Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 24 (1974), No. 1, 107–113

Persistent URL: http://dml.cz/dmlcz/101221

Czechoslovak Mathematical Journal, 24 (99) 1974, Praha

ON A PARTIAL PRODUCT STRUCTURE

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(Received February 21, 1973)

In his paper published in Annali di Mat. (vol. *11*, 1932, 17–90), E. CARTAN solved the equivalence problem for real hypersurfaces in $\mathscr{C}^2$. Unfortunately, his approach is not very precise and effective. Because of this, I solve the equivalence problem using other more direct methods. In what follows, the equivalence problem has been solved for a class of partial product structures; the results are, evidently, equivalent to those of E. Cartan. The theory of real hypersurfaces in $\mathscr{C}^n$ will be treated in another paper.

1. Be given a 3-dimensional differentiable manifold endowed with a structure consisting of the choice of two tangent directions at each of its points. Such a structure gives rise to a G-structure B_G as follows: the frame (v_1, v_2, v_3), $v_i \in T_m(M)$, belongs to B_G if and only if v_1 and v_2 span the given directions. If $(v_1, v_2, v_3) \subset B_G$ and $(w_1, w_2, w_3) \subset B_G$ are two (local) sections of B_G, there are functions $\alpha, \beta, \gamma, \delta, \varphi$ such that

$$(1.1) \qquad v_1 = \alpha w_1\,, \quad v_2 = \beta w_2\,, \quad v_3 = \gamma w_1 + \delta w_2 + \varphi w_3\,; \qquad \alpha\beta\varphi \neq 0\,.$$

We have

$$(1.2) \quad \begin{aligned} [v_1, v_2] &= a_1 v_1 + a_2 v_2 + a_3 v_3\,, & [w_1, w_2] &= A_1 w_1 + A_2 w_2 + A_3 w_3\,,\\ [v_1, v_3] &= b_1 v_1 + b_2 v_2 + b_3 v_3\,, & [w_1, w_3] &= B_1 w_1 + B_2 w_2 + B_3 w_3\,,\\ [v_2, v_3] &= c_1 v_1 + c_2 v_2 + c_3 v_3\,, & [w_2, w_3] &= C_1 w_1 + C_2 w_2 + C_3 w_3\,; \end{aligned}$$

the functions $a_1, \ldots, C_3$ satisfy the Jacobi identities

$$(1.3) \qquad \begin{gathered} [v_1, [v_2, v_3]] + [v_2, [v_3, v_1]] + [v_3, [v_1, v_2]] = 0\,,\\ [w_1, [w_2, w_3]] + [w_2, [w_3, w_1]] + [w_3, [w_1, w_2]] = 0\,. \end{gathered}$$

Obviously,

$$[v_1, v_2] = [\alpha w_1, \beta w_2] = (\alpha\beta A_1 - v_2\alpha)\, w_1 + (\alpha\beta A_2 + v_1\beta)\, w_2 + \alpha\beta A_3 w_3 =$$
$$= a_1\alpha w_1 + a_2\beta w_2 + a_3(\gamma w_1 + \delta w_2 + \varphi w_3)\,,$$

$$[v_1, v_3] = [\alpha w_1, \gamma w_1 + \delta w_2 + \varphi w_3] = (.)\, w_1 + (.)\, w_2 + (v_1\varphi + \alpha\delta A_3 + \alpha\varphi B_3)\, w_3 =$$
$$= (.)\, w_1 + (.)\, w_2 + b_3\varphi w_3\,,$$

$$[v_2, v_3] = [\beta w_2, \gamma w_1 + \delta w_2 + \varphi w_3] =$$
$$= (.)\, w_1 + (.)\, w_2 + (v_2\varphi - \beta\gamma A_3 + \beta\varphi C_3)\, w_3 = (.)\, w_1 + (.)\, w_2 + c_3\varphi w_3\,,$$

i.e.,

$$(1.4)\qquad \alpha\beta A_1 - v_2\alpha = \alpha a_1 + \gamma a_3\,,\quad \alpha\beta A_2 + v_1\beta = \beta a_2 + \delta a_3\,,\quad \alpha\beta A_3 = \varphi a_3\,,$$
$$v_1\varphi + \alpha\delta A_3 + \alpha\varphi B_3 = \varphi b_3\,,\quad v_2\varphi - \beta\gamma A_3 + \beta\varphi C_3 = \varphi c_3\,.$$

