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# GENERIC PROPERTIES OF THE ROTATION NUMBER OF ONE-PARAMETER DIFFEOMORPHISMS OF THE CIRCLE

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## 1

In this paper, we consider $C^r$ $(2 \leqq r \leqq \infty)$ maps $f : P \times R^1 \to R^1$, where $P$ is a one-dimensional manifold and for every fixed $\mu \in P$ the map $f_\mu : R^1 \to R^1$ defined as $f_\mu(x) = f(\mu, x)$ is a diffeomorphism satisfying

$$f_\mu(x + 1) = f_\mu(x) + 1 . \tag{1}$$

The space of such maps, endowed by the $C^r$ Whitney topology we denote by $\mathscr{F}$.

The maps $f$ from $\mathscr{F}$ are obtained as liftings of one-parameter diffeomorphisms of $S^1$ induced by the covering of $S^1$ by $R^1$. By $\hat{f}$ we shall denote the map the lifting of which is $f$; similarly, we shall mark by hats the projections of points and subsets of $R^1$ to $S^1$.

Our main subject of interest will be the behavior of the rotation number of $f$ for varying $\mu$ and its stability under small perturbation of $f$ in $\mathscr{F}$. In addition, we obtain some results about the nature and stability of the loci of periodic points of $f$. Related problems (dependence of the rotation number on the changes of the diffeomorphism, problems of structural stability and topological classification of the diffeomorphisms of $S^1$) have been studied by several authors (cf. [1]–[5]). The understanding of the topology of the parameter-dependent diffeomorphisms of the circle is important for the problem of structural stability of one-parameter two-dimensional flows (cf. [6]) as well as for the problem of bifurcation of periodic problems of one-parameter flows (cf. [7], [8]).

We shall carry out the proofs only for the case $P = R^1$, since the case $P = S^1$ requires only minor adjustments of the proofs and $R^1$ and $S^1$ are the only possible connected components of one-dimensional manifolds.

For $f \in \mathscr{F}$, we denote by $\sigma_f : P \to R^1$ the function assigning to every $\mu \in P$ the rotation number of $f_\mu$. We shall call $(\mu, y)$ an $l$-periodic point of $f$, if $\hat{y}$ is an $l$-periodic point of $\hat{f}_\mu$ in the usual sense, (i.e. $f^l_\mu(y) - y$ is integer, but $f^j_\mu(y) - y$ is not integer for $0 < j < l$). The set of $l$-periodic points of $f$ in $P \times R^1$ will be denoted by $Z_l(f)$. Given $f$, by the orbit of a point $(\mu, y)$ we shall understand the set of points $\{(\mu, f^j_\mu(y)) \mid j \text{ integer}\}$.

We shall restrict ourselves to the study of typical (or *generic*) properties of $\sigma_f$ and $Z_l$. We shall call a property of maps from $\mathscr{F}$ generic, if it is valid for every $f$ from some residual subset of $\mathscr{F}$ (i.e. a set which is a countable intersection of open dense subsets of $\mathscr{F}$).

For convenience, we conclude this section by listing some of the well known properties of diffeomorphisms of $S^1$ and their rotation numbers, which we shall need in the sequel.

Let $f$ be a $C^2$ diffeoemorphism $R^1 \to R^1$ which is a lifting of a diffeoemorphism $\hat{f}: S^1 \to S^1$, i.e. $f$ satisfies $f(y + 1) = f(y) + 1$ for all $y$. Denote by $\varrho(f)$ the rotation number of $f$. Then

(i) $\varrho(f)$ is a topological invariant of $f$ and is determined by $\hat{f}$ up to an additive integer,
(ii) if $\varrho(f)$ is irrational, then $f$ is topologically equivalent to the shift $y \mapsto y + \varrho(f)$,
(iii) if $\varrho(f) = kl^{-1}$, where $k$, $l$ are relatively prime integers and $l > 0$ (if we express a rational as a fraction we shall allways implicitely assume this), then there exists at least one point $y$ such that $f^l(y) = y + k$ and there is no $j$-periodic point with $0 < j < l$ of $f$,
(iv) for any $n$, $\varepsilon_n + \delta_n < 1$, and $\varepsilon_n \geqq 0$, $\delta_n \geqq 0$, where $\varepsilon_n = \max_y [f^n(y) - y] - n\,\varrho(f)$, $\delta_n = n\varrho(f) - \min_y (f^n(y) - y)$.

The proof of these properties can be found e.g. in [9].

Further, we note that since $f$ is a diffeomorphism,

(v) $f'$ does not change sign and if it is negative, $\varrho(f)$ must be an integer,
(vi) if $f'$ is positive and $\varrho(f) = kl^{-1}$, then all periodic points of $f$ are $l$-periodic.

To prove (vi) assume that there exists a point $y$ which is $m$-periodic, $m \neq l$. Since $y$ is not $l$-periodic, we have $f^l(y) \neq k$. Assume e.g. $f^l(y) < y + k$. Since $f^l$ is increasing, we have thus $f^{ml}(y) - y < mk$ and since $y \in Z_m(f)$, $f^{ml}(y) - y$ is integer. Therefore, $f^{ml}(y) - y \leqq mk - 1$, which contradicts property (iv).

Note also that since allways $(f^2)' > 0$, it follows from property (vi) that if $f' < 0$ all periodic points of $f$ are 1- or 2-periodic.

## 2

We know from [1] and [2] that $\sigma_f$ is continuous but not Lipschitz continuous in general (later we prove that generically it is of bounded variation). From the results of [10] it follows that, generically, there is an open dense subset $\mathscr{F}'_{1L}$ of $\mathscr{F}$ such that for every $f \in \mathscr{F}'_{1L}$, and $l \leqq L$, $Z_l(f)$ are one-dimensional imbedded $C^r$ submanifolds of $P \times R^1$, intersecting every set $\Sigma_\mu = \{\mu\} \times R^1$ in isolated points. Furthermore, the points $(\mu, y) \in Z_l(f)$ at which $Z_l(f)$ does not intersect $\Sigma_\mu$ transversally, are isolated and satisfy $(f^l_\mu)''(y) \neq 0$, $(\partial/\partial\mu)(f^l_\mu)(y) \neq 0$.*)

