheta(A,f)-1} \to B^0$ which is a restriction of a surjection $\varphi : E(A, f) \to B^0$ by 4.2 (a). We put $I = \{1\}$, $A_1 = A, f_1 = f, (C, h) = \bigcup_{i \in I} (A_i, f_i) \underset{\varphi}{\oplus} (B, g)$.

By 4.2 (b), (C, h) is a connected unary algebra. Clearly, $\vartheta(B, g) = 1$ which implies $\vartheta(C, h) = \vartheta(A, f) + \vartheta(B, g) = (\alpha - 1) + 1 = \alpha$ by 4.2 (c).

Further, $C^{\varkappa} = A^{\varkappa}$ for each $\varkappa < \alpha - 1$, $C^{\alpha-1} = B^0$, $C^{\infty} = B^{\infty}$ by 4.2 (d) because $\vartheta(C, h) = \alpha$. It follows $|C^{\varkappa}| = m_{\varkappa}$ for each $\varkappa \in W_{\alpha} \cup \{\infty\}$.

(II) Suppose that α is a limit ordinal. We put $I = W_{\operatorname{cf}\alpha}$. Then there exists an increasing sequence of positive ordinals $(\beta_i)_{i \in I}$ such that $\sup_{i \in I} \beta_i = \alpha$. For each $i \in I$ there exists an ∞-algebra (A_i, f_i) having the property (β_i) with respect to the given sequence. We can suppose, without loss of generality, that the ∞-algebras (A_i, f_i) are mutually disjoint.

The set $\{(A_i, f_i);\ i \in I\}$ of ∞-algebras has the following properties: $\vartheta(A_i, f_i) = \beta_i > 0$ for each $i \in I$; $\vartheta_I = \sup_{i \in I} \vartheta(A_i, f_i) = \sup_{i \in I} \beta_i = \alpha$; if we put $I(\varkappa) = \{i \in I;\ \varkappa \in W_{\vartheta(A_i, f_i)}\}$ for each $\varkappa < \alpha$ (see 4.3) then, for each $i \in I(\varkappa)$, we have $|A_i^{\varkappa}| = m_{\varkappa}$ because (A_i, f_i) is an ∞-algebra having the property (β_i) with respect to the given sequence.

Two cases can occur:

(i) Let us have $m_{\infty} = 1$. Since $(m_{\varkappa})_{\varkappa \in W_{\alpha} \cup \{\infty\}}$ is a suitable sequence then $\operatorname{crit}(m_{\varkappa})_{\varkappa \in W_{\alpha} \cup \{\infty\}} = W_{\alpha} \cup \{\alpha\}$ and, by 3.7 (3), we have $m_{\varkappa} \geqq |\operatorname{cf}\alpha| = |I|$ for each $\varkappa \in W_{\alpha}$. Thus, for each $\varkappa < \alpha = \vartheta_I$, we have $m_{\varkappa} \geqq \max\{|I|, \aleph_0\}$ because $|I| \geqq \aleph_0$.

We take a cone (B, g) disjoint with all ∞-algebras (A_i, f_i) such that $B = B^0 \cup B^{\infty}$ where $|B^0| = |I| = |\operatorname{cf}\alpha|$, $|B^{\infty}| = 1$.

Thus, $B - B^{\infty} = B^0$ and $|B - B^{\infty}| = |I|$.

By 4.3 (a), there exists a surjection $\varphi : \bigcup_{i\in I} E(A_i, f_i) \to B - B^\infty$ such that, for each $x \in B - B^\infty$, there is (precisely one) $i \in I$ such that $\varphi^{-1}(x) = E(A_i, f_i)$. We put $(C, h) = \bigcup_{i\in I} (A_i, f_i) \oplus_\varphi (B, g)$.

Then (C, h) is a connected unary algebra by 4.3 (b). Further, $\vartheta(C, h) = \vartheta_I = \alpha$ by 4.3 (c).

Finally, $|C^\varkappa| = m_\varkappa$ for each $\varkappa < \vartheta_I = \alpha$, $|C^\infty| = |B^\infty| = 1 = m_\infty$ by 4.3 (d).