Let us restrict ourselves to the case of the non-integrability of the field of planes spanned by the vectors v_1 and v_2, i.e.,

$$(1.5)\qquad a_3 \neq 0\,.$$

To a given section (w_1, w_2, w_3) of B_G there exist functions $\alpha, \ldots, \varphi$ such that $a_3 = 1$, $a_1 = a_2 = b_3 = c_3 = 0$; from (1.3), we get $b_1 + c_2 = 0$, $v_1c_1 = v_2b_1$, $v_1b_1 = -v_2b_2$. In B_G, there are always sections (v_1, v_2, v_3) satisfying

$$(1.6)\qquad [v_1, v_2] = v_3\,,\quad [v_1, v_3] = av_1 + bv_2\,,\quad [v_2, v_3] = cv_1 - av_2\,,$$

$$(1.7)\qquad v_2b = -v_1a\,,\quad v_1c = v_2a\,.$$

Let the section (w_1, w_2, w_3) of B_G satisfy

$$(1.8)\qquad [w_1, w_2] = w_3\,,\quad [w_1, w_3] = Aw_1 + Bw_2\,,\quad [w_2, w_3] = Cw_1 - Aw_2\,,$$

$$(1.9)\qquad w_2B = -w_1A\,,\quad w_1C = w_2A\,.$$

Then

$$(1.10)\qquad \varphi = \alpha\beta\,,$$

$$(1.11)\qquad v_1\alpha = -2\alpha\beta^{-1}\delta\,,\quad v_2\alpha = -\gamma$$
$$v_1\beta = \delta\,,\quad v_2\beta = 2\alpha^{-1}\beta\gamma\,,$$
$$v_2\gamma = \alpha c - \alpha\beta^2 C\,,\quad v_1\delta = \beta b - \alpha^2\beta B\,,$$
$$v_1\gamma - v_3\alpha = \alpha a - \alpha^2\beta A\,,\quad v_2\delta - v_3\beta = -\beta a + \alpha\beta^2 A\,.$$

The integrability conditions of $(1.11_{1,2})$ and $(1.11_{3,4})$ being

$$v_3\alpha = -v_1\gamma + 2\alpha\beta^{-1}v_2\delta - 6\beta^{-1}\gamma\delta\,, \quad v_3\beta = 2\alpha^{-1}\beta v_1\gamma - v_2\delta + 6\alpha^{-1}\gamma\delta\,,$$

there is the function $\varkappa$ such that

$$\begin{aligned}
v_3\alpha &= \alpha\varkappa - \tfrac{3}{2}\beta^{-1}\gamma\delta - \tfrac{3}{4}\alpha a + \tfrac{3}{4}\alpha^2\beta A\,,\\
v_3\beta &= \beta\varkappa + \tfrac{3}{2}\alpha^{-1}\gamma\delta + \tfrac{3}{4}\beta a - \tfrac{3}{4}\alpha\beta^2 A\,,\\
v_1\gamma &= \alpha\varkappa - \tfrac{3}{2}\beta^{-1}\gamma\delta + \tfrac{1}{4}\alpha a - \tfrac{1}{4}\alpha^2\beta A\,,\\
v_2\delta &= \beta\varkappa + \tfrac{3}{2}\alpha^{-1}\gamma\delta - \tfrac{1}{4}\beta a + \tfrac{1}{4}\alpha\beta^2 A\,.
\end{aligned} \tag{1.12}$$