**Lemma 1.** *Let $I, J$ be intervals of $R^1$, $J$ compact. Let $f : I \times J \to R^1$ be $C^1$. Denote $\varphi(\mu) = \min_{y \in J} f(\mu, y)$ for $\mu \in I$. Then, at every $\mu \in \operatorname{int} I$ both the right derivative $\varphi'^{+}(\mu)$ and the left derivative $\varphi'^{-}(\mu)$ exist and are equal to* $\min \{(\partial f/\partial\mu)(\mu, y) \mid y \in M(\mu)\}$ *and* $\max \{(\partial f/\partial\mu)(\mu, y) \mid y \in M(\mu)\}$, *respectively, where $M(\mu) = \{y \mid f(\mu, y) = \varphi(\mu)\}$.*

Proof. Denote $\psi(\mu) = \min_{y \in M(\mu)} (\partial f/\partial\mu)(\mu, y)$, $N(\mu) = \{y \in M(\mu) \mid (\partial f/\partial\mu)(\mu, y) = \psi(\mu)\}$. Both $M(\mu)$ and $N(\mu)$ are compact.

We prove first that if $h \to 0+$, $y_h \in M(\mu + h)$, then $y_h \to N(\mu)$. Assume the contrary. Then, there exists a sequence $h_n \to 0$, $y_{h_n} \in M(\mu + h_n)$ such that $y_{h_n} \to y^* \notin N(\mu)$. Obviously, $y^* \in M(\mu)$ which implies that for sufficiently large $n$, there exists a positive $\eta > 0$ such that $(\partial f/\partial\mu)(\mu, y_{h_n}) > \psi(\mu) + \eta$. Consequently, for sufficiently large $n$ we have for any $y_0 \in N(\mu)$

$$f(\mu + h_n, y_{h_n}) - f(\mu + h_n, y_0) = h_n \left[\frac{\partial f}{\partial\mu}(\mu + v_1 h_n, y_{h_n}) - \frac{\partial f}{\partial\mu}(\mu + v_2 h_n, y_0)\right] > 0$$

$(0 \leqq v_1, v_2 \leqq 1)$, which contradicts our assumption.

Now we have for sufficiently small $h > 0$:

$$\varphi(\mu + h) - \varphi(\mu) = \min_{y \in J} f(\mu + h, y) - \min_{y \in J} f(\mu, y) = f(\mu + h, y_h) - f(\mu, y_0) =$$
$$= f(\mu + h, y_h) - f(\mu, y_h) + f(\mu, y_h) - f(\mu, y_0) \geqq \frac{\partial f}{\partial\mu}(\mu + vh, y_h)\, h\,,$$

where $y_0 \in N(\mu)$, $y_h \in M(\mu + h)$, $v \in [0, 1]$, and, therefore, $y_h \to N(\mu)$ as $h \to 0$.

Consequently,

$$\liminf_{h \to 0+} h^{-1}(\varphi(\mu + h) - \varphi(\mu) - \psi(\mu)\, h) \geqq 0\,. \tag{2}$$

On the other hand we have

$$\varphi(\mu + h) - \varphi(\mu) = f(\mu + h, y_h) - f(\mu, y) =$$
$$= f(\mu + h, y_h) - f(\mu + h, y_0) + f(\mu + h, y_0) - f(\mu, y_0) \leqq \frac{\partial f}{\partial\mu}(\mu, y_0)\, h + o(h)$$

*) Actually, in [10, I] the openness of $\mathscr{F}'_{1L}$ is not explicitely stated for $L > 1$ (cf. Theorem 1 and the following Remark 2). However, the openness of $\mathscr{F}'_{1L}$ (which is true in general), follows easily in our case, since by properties 5, 6 of §1, $Z_l$ are closed and isolated from $\bigcup_{j<l} Z_j$.

from which

$$\limsup_{h\to 0+} h^{-1}(\varphi(\mu + h) - \varphi(\mu) - \psi(\mu)\, h) \leqq 0$$

which together with (2) proves $\psi(\mu) = \varphi'^{+}(\mu)$. The corresponding statement for the left derivative is obtained by replacing $\mu$ by $-\mu$.

For $f \in \mathscr{F}$, denote $f^l(\mu, x) = f^l_\mu(x)$. Obviously, $f^l \in \mathscr{F}$ for all $l$. In several proofs we shall use the following fact which we shall formulate as

**Lemma 2.** *Given $f \in \mathscr{F}$ and neighbourhoods $W_l$ of $f^l$ for $0 \leqq l \leqq L$, there exists a neighbourhood $W$ of $f$ such that $f^l \in W_l$ for $0 \leqq l \leqq L$ as soon as $f \in W$.*

The proof follows from the fact that the operation of composition is continuous on $\mathscr{F}$.

A closed interval of $R^1$ will be called non-trivial if it has a non-empty interior.

**Lemma 3.** *Let $f \in \mathscr{F}'_{1L}$. Then for every $\varrho = kl^{-1}$ with $l \leqq L$, $\sigma_f^{-1}(\varrho)$ is disjoint union of non-trivial closed intervals only finitely many of which intersect any compact interval $I$. If $\mu_0$ is a right (left) endpoint of an interval of $\sigma_f^{-1}(\varrho)$, and $(\mu_0, y_0) \in Z_l(f)$, then $(f^l_{\mu_0})'(y_0) = 1$ and $(\partial f/\partial\mu)(f^l_{\mu_0})(y_0)$* and *$(f^l_{\mu_0})''(y_0)$ are distinct from zero and have the same (the opposite) sign.*

Proof. First we prove that the number connected components of $\sigma_f^{-1}(\varrho)$ intersecting $I$ is finite. If this was not true, there would exist a sequence of points $(\mu_n, y_n)$ from mutually disjoint components of $Z_l(f)$ such that $\mu_n \in I$ and $\mu_n \to \mu^*$. Because of compactness of $S^1$ we can assume $(\mu_n, y_n) \to (\mu^*, \hat{y}^*) \in \hat{Z}_l(\hat{f})$, which contradicts to the fact that $\hat{Z}_l(\hat{f})$ are imbedded submanifolds of $P \times S^1$.

Let $\mu \in \sigma_f^{-1}(\varrho)$. Then, $Z_l(f) \cap \Sigma_\mu \neq 0$ and, since $Z_l(f)$ is a one-dimensional manifold, the intersection of which with $\Sigma_\mu$ consists of isolated points, it must intersect $\Sigma_{\mu'}$ for all $\mu'$ from some right or left neighbourhood $U$ of $\mu$, which implies $\sigma_f(\mu') = \varrho$ for $\mu' \in U$. Since $\sigma_f$ is continuous and, thus, $\sigma_f^{-1}(\varrho)$ is closed, this proves the first part of the lemma.