Thus, we have constructed a connected unary algebra (C, h) such that $\vartheta(C, h) = \alpha$ and $|C^\varkappa| = m_\varkappa$ for each $\varkappa \in W_\alpha \cup \{\infty\}$.

(ii) Let us have $m_\infty = 0$. Since $(m_\varkappa)_{\varkappa\in W_\alpha \cup\{\infty\}}$ is a suitable sequence we have $\operatorname{cf} \alpha = \omega_0$ by 3.7.

Two cases are possible:

(1) Suppose $m_\varkappa \geqq \aleph_0$ for each $\varkappa \in W_\alpha$.

Then, for each $\varkappa < \alpha = \vartheta_I$, we have $m_\varkappa \geqq \max\{|I|, \aleph_0\}$ because $|I| = |\operatorname{cf} \alpha| = |\omega_0| = \aleph_0$.

Let (B, g) be the cone (constructed in 2.5, 2) such that $|B^n| = 1$ for each $n \in N$. Suppose that (B, g) is disjoint with all ∞-algebras (A_i, f_i).

Thus, $B^\infty = \emptyset$ and $|B - B^\infty| = |B| = \aleph_0 = |I|$.

We take, by 4.3 (a), a surjection $\varphi : \bigcup_{i\in I} E(A_i, f_i) \to B$ such that $\varphi(E(A_i, f_i)) = B^i$ for each $i \in I = W_{\omega_0} = N$. We put $(C, h) = \bigcup_{i\in I} (A_i, f_i) \oplus_\varphi (B, g)$.

Then (C, h) is a connected unary algebra by 4.3 (b) and $\vartheta(C, h) = \vartheta_I = \alpha$ by 4.3 (c).

Further, $|C^\varkappa| = m_\varkappa$ for each $\varkappa < \alpha$, $|C^\infty| = |B^\infty| = 0 = m_\infty$ by 4.3 (d).

Thus, we have constructed a connected unary algebra (C, h) such that $\vartheta(C, h) = \alpha$ and $|C^\varkappa| = m_\varkappa$ for each $\varkappa \in W_\alpha \cup \{\infty\}$.

(2) Suppose the existence of $\varkappa_0 \in W_\alpha$ such that $m_{\varkappa_0} < \aleph_0$.

Clearly, $\varkappa \in W_\alpha$ implies $\varkappa \in \operatorname{crit} (m_\varkappa)_{\varkappa\in W_\alpha\cup\{\infty\}}$. If there is a limit ordinal $\varkappa$, $\varkappa_0 < \varkappa < \alpha$, then $m_{\varkappa_0} \geqq |\operatorname{cf} \varkappa|$ by 3.7 (3) which is a contradiction to the finiteness of $m_{\varkappa_0}$. Thus, each $\varkappa$ with the property $\varkappa_0 < \varkappa < \alpha$ is isolated.

We take an arbitrary λ, $\varkappa_0 < \lambda < \alpha$. Thus, there is an ∞-algebra (A, f) having the property (λ). Thus $\vartheta(A, f) = \lambda > 0$ is an isolated ordinal.

By 3.7 (2) (b), there is $\mu \in W_\alpha$ such that $m_\mu = 1$. It follows the existence of a cone (B, g) such that $|B^n| = m_{\lambda+n}$ for each $n \in N$ by 2.5.

Then $|A^{\vartheta(A,f)-1}| = |A^{\lambda-1}| = m_{\lambda-1} \geqq m_\lambda = |B^0|$.

By 4.2 (a), there exists a surjection $\psi : A^{\vartheta(A,f)-1} \to B^0$ which is a restriction of a surjection $\varphi : E(A, f) \to B^0$.

We put $J = \{1\}$, $A_1 = A$, $f_1 = f$ and $(C, h) = \bigcup_{i\in J} (A_i, f_i) \oplus_\varphi (B, g)$.

By 4.2 (b), (C, h) is a connected unary algebra and $\vartheta(C, h) = \vartheta(A, f) + \vartheta(B, g) = \lambda + \omega_0 = \alpha$ by 4.2 (c) because $\vartheta(B, g) = \omega_0$ and $\lambda + \omega_0$, α are both equal to the least limit ordinal greater than λ.