The integrability conditions of the equations $(1.11_1) + (1.12_1)$, $(1.11_3) + (1.12_2)$, $(1.11_4) + (1.12_2)$ and $(1.11_2) + (1.12_1)$ being

$$\begin{aligned}
\alpha v_1\varkappa + 2\alpha\beta^{-1}v_3\delta &= \tfrac{7}{2}\alpha\beta^{-1}\delta\varkappa + \tfrac{9}{4}\beta^{-1}\gamma\delta^2 - \tfrac{1}{8}\alpha\beta^{-1}\delta a + \tfrac{1}{2}\gamma b + \tfrac{3}{4}\alpha v_1 a -\\
&\quad - \tfrac{9}{8}\alpha^2\delta A - \tfrac{3}{2}\alpha^2\gamma B - \tfrac{3}{4}\alpha^3\beta w_1 A\,,\\
\beta v_1\varkappa - v_3\delta &= -\tfrac{5}{2}\delta\varkappa - \tfrac{3}{4}\alpha^{-1}\beta^{-1}\gamma\delta^2 - \tfrac{1}{8}\delta a + \tfrac{1}{2}\alpha^{-1}\beta\gamma b - \tfrac{3}{4}\beta v_1 a +\\
&\quad + \tfrac{3}{8}\alpha\beta\delta A + \tfrac{3}{2}\alpha\beta\gamma B + \tfrac{3}{4}\alpha^2\beta^2 w_1 A\,,\\
\beta v_2\varkappa - 2\alpha^{-1}\beta v_3\gamma &= -\tfrac{7}{2}\alpha^{-1}\beta\gamma\varkappa + \tfrac{9}{4}\alpha^{-1}\gamma^2\delta - \tfrac{1}{8}\alpha^{-1}\beta\gamma a - \tfrac{1}{2}\delta c - \tfrac{3}{4}\beta v_2 a -\\
&\quad - \tfrac{9}{8}\beta^2\gamma A + \tfrac{3}{2}\beta^2\delta C + \tfrac{3}{4}\alpha\beta^3 w_2 A\,,\\
\alpha v_2\varkappa + v_3\gamma &= \tfrac{5}{2}\gamma\varkappa - \tfrac{3}{4}\alpha^{-1}\beta^{-1}\gamma^2\delta - \tfrac{1}{8}\gamma a - \tfrac{1}{2}\alpha\beta^{-1}\delta c + \tfrac{3}{4}\alpha v_2 a +\\
&\quad + \tfrac{3}{8}\alpha\beta\gamma A - \tfrac{3}{2}\alpha\beta\delta C - \tfrac{3}{4}\alpha^2\beta^2 w_2 A\,,
\end{aligned}$$

we obtain

$$\begin{aligned}
v_3\gamma &= 2\gamma\varkappa - \alpha^{-1}\beta^{-1}\gamma^2\delta + \tfrac{1}{2}\alpha v_2 a + \tfrac{1}{2}\alpha\beta\gamma A - \alpha\beta\delta C - \tfrac{1}{2}\alpha^2\beta^2 w_2 A\,,\\
v_3\delta &= 2\delta\varkappa + \alpha^{-1}\beta^{-1}\gamma\delta^2 + \tfrac{1}{2}\beta v_1 a - \tfrac{1}{2}\alpha\beta\delta A - \alpha\beta\gamma B - \tfrac{1}{2}\alpha^2\beta^2 w_1 A\,,\\
v_1\varkappa &= -\tfrac{1}{2}\beta^{-1}\delta\varkappa + \tfrac{1}{4}\alpha^{-1}\beta^{-2}\gamma\delta^2 - \tfrac{1}{8}\beta^{-1}\delta a + \tfrac{1}{2}\alpha^{-1}\gamma b - \tfrac{1}{4}v_1 a -\\
&\quad - \tfrac{1}{8}\alpha\delta A + \tfrac{1}{2}\alpha\gamma B + \tfrac{1}{4}\alpha^2\beta w_1 A\,,\\
v_2\varkappa &= \tfrac{1}{2}\alpha^{-1}\gamma\varkappa + \tfrac{1}{4}\alpha^{-2}\beta^{-1}\gamma^2\delta - \tfrac{1}{8}\alpha^{-1}\gamma a - \tfrac{1}{2}\beta^{-1}\delta c + \tfrac{1}{4}v_2 a -\\
&\quad - \tfrac{1}{8}\beta\gamma A - \tfrac{1}{2}\beta\delta C - \tfrac{1}{4}\alpha\beta^2 w_2 A\,.
\end{aligned} \tag{1.13}$$