The second part of the lemma follows from the fact that if $\mu_0$ is a right endpoint of some interval of $\sigma_f^{-1}(\varrho)$, the components of $Z_l(f)$ intersecting $\Sigma_{\mu_0}$ all lie to the left of it. Consequently, any $(\mu_0, y_0) \in Z_l(f)$ is a collapsation point (in the terminology of [10]) and the inequalities follows from [10, Lemma 3]. The proof for the teft endpoint is similar.

If their modifications to other generically possible situations (left endpoint, different signs of the inequalities) is straightforward, we shall formulate the statements concerning the endpoints of the intervals of $\sigma_f^{-1}(\varrho)$ only for those right ones, satisfying $(\partial/\partial\mu)(f^l_{\mu_0})(y_0) > 0$, $(f^l_{\mu_0})''(y_0) > 0$.

**Lemma 4.** *Let $f \in \mathscr{F}'_{1L}$, $\mu_0$ be a right endpoint of an interval of $\sigma_f^{-1}(kl^{-1})$ and let $(\partial/\partial\mu)(f^l_{\mu_0})(y_0) > 0$, $(f^l_{\mu_0})''(y_0) > 0$ for some periodic point $(\mu_0, y_0)$. Then,*

$(\partial/\partial\mu) f^l_{\mu_0}(y) > 0$, $(f^l_{\mu_0})''(y) > 0$ *for every periodic point* $(\mu_0, y)$ *and* $f^l_\mu(y) \succeq k$ *for all* $y$ *and all* $\mu > \mu_0$ *sufficiently close to* $\mu_0$.

Proof. Let $f \in \mathscr{F}'_{1L}$. If there was a periodic point $(\mu_0, y^*)$ with $(f^l_{\mu_0})''(y^*) \prec 0$, $y^*$ would be a local maximum of the function $\varphi(y) = f^l_{\mu_0}(y) - y$, while the point $(\mu_0, y_0)$ is a local minimum of $\varphi$. Since $\varphi$ has both a local maximum and a local minimum of value $k$, there must also be a point $y_1$ with $\varphi(y_1) = k$ which is not a local extremum of $\varphi$. Thus, $(\mu_0, y_1)$ would be a periodic point not satisfying lemma 3.

Since by lemma 3, sign $(\partial/\partial\mu)(f^l_{\mu_0})(y)$ has to be equal to sign $(f^l_{\mu_0})''(y)$ for every periodic point $(\mu_0, y)$, the lemma is proven.

**Lemma 5.** *The subset* $\mathscr{F}_{1L}$ *of* $\mathscr{F}'_{1L}$ *of those* $f \in \mathscr{F}'_{1L}$ *which for every* $\varrho \doteq kl^{-1}$ *with* $l \leqq L$ *satisfy*

(i) *if* $\mu_0$ *is an interior point of* $\sigma_f^{-1}(\varrho)$, *then* $\min_y [f^l_{\mu_0}(y) - y] < k < \max_y [f^l_{\mu_0}(y) - y]$,

(ii) *if* $\mu_0$ *is a boundary point of* $\sigma_f^{-1}(kl^{-1})$, *then the projections* $(\mu_0, \hat{y}_i)$ *of the periodic points* $(\mu_0, y_i)$ *belong to one orbit of* $\hat{f}$

*is open dense in* $\mathscr{F}$.

For the proof of this lemma the following lemma will be useful:

**Lemma 6.** *Let* $f \in \mathscr{F}'_{1L}$, $\sigma_f(\mu_0) = kl^{-1}$. *Then* $\min_y f^l_{\mu_0}(y) - y < k < \max_y f^l_{\mu_0}(y) - y$ *if and only if there exists an* $y_0$ *such that* $Z_l(f)$ *intersects* $\Sigma_{\mu_0}$ *transversally at* $(\mu_0, y)$ (*i.e.* $(f^l_{\mu_0})'(y) \neq 1$ – cf. [10, I, p. 561]).

The proof follows easily from the fact that at points $(\mu_0, y) \in Z_l(f)$ with $(f^l_{\mu_0})'(y) = 1$, $(f^l_{\mu_0})''(y)$ has to be different from zero and, therefore, $f^l_{\mu_0}(y) - y - k$ cannot change signs at such points.

**Corollary 1.** *If* $f \in \mathscr{F}'_{1L}$, $\mu \in \sigma_f^{-1}(kl^{-1})$, *then* $\min_y f^l_\mu(y) - y < k < \max_y f^l_\mu(y) - y$ *if and only if* $\mu$ *is an interior point of the projection of some component of* $Z_l(f)$ *into* $P$.

Proof of lemma 5. Openness. Let $f \in \mathscr{F}_{1L}$. Then, by Corollary 1, the interiors of the projections into $P$ of the components of $Z_l(f)$ cover the interior of $\sigma_f^{-1}(kl^{-1})$ and the components of $\hat{Z}_l(\hat{f})$ whose projections contain the same boundary point of $\sigma_f^{-1}(kl^{-1})$, belong to one orbit. The manifolds $Z_l(f)$, $l \leqq L$ are obtained as pre-images of points of intersection of certain maps associated with the maps $f^l_\mu$ (cf. [10, I, Lemma 1]) with the diagonal in $R^2$, which for $f \in \mathscr{F}_{1L}$ is transversal. Therefore, in virtue of lemma 2, the transversal isotopy theorem [11, 20. 2][1]) implies that for

[1]) The compactness of $X$ in [11, 20, 2] is not needed if the Whitney topology is used.

every $\tilde{f}$ from some neighbourhood of $W$ of $f$, $Z_l(\tilde{f})$ are isotopic to $Z_l(f)$ for $l \leqq L$. Also, it follows from the proof of [11, 20, 2] that $Z_l(\tilde{f})$ can be made arbitrary close to $Z_l(f)$, $l \leqq L$ provided $W$ is choosen small enough. In particular, $W$ can be choosen so small that $W \in \mathscr{F}'_{1L}$ and that the interiors of the projections of those components of $Z_l(\tilde{f})$, which are isotopic to the components of $Z_l(f)$ the projections of which cover some interval $J$ of $\sigma_f^{-1}(kl^{-1})$, cover an interval $\tilde{J}$ in such a way that each of the end-points of $\tilde{J}$ belongs to the projection of components of $\hat{Z}_l(\hat{f})$ belonging to one orbit. Thus, $W \in \mathscr{F}_{1L}$.