Further, $C^\varkappa = A^\varkappa$ for each $\varkappa < \lambda$ and $C^\varkappa = B^{n(\varkappa)}$ for each $\varkappa$, $\lambda \leqq \varkappa < \alpha$ where $n(\varkappa)$ is the only element of N such that $\varkappa = \lambda + n(\varkappa)$ (see 4.1 (a)), $C^\infty = B^\infty = \emptyset$ by 4.2 (d). It follows $|C^\varkappa| = |A^\varkappa| = m_\varkappa$ for each $\varkappa < \lambda$, $|C^\varkappa| = |B^{n(\varkappa)}| = m_{\lambda+n(\varkappa)} = m_\varkappa$ for each $\varkappa$, $\lambda \leqq \varkappa < \alpha$ and $|C^\infty| = 0 = m_\infty$. Thus, $|C^\varkappa| = m_\varkappa$ for each $\varkappa \in W_\alpha \cup \{\infty\}$.

4.6. Corollary. *Let* $\alpha \in \mathrm{Ord}$, *let* $(m_\varkappa)_{\varkappa \in W_\alpha \cup \{\infty\}}$ *be a suitable sequence of cardinals such that* $m_\infty = 1$. *Then there is a connected unary algebra* (A, f) *such that* $\vartheta(A, f) = \alpha$ *and* $|A^\varkappa| = m_\varkappa$ *for each* $\varkappa \in W_\alpha \cup \{\infty\}$.

Proof. For each ordinal, we denote by $V(\alpha)$ the following assertion: If $(m_\varkappa)_{\varkappa \in W_\alpha \cup \{\infty\}}$ is an arbitrary suitable sequence of cardinals such that $m_\infty = 1$ then there is a connected unary algebra (A, f) such that $\vartheta(A, f) = \alpha$ and $|A^\varkappa| = m_\varkappa$ for each $\varkappa \in W_\alpha \cup \{\infty\}$.

If we put $A = A^\infty$ where $|A^\infty| = 1 = m_\infty$ then we see that $V(0)$ holds. Similarly, if we define the cone $A = A^0 \cup A^\infty$ where $|A^0| = m_0$, $|A^\infty| = 1 = m_\infty$ then we see that $V(1)$ holds.

Let us have $\beta \geqq 2$ and suppose that $V(\gamma)$ holds for each $\gamma < \beta$. Let $(m_\varkappa)_{\varkappa \in W_\beta \cup \{\infty\}}$ be a suitable sequence of cardinals such that $m_\infty = 1$. If $\gamma \in W_\beta$ then the sequence $(m_\varkappa)_{\varkappa \in W_\gamma \cup \{\infty\}}$ is a suitable sequence of cardinals such that $m_\infty = 1$ by 3.9. Thus, by the induction hypothesis, there is a connected unary algebra (A_γ, f_γ) such that $\vartheta(A_\gamma, f_\gamma) = \gamma$ and $|A_\gamma^\varkappa| = m_\varkappa$ for each $\varkappa \in W_\gamma \cup \{\infty\}$. Thus, for each $\gamma \in W_\beta$, (A_γ, f_γ) is an ∞-algebra having the property (γ) with respect to the sequence $(m_\varkappa)_{\varkappa \in W_\beta \cup \{\infty\}}$ (cf. 4.4). By 4.5, there is a connected unary algebra (A, f) such that $\vartheta(A, f) = \beta$ and $|A^\varkappa| = m_\varkappa$ for each $\varkappa \in W_\beta \cup \{\infty\}$. Thus, $V(\beta)$ holds.

It follows by transfinite induction that $V(\alpha)$ holds for each ordinal α which is our assertion.

4.7. Corollary. *Let* $\alpha \in \mathrm{Ord}$, *let* $(m_\varkappa)_{\varkappa \in W_\alpha \cup \{\infty\}}$ *be a suitable sequence of cardinals such that* $m_\infty = 0$. *Then there is a connected unary algebra* (A, f) *such that* $\vartheta(A, f) = \alpha$ *and* $|A^\varkappa| = m_\varkappa$ *for each* $\varkappa \in W_\alpha \cup \{\infty\}$.