The integrability conditions of the equations $(1.11_5) + (1.13_1)$ and $(1.11_6) + (1.13_2)$ are

$$k_1 = \alpha^3\beta K_1\,, \quad k_2 = \alpha\beta^3 K_2\,, \tag{1.14}$$

where

$$k_1 = v_1 v_1 a - 2v_3 b - 3ab\,, \qquad k_2 = v_2 v_2 a - 2v_3 c + 3ac\,,$$

$$K_1 = w_1 w_1 A - 2w_3 B - 3AB\,, \quad K_2 = w_2 w_2 A - 2w_3 C + 3AC\,.$$

If $k_1 k_2 \neq 0$, there exists a section (v_1, v_2, v_3) such that $k_1 = k_2 = 1$. Of course, $\alpha = \varepsilon = \pm 1$, $\beta = \varepsilon$ and, as a consequence of $(1.11_{2,3}) + (1.10)$, $\gamma = \delta = 0$, $\varphi = 1$. The next result follows: *In B_G, choose a section (v_1, v_2, v_3) satisfying* (1.6). *If $k_1 k_2 \neq 0$, there are exactly two sections (v_1, v_2, v_3) such that we have* (1.6), (1.7) *and*

$$(1.15) \qquad k_1 = v_1 v_1 a - 2v_3 b - 3ab = 1\,, \quad k_2 = v_2 v_2 a - 2v_3 c + 3ac = 1\,.$$

(v_1, v_2, v_3) being one of these sections, the other one is $w_1 = -v_1$, $w_2 = -v_2$, $w_3 = v_3$.

2. Next, suppose

$$(2.1) \qquad k_1 = k_2 = 0\,.$$

Consider the system

$$(2.2) \quad v_1\alpha = -2\alpha\beta^{-1}\delta\,, \quad v_2\alpha = -\gamma\,, \quad v_3\alpha = \alpha\varkappa - \tfrac{3}{2}\beta^{-1}\gamma\delta - \tfrac{3}{4}\alpha a\,,$$

$$v_1\beta = \delta\,, \quad v_2\beta = 2\alpha^{-1}\beta\gamma\,, \quad v_3\beta = \beta\varkappa + \tfrac{3}{2}\alpha^{-1}\gamma\delta + \tfrac{3}{4}\beta a\,,$$

$$v_1\gamma = \alpha\varkappa - \tfrac{3}{2}\beta^{-1}\gamma\delta + \tfrac{1}{4}\alpha a\,, \quad v_2\gamma = \alpha c\,, \quad v_3\gamma = 2\gamma\varkappa - \alpha^{-1}\beta^{-1}\gamma^2\delta + \tfrac{1}{2}\alpha v_2 a\,,$$

$$v_1\delta = \beta b\,, \quad v_2\delta = \beta\varkappa + \tfrac{3}{2}\alpha^{-1}\gamma\delta - \tfrac{1}{4}\beta a\,, \quad v_3\delta = 2\delta\varkappa + \alpha^{-1}\beta^{-1}\gamma\delta^2 + \tfrac{1}{2}\beta v_1 a\,,$$

$$v_1\varkappa = -\tfrac{1}{2}\beta^{-1}\delta\varkappa + \tfrac{1}{4}\alpha^{-1}\beta^{-2}\gamma\delta^2 - \tfrac{1}{8}\beta^{-1}\delta a + \tfrac{1}{2}\alpha^{-1}\gamma b - \tfrac{1}{4}v_1 a\,,$$