Density. The density of $\mathscr{F}_{1L}$ in $\mathscr{F}$ will be proven if we show that for any $f \in \mathscr{F}_{1L}$, if $\mu_0$ is a point not satisfying (i) or (ii), then by an arbitrarily small perturbation of $f$ with an arbitrary small support a map from $\mathscr{F}$ can be obtained satisfying (i), (ii) for all $\mu_0$ contained in the projection of this support into $P$.

Let $\mu_0$ be an interior point of $\sigma_f^{-1}(kl^{-1})$ not satisfying (i). Let e.g. $f^l_{\mu_0}(y) \geqq k$ for all $y$. Let $\varphi : P \to R^1$ be a bump function equal 1 in some neighbourhood of $\mu_0$, the support of which contains no boundary points of $\sigma_f^{-1}(kl^{-1})$ as well as no point except of $\mu_0$ in which (i) would not be satisfied (note that by Corollary 1 the points not satisfying (i) are isolated). For sufficiently small $\varepsilon > 0$, the function $f - \varepsilon\varphi$ will be in $\mathscr{F}'_{1L}$ and satisfy (i), (ii) for $\mu \in U$.

Let now $\mu_0$ be a right endpoint of an interval of $\sigma_f^{-1}(kl^{-1})$ satisfying $f^l_{\mu_0}(y) \geqq k$ for all $y$, which implies

$$(3) \qquad (f^l_{\mu_0})''(y) > 0\,, \quad \frac{\partial}{\partial\mu}(f^l_{\mu_0})(y) > 0$$

for all periodic points $(\mu_0, y)$. Let $\{(\mu_0, \hat{z}_i)\}$ be a periodic orbit. Choose neighbourhoods $U$ of $\mu_0$, $\hat{V}$ of $\hat{z}_0$ and an $\varepsilon > 0$ so small that $\sigma_{f+\varepsilon}(\mu)$ is not rational with denominater less than $L$ for $\mu \in U$, $\mu > \mu_0$ and $\{\mu_0\} \times \hat{V}$ does not contain any other periodic point except of $(\mu_0, \hat{z}_0)$. This is possible due to Corollary 1, the continuous and monotonic dependence of $\sigma_{f+\varepsilon}$ on $\varepsilon$ and the fact that by lemma 4 and (3), $\min_y f^l_{\mu_0}(y) - y > k$ for $\mu > \mu_0$ sufficiently close to $\mu_0$. Further, let $\varphi : P \times R^1 \to R^1$ be a bump function which is 1-periodic, equal to 1 in some neighbourhood of $(\mu_0, z_0)$ and 0 outside $U \times V$. Denote $\tilde{f} = f + \varepsilon\varphi$. From (4) it follows that for all points $(\mu, y) \in U \times V$ with $\mu \geqq \mu_0$ we have $\tilde{f}^l_\mu(y) > k$. In this way, we can destroy any periodic orbit of $\hat{f}_{\mu_0}$ and leave only one to suit (ii).

From this lemma we obtain

**Proposition 1.** *There is a residual set $\mathscr{F}_1$ of $\mathscr{F}$ such that for $f \in \mathscr{F}_1$ the following is valid*:

*For any $\varrho = kl^{-1}$, $\sigma_f^{-1}(\varrho)$ is a disjoint union of closed non-trivial intervals only finite of them intersecting any compact subset of $P$. If $\mu_0$ is an interior point of $\sigma_f^{-1}(\varrho)$, then $\min_y f^l_{\mu_0}(y) - y < k < \max_y f^l_{\mu_0}(y) - y$. If $\mu_0$ is a right (left) endpoint of an interval of $\sigma_f^{-1}(\varrho)$, then there is exactly one orbit of periodic points of $\hat{f}_{\mu_0}$, for every point $\hat{y}_0$*

*of which*

$$(f^l_{\mu_0})'(y_0) = 1\,, \quad \frac{\partial}{\partial\mu} f^l_{\mu_0}(y_0) \neq 0\,, \quad (f^l_{\mu_0})''(y_0) \neq 0$$

*and the latter two quantities have the same (the opposite) sign.*

Remark 1. In case $f' < 0$ it follows from 5, §1 and the continuity of $\sigma_f$, that $\sigma_f$ is constant over $P$.

Next we shall examine the behavior of the rotation number in the neighbourhood of a boundary point $\mu$ of $\sigma_f^{-1}(\varrho)$, $\varrho$ rational, outside of $\sigma_f^{-1}(\varrho)$. The difficulty here is that in any neighbourhood of $\mu$, $\sigma_f$ can have rational values with an arbitrarily large denominator. We prove the following

**Proposition 2.** *Let $f \in \mathscr{F}$, $\sigma_f(\mu_0) = kl^{-1}$ and let the set of periodic points of $f$ with $\mu = \mu_0$ form an increasing sequence $(\mu_0, y_i)$ such that $(\mu_0, \hat{y}_i)$ belong to one orbit of $\hat{f}$. Let*

$$(f^l_{\mu_0})'(y_i) = 1\,, \quad \frac{\partial}{\partial\mu} f^l_{\mu_0}(y_i) > 0\,, \quad (f^l_{\mu_0})''(y_i) > 0 \tag{4}$$

*for all $i$. Then, there is a neighbourhood $U$ of $\mu_0$ in which $\sigma_f$ is non-decreasing. Furthermore, $\sigma_f$ is not constant on any neighbourhood of $\mu_0$ and $\sigma_f^{-1}(\varrho) \cap U$ is at most a one point set for $\varrho$ irrational.*

The prcof of this proposition is based upon the following lemma, which appears to be crucial for the results of this paper:

**Lemma 7.** *Let $f$, $\mu_0$ satisfy the assumptions of proposition 2. Then, there are $\eta > 0$, $n$ such that for all $v > n$, $\mu_0 \leqq \mu_1 < \mu_2 \leqq \mu_0 + \eta$, $y \in R^1$,*

$$f^{vl}_{\mu_1}(y) < f^{vl}_{\mu_2}(y)\,. \tag{5}$$

Remark 2. It is easy to check that if (4) is satisfied for some $i$, then it is satisfied for all $i$.