Proof. Since $m_\infty = 0$ the ordinal α is a limit ordinal by 3.7 which implies $\alpha \geqq 2$. We put $m'_\varkappa = m_\varkappa$ for each $\varkappa \in W_\alpha$, $m'_\infty = 1$. If $\beta \in W_\alpha$ then $(m'_\varkappa)_{\varkappa \in W_\beta \cup \{\infty\}}$ is a suitable sequence with the property $m'_\infty = 1$ by 3.10. By 4.6, there is an ∞-algebra (A_β, f_β) such that $\vartheta(A_\beta, f_\beta) = \beta$ and $|A_\beta^\varkappa| = m'_\varkappa = m_\varkappa$ for each $\varkappa \in W_\beta$. Thus, for each $\beta \in W_\alpha$, (A_β, f_β) has the property (β) with respect to the sequence $(m_\varkappa)_{\varkappa \in W_\alpha \cup \{\infty\}}$ (cf. 4.4). The assertion follows by 4.5.

4.8. Lemma. *Let $m > 0$ be a cardinal, $R \in N$ an ordinal such that $m < \aleph_0$ implies $R = m$. Then there is a connected unary algebra (A, f) such that $A = A^\infty$, $|A| = |A^\infty| = m$ and $R(A, f) = R$.*

Proof. Let A be an arbitrary set such that $|A| = m$. We take an arbitrary subset $B \subseteq A$ such that $|B| = R$. We have the following possibilities:

(I) $m < \aleph_0$.

Then $R = m$ and $B = A$. We put $A = \{a_1, a_2, \ldots, a_m\}$. We put $f(a_i) = a_{i+1}$ for each i, $1 \leqq i \leqq m - 1$, $f(a_m) = a_1$. Then (A, f) is a connected unary algebra such that $A = A^\infty = Z(A, f)$ which implies $|A| = |A^\infty| = m = R = |Z(A, f)| = R(A, f)$.

(II) $m \geqq \aleph_0$.

Then $|A - B| = m = \aleph_0 m$. We take an arbitrary set K such that $|K| = m$ and, for each $\varkappa \in K$, we define a subset $B_\varkappa \subseteq A - B$ such that $|B_\varkappa| = \aleph_0$, $A - B = \bigcup_{\varkappa \in K} B_\varkappa$ with disjoint summands. We have $A = B \cup \bigcup_{\varkappa \in K} B_\varkappa$. Two cases can occur:

(1) $R \neq 0$.

Then we put $B = \{a_1, a_2, \ldots, a_R\}$, $B_\varkappa = \{a_i^\varkappa;\ i \in N\}$ for each $\varkappa \in K$. We define $f(a_i) = a_{i+1}$ for i, $1 \leqq i \leqq R - 1$, $f(a_R) = a_1$, $f(a_i^\varkappa) = a_{i-1}^\varkappa$ for each $\varkappa \in K$, $i \in N - \{0\}$, $f(a_0^\varkappa) = a_1$ for each $\varkappa \in K$. Then (A, f) is a connected unary algebra, $R(A, f) = |Z(A, f)| = |B| = R$, $A^\infty = A$ which implies $|A| = |A^\infty| = m$.

(2) $R = 0$.

Then we have $B = \emptyset$. We put $B_\varkappa = \{a_i^\varkappa;\ i \in N\}$ for each $\varkappa \in K$, we take an arbitrary $\varkappa_0 \in K$ and we define $f(a_i^{\varkappa_0}) = a_{i+1}^{\varkappa_0}$ for each $i \in N$, $f(a_i^\varkappa) = a_{i-1}^\varkappa$ for each $\varkappa \in K - \{\varkappa_0\}$ and each $i \in N - \{0\}$, $f(a_0^\varkappa) = a_0^{\varkappa_0}$ for each $\varkappa \in K - \{\varkappa_0\}$.