$$v_2\varkappa = \tfrac{1}{2}\alpha^{-1}\gamma\varkappa + \tfrac{1}{4}\alpha^{-2}\beta^{-1}\gamma^2\delta - \tfrac{1}{8}\alpha^{-1}\gamma a - \tfrac{1}{2}\beta^{-1}\delta c + \tfrac{1}{4}v_2 a$$

obtained from $(1.11) + (1.12) + (1.13)$ by means of the substitution $A = B = C = 0$. The integrability conditions of $(2.2_{8,9})$ and $(2.2_{10,12})$ are $\alpha k_2 = 0$, $\beta k_1 = 0$, and they are satisfied. The integrability condition of $(2_{7,9})$ is

$$(2.3) \qquad v_3\varkappa = \varkappa^2 + \tfrac{1}{4}\alpha^{-2}\beta^{-2}\gamma^2\delta^2 - \tfrac{1}{16}a^2 - bc + \tfrac{1}{4}\alpha^{-1}\gamma v_1 a - \tfrac{1}{4}\beta^{-1}\delta v_2 a + {} + \tfrac{1}{4}v_3 a + \tfrac{1}{2}v_2 v_1 a\,,$$

this being the integrability condition of $(2_{11,12})$ as well. The integrability condition of $(2.2_{13,14})$ is satisfied identically. Finally, the integrability conditions $\alpha^{-1}\gamma k_1 + 2v_2 k_1 = 0$, $\beta^{-1}\delta k_2 - 2v_1 k_2 = 0$ of $(2.2_{11}) + (2.3)$ and $(2.2_{12}) + (2.3)$ are satisfied. The system $(2.2) + (2.3)$ being completely integrable, we obtain the following result: *In B_G, choose a section (v_1, v_2, v_3) satisfying* (1.6). *If $k_1 = k_2 = 0$, there are, in B_G, sections (v_1, v_2, v_3) satisfying*

$$(2.4) \qquad [v_1, v_2] = v_3\,, \quad [v_1, v_3] = 0\,, \quad [v_2, v_3] = 0\,.$$

3. Finally, suppose $k_1 \neq 0$, $k_2 = 0$, the case $k_1 = 0$, $k_2 \neq 0$ is symmetric. In B_G, there are sections (v_1, v_2, v_3), satisfying (1.6), (1.7) and

$$(3.1) \qquad k_1 = v_1 v_1 a - 2v_3 b - 3ab = 1\,, \quad k_2 = v_2 v_2 a - 2v_3 c + 3ac = 0\,.$$

(v_1, v_2, v_3) and (w_1, w_2, w_3) being two such sections, we get $\beta = \alpha^{-3}$ from (1.13). From $(1.11_{3,4}) + (1.12_2)$,

$$(3.2) \qquad \begin{aligned} v_1\alpha &= -\tfrac{1}{3}\alpha^4\delta\,, \quad v_2\alpha = -\tfrac{2}{3}\gamma\,, \\ v_3\alpha &= -\tfrac{1}{3}\alpha\varkappa - \tfrac{1}{2}\alpha^3\gamma\delta - \tfrac{1}{4}\alpha a + \tfrac{1}{4}\alpha^{-1}A\,. \end{aligned}$$

Comparing with $(1.11_{1,2}) + (1.12_1)$ we get $\gamma = \delta = 0$ and

$$(3.3) \qquad \varkappa = \tfrac{3}{8}a - \tfrac{3}{8}\alpha^{-2}A\,.$$

The system (3.2) reduces to

$$(3.4) \qquad v_1\alpha = 0\,, \quad v_2\alpha = 0\,, \quad v_3\alpha = -\tfrac{3}{8}\alpha a + \tfrac{3}{8}\alpha^{-1}A\,,$$