Proof. Denote $g_\mu(x) = f^l_\mu(x) - k$. Obviously, $g_{\mu_0}$ has rotation number 0 and $g$ has a sequence of periodic points $(\mu_0, y_i)$, exactly $l$ of which lie in $[0, 1)$. Further, we have

$$\frac{\partial}{\partial\mu} g(\mu_0, y_i) > 0\,, \quad g'_{\mu_0}(y_i) = 1\,, \quad g''_{\mu_0}(y_i) > 0\,. \tag{6}$$

Rewriting (5) in terms of $g$, we have the inequality $g^v_{\mu_1}(y) < g^v_{\mu_2}(y)$. Below in the proof, if the index $i$ occurs in some statement or formula, it should allways be read "for all $i$".

1° Denote $N = 2\max\limits_y [g'_{\mu_0}(y)]$, $m = \frac{1}{2}\min\limits_y [g'_{\mu_0}(y)]$. From (6) it follows that $m > 0$ and that $y_i$ are minima of the function $g_{\mu_0}(y) - y$. Consequently, $g_{\mu_0}(y) > 0$ for

$y \neq y_i$ from which it follows

$$\lim_{\nu\to\infty} g^{\nu}_{\mu_0}(y) = y_i \quad \text{for all} \quad y \in (y_{i-1}, y_i]\,. \tag{7}$$

2° We prove that for every $q$ positive integer, there exist $\zeta_0 > 0$ and $\eta > 0$ such that if $\zeta \leqq \zeta_0$, $\mu_0 \leqq \mu \leqq \mu_0 + \eta$ then

$$g^q_\mu(y_i - \zeta) < y_i - \tfrac{1}{2}\zeta \tag{8}$$

$$g^q_\mu(y_i + \zeta/2) < y_i + \zeta\,. \tag{9}$$

Since $g_{\mu_0}(y_i) = 1$, there is a $\zeta_0$ such that for $|y - y_i| \leqq \zeta_0$, $g'_{\mu_0}(y) - 1 > -(2q)^{-1}$. If $\zeta \leqq \zeta_0$, we have

$$g^q(y_i - \zeta) - (y_i - \zeta) - \sum_{j=0}^{q-1}[g^{j+1}(y_i - \zeta) - g^j(y_i - \zeta)] =$$
$$= \sum_{j=0}^{q-1}[g(g^j(y_i - \zeta)) - g^j(y_i - \zeta)] \leqq (2q)^{-1}\sum_{j=0}^{q-1}[y_i - g^j(y_i - \zeta)] < \tfrac{1}{2}\zeta\,.$$

The inequality (9) for $\mu = \mu_0$ is established similarly. Because of continuous dependence of $g_\mu$ on $\mu$, inequalities (9), (8), remain valid for $|\mu - \mu_0| \leqq \eta$ provided $\eta$ is sufficiently small.

3° Since $g$ is $C^1$ in $\mu$, for $|\mu - \mu_0| \leqq \eta$ there exists a $\Gamma > -\infty$ such that

$$g_{\mu_2}(y) - g_{\mu_1}(y) > \Gamma(\mu_2 - \mu_1) \quad \text{for all } y\,. \tag{i}$$

It follows from (6) that we can take $\eta > 0$, $0 < \xi < \frac{1}{2}\min(y_i - y_{i-1})$, $\xi > 0$ so small that

$$\begin{aligned} &g_{\mu_2}(y) - g_{\mu_1}(y) > \gamma(\mu_2 - \mu_1) \quad \text{for all} \quad y \in [y_i - \xi,\ y_i + \zeta] \\ &g'_\mu(y) < \varkappa < 1 \quad \text{for} \quad y \in [y_i - \xi,\ y_i - \tfrac{1}{2}\xi] \\ &g'_\mu(y) > 1 \quad \text{for} \quad y \in [y_i - \tfrac{1}{2}\zeta,\ y_i + \xi] \\ &g_\mu(y_{i-1} + \xi) < y_i - \xi\,. \end{aligned} \tag{ii}$$

From (7) it follows that there is a $q_1$ such that

$$g^{q_1}_\mu(y_{i-1} + \xi) > y_i - \xi \tag{iii}$$

and that for sufficiently small $\zeta$ we have

$$g^{q_1}(y_i - \xi) < y_i - \zeta \tag{iv}$$

for all $|\mu - \mu_0| < \eta$, provided $\eta > 0$ is sufficiently small.

According to 2°, $\eta$ and $\zeta$ can be choosen so small that

$$\begin{aligned} &g^{q_2}_\mu(y_i - \xi) < y_i - \tfrac{1}{2}\zeta\,, \\ &g^{q_3}_\mu(y_i + \tfrac{1}{2}\xi) < y_i + \zeta \end{aligned} \tag{v}$$

where $q_2 > \gamma^{-1}(N-1)^{-1}(N^{q_1}-1)|\Gamma|$ and $q_3 > \gamma^{-1}m^{-q_1}(m-1)^{-1}(m^{q_1-1})|\Gamma|$ are fixed.

It follows from (ii) that

(vi) $g_{\mu_2}(y_2) \geqq g_{\mu_1}(y_1)$ as soon as $y_2 \in [y_i - \xi, y_i + \xi]$ $y_1 \leqq y_2$ and $\mu_0 \leqq \mu_1 < < \mu_2 \leqq \mu_0 + \eta$.

By lemma 4 we have $\min_y g_\mu(y) - y > 0$ for $\mu_0 < \mu \leqq \mu_0 + \eta$ from which it follows

(vii) $\lim_{\nu\to\infty} g_\mu^\nu(x) = \infty$ for all $\mu_0 < \mu \leqq \mu_0 + \eta$ provided $\eta$ is suitably restricted.

Throughout the rest of this proof we shall assume that $\eta, \xi, \zeta$ are choosen so as to suit (i)–(vii) and that $\mu_0 \leqq \mu_1 < \mu_2 \leqq \mu_0 + \eta$; for the purpose of later reference, we shall sometimes write $\eta = \eta(f, \mu_0)$.