Then, clearly, (A, f) is a connected unary algebra such that $Z(A, f) = \emptyset$ which implies $R(A, f) = 0 = R$. Further, $A^\infty = A$ and $|A| = |A^\infty| = m$.

4.9. Theorem. *Let $\alpha \in \mathrm{Ord}$, let $(m_\varkappa)_{\varkappa \in W_\alpha \cup \{\infty\}}$ be a suitable sequence of cardinals, let $R \in N$ be such that $m_\infty < \aleph_0$ implies $R = m_\infty$. Then there is a connected unary algebra (A, f) such that $R(A, f) = R$, $\vartheta(A, f) = \alpha$ and $|A^\varkappa| = m_\varkappa$ for each $\varkappa \in W_\alpha \cup \{\infty\}$.*

Proof. (I) If $m_\infty = 0$ then there is a connected unary algebra (A, f) such that $\vartheta(A, f) = \alpha$ and $|A^\varkappa| = m_\varkappa$ for each $\varkappa \in W_\alpha \cup \{\infty\}$ by 4.7. Further, $Z(A, f) \subseteq A^\infty$ by 1.15 which implies $R(A, f) = |Z(A, f)| \leqq |A^\infty| = m_\infty = 0 = R$; thus, $R(A, f) = R$.

(II) If $m_\infty \neq 0$ then we put $m'_\varkappa = m_\varkappa$ for each $\varkappa \in W_\alpha$ and $m'_\infty = 1$. By 3.11, $(m'_\varkappa)_{\varkappa \in W_\alpha \cup \{\infty\}}$ is a suitable sequence with the property $m'_\infty = 1$. By 4.6, there is a connected unary algebra (A, f) such that $\vartheta(A, f) = \alpha$ and $|A^\varkappa| = m'_\varkappa = m_\varkappa$ for each $\varkappa \in W_\alpha$. By 4.8, there is a cone (B, g) such that $B = B^\infty$, $|B| = |B^\infty| = m_\infty$ and $R(B, g) = R$. We can suppose, without loss of generality, that (A, f), (B, g) are mutually disjoint.

Two cases can occur:

(1) If $\alpha = 0$ then $W_\alpha \cup \{\infty\} = \{\infty\}$ and (B, g) has the properties $R(B, g) = R$, $|B^\infty| = m_\infty$.

(2) If $\alpha > 0$ then $\emptyset \neq A^0 \subseteq A - A^\infty$ which implies $E(A, f) \neq \emptyset$ by 2.3 (a). Let $\varphi : E(A, f) \to B$ be an arbitrary map. We put $I = \{1\}$, $A_1 = A$, $f_1 = f$, $(C, h) = \bigcup_{i \in I} (A_i, f_i) \oplus_{\varphi} (B, g)$.

Then (C, h) is connected unary algebra by 2.9. If ϑ_I and n^* are defined by 2.12 then $\vartheta_I = \vartheta(A, f)$. Clearly, $\vartheta(B, g) = 0$ which implies $n^* = 0$. By 2.13, $\vartheta(C, h) = \vartheta_I + (-n^* + \vartheta(B, g)) = \vartheta(A, f) = \alpha$. If $\varkappa \in W_\alpha$ then $\varkappa < \vartheta(A, f)$ which implies $C^\varkappa \subseteq (B - B^\infty) \cup (A - A^\infty) = A - A^\infty$ by 2.10 (d). It follows $C^\varkappa = A^\varkappa$ for each $\varkappa < \vartheta(A, f)$ by 2.10 (h). Thus, $|C^\varkappa| = m_\varkappa$ for each $\varkappa \in W_\alpha$. By 2.10 (b), (c), we have $C^\infty = B^\infty$ and $Z(C, h) = Z(B, g)$ which implies $|C^\infty| = |B^\infty| = m_\infty$, $R(C, h) = |Z(C, h)| = |Z(B, g)| = R(B, g) = R$. Thus, we have constructed a connected unary algebra (C, h) such that $R(C, h) = R$, $\vartheta(C, h) = \alpha$, $|C^\varkappa| = m_\varkappa$ for each $\varkappa \in W_\alpha \cup \{\infty\}$.