the integrability condition of $(3.4_{1,2})$ being

$$(3.5) \qquad \alpha a = \alpha^{-1}A\,.$$

Suppose $a \neq 0$. Then there are, in B_G, sections (v_1, v_2, v_3) satisfying

$$(3.6) \qquad [v_1, v_2] = v_3\,, \quad [v_1, v_3] = \varepsilon v_1 + bv_2\,, \quad [v_2, v_3] = cv_1 - \varepsilon v_2\,; \quad \varepsilon = \pm 1\,;$$

from (3.5), we get $\alpha^2 = 1$. From (1.7) and (3.1),

$$(3.7) \qquad v_2 b = 0\,, \quad v_3 b = -\tfrac{1}{2} - \tfrac{3}{2}\varepsilon b\,, \quad v_1 c = 0\,, \quad v_3 c = \tfrac{3}{2}\varepsilon c\,.$$

The integrability conditions of $(3.7_{1,2})$ and $(3.7_{3,4})$ are $cv_1 b = 0$ and $bv_2 c = 0$ resp. Suppose $v_1 b = 0$. From (3.7), $b = -\frac{1}{3}\varepsilon$ and $v_2 c = 0$; from (3.7_3), $c = 0$. In B_G, choose a section (v_1, v_2, v_3) satisfying (1.6). If $k_1 = 1$, $k_2 = 0$, $a \neq 0$, there exist exactly two sections (v_1, v_2, v_3), $(-v_1, -v_2, v_3)$ satisfying

$$(3.8) \qquad [v_1, v_2] = v_3\,, \quad [v_1, v_3] = \varepsilon v_1 + bv_2\,, \quad [v_2, v_3] = \varepsilon v_2\,; \quad \varepsilon = \pm 1\,;$$

$$(3.9) \qquad v_2 b = 0\,, \quad v_3 b = -\tfrac{1}{2} - \tfrac{3}{2}\varepsilon b\,.$$

Suppose $a = 0$. The system (3.4) reduces to $v_1\alpha = v_2\alpha = v_3\alpha = 0$. In B_G, there exist sections (v_1, v_2, v_3) satisfying

$$(3.10) \qquad [v_1, v_2] = v_3\,, \quad [v_1, v_3] = bv_2\,, \quad [v_2, v_3] = cv_1\,,$$

$$(3.11) \qquad v_2 b = 0\,, \quad v_3 b = -\tfrac{1}{2}\,, \quad v_1 c = 0\,, \quad v_3 c = 0\,.$$

From $(1.11_{5,6})$,

$$c = \alpha^{-4} C\,, \quad b = \alpha^2 B\,. \tag{3.12}$$

The integrability conditions of the system (3.11) are $cv_1 b = 0$, $bv_2 c = 0$. From $v_1 b = 0$ and (3.11_1) it follows $v_3 b = 0$, this being a contradiction. Thus $v_1 b \neq 0$ and $c = 0$. From (3.12), $v_1 b = \alpha^2 v_1 B = \alpha^3 w_1 B$, i.e.,

$$b(v_1 b)^{-2/3} = B(w_1 B)^{-2/3}\,. \tag{3.13}$$

The following result follows: *In B_G, choose a section (v_1, v_2, v_3) satisfying (1.6). If $k_1 = 1$, $k_2 = 0$, $a = 0$, there are sections satisfying*

$$[v_1, v_2] = v_3\,, \quad [v_1, v_3] = bv_2\,, \quad [v_2, v_3] = 0\,; \tag{3.14}$$
$$v_2 b = 0\,, \quad v_3 b = -\tfrac{1}{2}\,.$$

The section (w_1, w_2, w_3) satisfying analoguous equations

$$[w_1, w_2] = w_3\,, \quad [w_1, w_3] = Bw_2\,, \quad [w_2, w_3] = 0\,, \tag{3.15}$$

we have

$$v_1 = \alpha w_1\,, \quad v_2 = \alpha^{-3} w_2\,, \quad v_3 = \alpha^{-2} w_3\,; \quad \alpha = \text{const.}\,; \tag{3.16}$$

and (3.13).