4° Let $y \in [y_i - \xi, y_i + \frac{1}{2}\zeta]$. We prove that there exists a $q$ such that

$$y_{i+1} - \xi \geqq g_{\mu_2}^q(y) \geqq y_i + \xi \tag{10}$$

and if $q^*$ is the smallest $q$ satisfying (10) then

$$g_{\mu_2}^{q^*}(y) > g_{\mu_1}^{q^*}(y) + (m-1)^{-1} m^{-q_1}(m^{q_1}-1)|\Gamma|(\mu_2-\mu_1)\,. \tag{11}$$

The existence of a $q$ satisfying (5) follows from (ii) and (vii). Let $q_4$ be such that $g_{\mu_2}^{q_4}(y) \geqq y_i + \frac{1}{2}\zeta$, $g_{\mu_2}^{q_4-1} \leqq y_i + \frac{1}{2}\zeta$. Then, from (v) it follows $q^* - q_4 \geqq q_3$ and by (vi) we have $g_{\mu_2}^{q^*}(y) - g_{\mu_1}^{q^*}(y) \geqq g_{\mu_2}^{q_3}(y) - g_{\mu_1}^{q_3}(x)$, where $x = g_{\mu_2}^{q_4}(y)$. But according to (ii) we have for all $v_1, v_2 \in [y_i + \frac{1}{2}\zeta, y_i + \xi]$

$$g_{\mu_2}(v_2) - g_{\mu_1}(v_1) \geqq g_{\mu_2}(v_2) - g_{\mu_1}(v_2) + g_{\mu_1}(v_2) - g_{\mu_1}(v_1) \geqq \gamma(\mu_2-\mu_1) + v_2 - v_1$$

from which we obtain by induction

$$g_{\mu_2}^{q^*}(y) - g_{\mu_1}^{q^*}(y) \geqq q_3\gamma(\mu_2-\mu_1)$$

from which (11) follows by (v).

5° Let $v_2 \in [y_{i-1} + \xi, y_i - \xi]$, $v_2 \geqq v_1 + (m-1)^{-1} m^{q_1}(m^{q_1}-1)|\Gamma|(\mu_2-\mu_1)$. Then, we prove $g_{\mu_2}^q(v_2) > g_{\mu_1}^q(v_1)$ for all $q \geqq 0$.

In virtue of 4° and (vi), it suffices to prove that

$$g_{\mu_2}^q(v_2) \geqq g_{\mu_1}^q(v_1) \quad \text{for all} \quad q \leqq q_1'\,, \tag{12}$$

where $q_1' \leqq q_1$ is such that $g_{\mu_2}^{q_1'}(v_2) \geqq y_i - \xi > g_{\mu_2}^{q_1'-1}(v_2)$ (the existence of such a $q_1'$

follows from (iii)). However, (12) follows by induction from the validity of

$$g_{\mu_2}(w_2) - g_{\mu_1}(w_1) \geqq g_{\mu_2}(w_2) - g_{\mu_1}(w_2) + g_{\mu_1}(w_2) - g_{\mu_1}(w_1) \geqq$$
$$\geqq -|\Gamma|(\mu_2 - \mu_1) + m(w_2 - w_1) \quad \text{for any} \quad w_2 \geqq w_1\,,$$
$$w_2 \in [y_{i-1} + \xi,\ y_i - \xi]\,.$$

6° From 4° and 5° it follows that for $y \in [y_i - \xi,\ y_i + \frac{1}{2}\zeta]$, $g^q_{\mu_2}(y) \geqq g^q_{\mu_1}(y)$ for all $q > 0$.

7° Let $y \in [y_{i-1} + \xi,\ y_i - \xi]$. Then, there is a $q_1' \leqq q_1$ such that $g^{q_1}_{\mu_2}(y) - g^{q_1}_{\mu_1}(y) \geqq$ $\geqq (N-1)^{-1}(N^{q_1} - 1)\,\Gamma(\mu_2 - \mu_1)$, $y_i - \xi \leqq g^{q_1'}_{\mu_i}(y) \leqq y_i - \zeta$ for $j = 1, 2$. This follows by induction from (iii), (iv) and the validity of

$$g_{\mu_2}(v_2) - g_{\mu_1}(v_1) = g_{\mu_2}(v_2) - g_{\mu_2}(v_1) + g_{\mu_2}(v_1) - g_{\mu_1}(v_1) \geqq$$
$$\geqq N(v_2 - v_1) + \Gamma(\mu_2 - \mu_1)$$

for all $v_1, v_2 \in [y_{i-1} + \xi,\ y_i - \xi]$, $v_2 \leqq v_1$.

8° Let $y$, $q_1'$ be as in 7°. We prove that then $g^{q_1'+q_2}_{\mu_2}(y) > g^{q_1'+q_2}_{\mu_1}(y)$.
It follows from (ii) that if $v_1 \geqq v_2 \in [y_i - \xi,\ y_i - \frac{1}{2}\zeta]$, then

$$g_{\mu_2}(v_2) - g_{\mu_1}(v_1) \geqq g_{\mu_2}(v_2) - g_{\mu_2}(v_1) + g_{\mu_2}(v_1) - g_{\mu_1}(v_1) \geqq$$
$$\geqq \varkappa(v_2 - v_1) + \gamma(\mu_2 - \mu_1)$$

from which we obtain by induction and (v)

$$g^{q_1'+q_2}_{\mu_2}(y) - g^{q_1'+q_2}_{\mu_1}(y) \geqq (N-1)^{-1}(N^{q_1} - 1)\,\Gamma(\mu_2 - \mu_1)\,\varkappa^{q_2} +$$
$$+ (\varkappa - 1)\,\varkappa^{q_2 - 1}\gamma(\mu_2 - \mu_1) \geqq \varkappa^{q_2}[(N-1)^{-1}(N^{q_1} - 1) + \gamma q_2](\mu_2 - \mu_1) > 0\,.$$

9° It follows now from 6° and 8° that

$$g^q_{\mu_2}(x) \geqq g^q_{\mu_1}(x) \geqq 0 \quad \text{for all} \quad x \in [y_{i-1} + \xi,\ y_i + \tfrac{1}{2}\zeta] \tag{13}$$

and $q > q_1 + q_2$. From (ii) it follows that $g_\mu(y) - y \geqq \lambda > 0$ for $\mu_0 \leqq \mu \leqq \mu_0 + \eta$, $y \in [y_i + \frac{1}{2}\zeta,\ y_i + \zeta]$ and some positive $\lambda$. Therefore, there exists a $q_5 > 0$ such that for every $\mu_0 \leqq \mu \leqq \mu_0 + \eta$ and $y \in [y_i - \frac{1}{2}\zeta,\ y_i + \zeta]$,

$$g^{q_5}_{\mu}(y) \geqq y_i + \zeta\,. \tag{14}$$

Let $q_5'$ be the smallest $q_5$ satisfying (13). Then, by (ii), $g^{q_5'}_{\mu_1}(y) \leqq g^{q_5'}_{\mu_2}(y)$ and applying (13) to $x = g^{q_5'}_{\mu_2}(y)$ we obtain $g^q_{\mu_1}(y) \leqq g^q_{\mu_2}(y)$ for all $q > q_1 + q_2 + q_5 > q_1 +$ $+ q_2 + q_5'$. This completes the proof.