4.10. Theorem. *Let A be a set, $S : A \to \mathrm{Ord} \cup \{\infty\}$ a map, $R \in N$. Let the following conditions be satisfied:*

(a) *If $|S^{-1}(\infty)| < \aleph_0$ then $R = |S^{-1}(\infty)|$.*

(b) *The sequence $(|S^{-1}(\varkappa)|)_{\varkappa \in S(A)}$ is suitable.*

Then there is a unary operation f on A such that (A, f) is a non empty connected unary algebra and $S(A, f) = S$, $R(A, f) = R$.

Proof. By 3.7 (1), there is $\alpha \in \mathrm{Ord}$ such that $S(A) = W_\alpha \cup \{\infty\}$. By 4.9, there is a connected unary algebra (A_*, f_*) such that $\vartheta(A_*, f_*) = \alpha$, $R(A_*, f_*) = R$ and $|A_*^\varkappa| = |S^{-1}(\varkappa)|$ for each $\varkappa \in W_\alpha \cup \{\infty\}$. We have $|A_*| = \sum_{\varkappa \in W_\alpha \cup \{\infty\}} |A_*^\varkappa| = \sum_{\varkappa \in W_\alpha \cup \{\infty\}} |S^{-1}(\varkappa)| = |A|$. Thus, there is a bijection $\varphi : A_* \to A$ such that $\varphi \mid A_*^\varkappa : A_*^\varkappa \to S^{-1}(\varkappa)$ is a bijection for each $\varkappa \in W_\alpha \cup \{\infty\}$. We put $f(x) = \varphi(f_*(\varphi^{-1}(x)))$ for each $x \in A$. Then f is a unary operation on A such that $\varphi^{-1}(f(x)) = f_*(\varphi^{-1}(x))$ for each $x \in A$. It follows that φ^{-1} is a bijective homorphism of (A, f) onto (A_*, f_*). Thus, (A, f), (A_*, f_*) are isomorphic, φ is an isomorphism of (A_*, f_*) onto (A, f). By 1.20, we have $\vartheta(A, f) = \vartheta(A_*, f_*)$, $\varphi(A_*^\varkappa) = A^\varkappa$ for each $\varkappa \in W_{\vartheta(A,f)} \cup \{\infty\}$ and $\varphi(Z(A_*, f_*)) = Z(A, f)$. Thus, $R(A, f) = |Z(A, f)| = |\varphi(Z(A_*, f_*))| = |Z(A_*, f_*)| = R(A_*, f_*) = R$. Further, for each $\varkappa \in W_\alpha \cup \{\infty\}$ we have $S^{-1}(A, f)(\varkappa) = A^\varkappa = \varphi(A_*^\varkappa) = \varphi(\varphi^{-1}(S^{-1}(\varkappa))) = S^{-1}(\varkappa)$ which implies $S(A, f) = S$.

If $|S^{-1}(\infty)| \neq 0$ then $\emptyset \neq A^\infty \subseteq A$; if $|S^{-1}(\infty)| = 0$ then α is infinite by 3.7 (2) which implies $|S^{-1}(0)| \neq 0$ by 3.7 (1) which implies $\emptyset \neq A^0 \subseteq A$. Thus, (A, f) is non-empty.

5. SOLUTION OF THE PROBLEM

5.1. Main Theorem. *Let A be a set, $S : A \to \mathrm{Ord} \cup \{\infty\}$ a map, $R \in N$ a finite ordinal. Then the following conditions are equivalent:*

(A) *There is a unary operation f on A such that (A, f) is a non empty connected unary algebra, $S(A, f) = S$, $R(A, f) \mathrel{\overline{\top}} R$.*

(B) *The following conditions are satisfied:*

(a) *If $|S^{-1}(\infty)| < \aleph_0$ then $R = |S^{-1}(\infty)|$.*

(b) *The sequence $(|S^{-1}(\varkappa)|)_{\varkappa \in S(A)}$ is suitable.*

It is a consequence of 3.8 and 4.10.

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