4. Let us consider the transitive G-structure B_G. First of all, suppose the case (1.6), (1.7) and (1.15). The functions a, b, c being now constant, we have $b = -\frac{1}{3}a$, $c = \frac{1}{3}a$ from (1.15). Next, let $k_1 = 1$, $k_2 = 0$, $a \neq 0$. From $b = \text{const.}$, we get $b = -\frac{1}{3}\varepsilon$ because of (3.9_2). Finally, consider the case (3.14). Applying v_2 to $b(v_1 b)^{-2/3} = \text{const.}$ and taking regard of $(3.14)_4$, we get $v_2 v_1 b = 0$. From (3.14_4), $v_1 v_2 b = 0$, i.e., $v_3 b = 0$, this being a contradiction. Our result is as follows: *Let B_G be transitive. Then there exist sections (v_1, v_2, v_3) of B_G such that*

$$[v_1, v_2] = v_3\,, \quad [v_1, v_3] = av_1 - \tfrac{1}{3}av_2\,, \quad [v_2, v_3] = \tfrac{1}{3}av_1 - av_2\,; \tag{4.1}$$
$$a = \text{const.}\,;$$

or

$$[v_1, v_2] = v_3\,, \quad [v_1, v_3] = \varepsilon v_1 - \tfrac{1}{3}\varepsilon v_2\,, \quad [v_2, v_3] = -\varepsilon v_2\,; \quad \varepsilon = \pm 1\,; \tag{4.2}$$

resp.

The problem of the construction of models of transitive G-structures turns now to be a (non-trivial!) exercise. Consider a flat product structure $\mathscr{R}^4 = \mathscr{R}_1^2 \oplus \mathscr{R}_2^2$ and its hypersurface $M^3 \subset \mathscr{R}^4$. On M^3, there is induced the G-structure of the considered type: let $m \in M^3$, the frame (v_1, v_2, v_3) belongs to B_G if and only if $v_p \in T_m(M^3) \cap S_p^2$; $p = 1, 2$; S_p^2 being determined by $m \in S_p^2 \parallel \mathscr{R}_p^2$. Now, be given a transitive G-structure on M. In local coordinates $(u^i;\ i = 1, 2, 3)$, let $v_1 = a^i(u) \,.\, \partial/\partial u^i$, $v_2 = b^i(u) \,.\, \partial/\partial u^i$ be the vector fields satisfying (4.1) or (4.2) resp. Let $(x^\alpha, y^\alpha;\ \alpha = 1, 2)$ be the coordinates in $\mathscr{R}^4$ such that $\mathscr{R}_1^2$ or $\mathscr{R}_2^2$ is given by $y^\alpha = 0$ or $x^\alpha = 0$ resp. Let $\Phi : M \to \mathscr{R}^4$ be an embedding given by $x^\alpha = x^\alpha(u^i)$, $y^\alpha = y^\alpha(u^i)$. Then

$$(4.3)\qquad \Phi_* v_1 = a^i(u) \frac{\partial x^\alpha(u)}{\partial u^i} \frac{\partial}{\partial x^\alpha} + a^i(u) \frac{\partial y^\alpha(u)}{\partial u^i} \frac{\partial}{\partial y^\alpha} = v_1 x^\alpha(u) \frac{\partial}{\partial x^\alpha} + v_1 y^\alpha \frac{\partial}{\partial y^\alpha},$$

$$\Phi_* v_2 = b^i(u) \frac{\partial x^\alpha(u)}{\partial u^i} \frac{\partial}{\partial x^\alpha} + b^i(u) \frac{\partial y^\alpha(u)}{\partial u^i} \frac{\partial}{\partial y^\alpha} = v_2 x^\alpha(u) \frac{\partial}{\partial x^\alpha} + v_2 y^\alpha \frac{\partial}{\partial y^\alpha}.$$

Our condition says

$$(4.4)\qquad v_1\, y^\alpha(u) = 0\,, \quad v_2\, x^\alpha(u) = 0\,; \qquad \alpha = 1, 2\,.$$

The problem is thus reduced to the exhibition of independent solutions of (4.4).

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