Remark 3. Going carefully through the estimates of 1°–3° one can check that for every $\varepsilon > 0$ there is a neighbourhood $W$ of $f$ such that for every $\tilde{f}$ in $W$, the quantity $\eta(\tilde{f}, \mu_0)$ (defined so as to satisfy the analogues of (i)–(vii) for $\tilde{f}$, $\mu_0$) can be choosen so that $\eta(\tilde{f}, \mu_0) > \eta(f, \mu_0) - \varepsilon$. For any $f$, $\mu_0$ we can define $\eta^-(f, \mu_0)$ as the quantity for which the analogues of (i)–(vii) are satisfied in $[\mu_0, \mu_0 - \eta]$; the analogy of the first part of this remark for $\eta^-$ is straightforward.

Proof of Proposition 2. From lemma 4 it follows that for $\mu_0 \leqq \mu_1 \leqq \mu_2 \leqq \mu_0 + \eta$

$$\sigma_f(\mu_1) = \lim_{n \to \infty} n^{-1} f^n_{\mu_1}(y) = \lim_{\nu \to \infty} (\nu l)^{-1} f^{\nu l}_{\mu_1}(y) \leqq \lim_{\nu \to \infty} (\nu l)^{-1} f^{\nu l}_{\mu_2}(y) = \sigma_f(\mu_2) .$$

Furthermore, $\sigma_f$ is constant in some left neighbourhood of $\mu_0$. Thus, $\sigma_f$ is non-decreasing in some neighbourhood of $\mu_0$. From Lemma 4 it follows that $\sigma_f$ is not constant in any (right) neighbourhood of $\mu_0$.

Now, let $\varrho$ be irrational and $\sigma_f(\mu) = \varrho$ for some $\mu \in U$. Then, by (i), (ii), §1 there exists an 1-periodic homeomorphism $h : R^1 \to R^1$ such that $F_\mu(y) = h^{-1} \circ f_\mu \circ \circ h(y) = y + \varrho$ for all $y$. From lemma 7 it follows that for all $\mu' > \mu$, $\mu' \in U$ it holds $F^{\nu l}_{\mu'}(y) = h^{-1} \circ f^{\nu l}_{\mu'} \circ h(y) > y + \nu\varrho$ for all $y$ and $\nu \geqq n$. Because of the periodicity of $F$, there exists a rational, $pq^{-1}$, such that $F^{nl}_{\mu'}(y) > y + pq^{-1} > y + nl\varrho = F^{nl}_\mu(y)$. This implies $F^{nlq}_{\mu'}(y) - y > p > F^{nlq}_\mu(y) - y$ and, by (i), §1, $\sigma_f(\mu') = \sigma_F(\mu') > > p(nlq)^{-1} > \sigma_F(\mu) = \sigma_f(\mu)$. In a similar way we can prove $\sigma_f(\mu') < \sigma_f(\mu)$ for $\mu' < \mu$.

Proposition 2 allows us to establish some generic global properties of the behavior of the rotation number as well as some stability results. For a more transparent formulation of these we introduce the following notation:

A closed interval $J$ will be called a plateau of a function $\sigma : P \to R^1$ if $f$ is constant on $J$ but on no toher interval which contains $J$ properly. A plateau will be called extremal (upper, lower) if its endpoints are local extrema of the same kind (local maxima, local minima). $\sigma(J)$ will be called the value of the plateau.

**Theorem 1.** *For every $f \in \mathscr{F}_1$, $\sigma_f$ has at most a countable number of extremal plateaus, only finitely many of them intersecting any compact subset of $P$. The values of all non-trivial plateaus of $\sigma_f$ are rational.*

Proof. From proposition 2 it follows that the intervals $[\mu_1 - \eta^-(f, \mu_1), \mu_2 + + \eta(f, \mu_2)]$ cover $P$, if $[\mu_1, \mu_2]$ runs over all plateaus with rational values. From this covering we can extract a locally finite subcovering. From proposition 2 it follows further that there is at most one extremal plateau contained in every element $[\mu_1 - - \eta^-(f, \mu_1), \mu_2 + \eta(f, \mu_2)]$ of this covering, namely $[\mu_1, \mu_2]$.

For the next theorem, we shall need two lemmas.

**Lemma 8.** *Let $f \in \mathscr{F}_{11}$, $f$, $\mu_0$ satisfy the assumptions of Proposition 2. Then, for every choice of sufficiently small neighbourhoods $U$ of $\mu_0$, $\hat{V}_i$ of $\hat{y}_i$, there is a neighbourhood $W$ of $f$ such that for every $\tilde{f} \in W$:*

1. *there is exactly one l-periodic point* $(\tilde{\mu}, \tilde{y}_i)$ *in* $U \times V_i$ *such that*

$$(15) \qquad (\tilde{f}^l_\mu)'(y) = 1$$

*is satisfied for* $\mu = \tilde{\mu}$, $y = \tilde{y}_i$, *where* $\tilde{\mu}$ *is the same for all i, and* $\tilde{\hat{y}}_i$ *belong to one orbit;*

2. *there are no other l-periodic points in* $U \times R^1$ *with* $\mu > \tilde{\mu}$ *except of the points* $(\tilde{\mu}, \tilde{y}_i)$, *while* $\sigma_{\tilde{f}}(\mu) = \varrho$ *for* $\mu \leqq \tilde{\mu}$, $\mu \in U$;

3. *it holds*

$$(16) \qquad \frac{\partial}{\partial \mu} \tilde{f}^l_\mu(y) > 0\,, \quad (\tilde{f}^l_\mu)''(y) > 0$$

*for* $\mu = \tilde{\mu}$ *and* $y = \tilde{y}_i$.

Proof. Let us choose $U$ and $V_i$ so that $f$ has no $l$-periodic points in $H = \bar{U} \times (R^1 \setminus \bigcup_i V_i)$, no periodic points satisfying (15) in $\bar{U} \times R^1$ except of $(\mu_0, y_i)$, and (16) is valid for $f = \tilde{f}$ and all $(\mu, y) \in G = \bar{U} \times \bigcup_i \bar{V}_i$. Since $\hat{H}$ and $\hat{G}$ are compact there exists a neighbourhood $W_1$ of $f$ such that every $\tilde{f} \in W_1$ is without periodic points in $H$, satisfies (18) in $G$ and

$$(17) \qquad \max_y (\tilde{f}^l_\mu(y) - y) < k + 1\,.$$

Denote $\tilde{F}(\mu, y) = \tilde{f}^l_\mu(y) - k$.

Then, the set of $l$-periodic points of $f$ in $G$ satisfying (15) can be written as $(J^1\tilde{F})^{-1}(Q) \cap G$, where $J^1\tilde{F}$ means the 1-jet of $\tilde{F}$ in $J^1(P \times R^1, R^1)$ and by $Q$ we have denoted the submanifold of points $(\mu, y, y, 0)$ in $J^1(P \times R^1, R^1) \simeq (P \times R^1 \times R^1 \times J^1(1, 1))$ (in the notation of [12]). From (16) it follows that $F$ intersects $Q$ transversally in $G$ at the points $(\mu_0, y_i)$. By [12, 8. 2, Proposition 1], there is a neighbourhood $W \subset W_1$ of $f$ such that for every $\tilde{f} \in W$, $J^1\tilde{F}$ will intersect $Q$ transversally in $G$. Because of our choice of $W_1$, (16) will be satisfied at these points. The unicity of the points of intersection (which we denote by $(\mu_i, y_i)$) in $U \times V_i$ follows from the validity of (16) in $U \times V_i$ and lemma 1. Because of unicity, the points $(\mu_i, y_i)$ must belong to one orbit, which implies the equality of $\mu_i$ for all $i$.

The equality $\sigma_{\tilde{f}}(\mu) = kl^{-1}$ for $\mu \leqq \tilde{\mu}$ in $U$ follows from the unicity of the periodic points of $f$ satisfying (15) in $U \times R^1$ (there should be a periodic point satisfying (15) with $\mu = \mu_1$, if $\mu_1$ is a left endpoint of an interval of $\sigma_f(kl^{-1})$). From (16) and (17) it follows that there are no $l$-periodic points of $\tilde{f}$ in $U \times V_i$ with $\mu \geqq \tilde{\mu}$. Since $\tilde{f}$ has no periodic points in $H$ as well this completes the proof.

**Lemma 9.** *Let* $f$, $\mu_0$ *satisfy the assumptions of proposition* 2. *Then, for every* $\varepsilon > 0$, *there exists a neighbourhood W of f such that for every* $\tilde{f} \in W$, *there is a* $\tilde{\mu}_0 \in (\mu_0 - \varepsilon, \mu_0 + \varepsilon)$ *such that* $\sigma_{\tilde{f}}(\mu) = kl^{-1}$ *for* $\mu \in (\mu_0 - \varepsilon, \tilde{\mu}_0)$ *and* $\eta(\tilde{f}, \tilde{\mu}_0) > \eta(f, \mu_0) - \varepsilon$.

This lemma is a consequence of Lemma 8 and Remark 3.

By the profile of a real function $\sigma$ with a countable number of extremal plateaus we shall understand the sequence $\{\sigma_j\}_{j=r}^{s}$ of values of the extremal plateaus ordered from left to right, where $r$ is 0 or $-\infty$ according to whether the sequence is finite from the left or not (note that in the infinite case the profile is defined up to shift). Two profiles are considered equal, if they can be brought to each other by an appropriate shift.

**Theorem 2.** *For every $f \in \mathscr{F}_1$, there is an open neighbourhood $W$ of $f$ such that for every $\tilde{f} \in W$ the profiles of $\sigma_f$ and $\sigma_{\tilde{f}}$ are equal.*

Proof. Let $J_i = [\mu_{i1}, \mu_{i2}]$ be a collection of plateaus of $\sigma_f$ with rational values $k_i/l_i$ choosen in such a way that only finitely many of them intersect any compact subset of $P$ and the intervals $(\mu_{i1} - \eta(f, \mu_{i1}), \mu_{i2} + \eta(f, \mu_{i2}))$ cover $P$. Then, by lemma 6, there exists a neighbourhood $W_i'$ of $f$ such that for every $\tilde{f} \in W_i'$, numbers $\mu_{ij}$ will exist, corresponding to $\mu_{ij}$ as $\tilde{\mu}_0$ to $\mu_0$ in lemma 9 and satisfying

$$|\tilde{\mu}_{i1} - \mu_{i1}| < \min\{\eta^-(f, \mu_{i1}), \tfrac{1}{2}(\mu_{i1} + \mu_{i2})\},$$
$$|\tilde{\mu}_{i2} - \mu_{i2}| < \min\{\eta(f, \mu_{i2}), \tfrac{1}{2}(\mu_{i1} + \mu_{i2})\},$$

such that

$$\sigma_{\tilde{f}}(\mu) = \sigma_f(J_i) \quad \text{for} \quad \mu \in [\tilde{\mu}_{i1}, \mu_{i1} + \varepsilon], \quad \mu \in [\mu_{i2} - \varepsilon, \tilde{\mu}_{i2}]$$

for some $\varepsilon > 0$ independent of $\tilde{f}$. By Lemma 6, $\min_y [f_\mu^{l_i}\mu(y) - y]$ and $\max_y [f^{l_i}(y) - y]$ must be bounded away from $k_i$ by some positive constant for $\mu \in [\mu_{i1} + \varepsilon, \mu_{i2} - \varepsilon]$. Therefore, there is a neighbourhood $W_i \subset W_i'$ of $f$ such that $\sigma_{\tilde{f}}(\mu) = \sigma_f(J_i)$ for $\mu \in [\mu_{i1} + \varepsilon, \mu_{i2} - \varepsilon]$ and, consequently, for $\mu \in J_i = [\tilde{\mu}_{i1}, \tilde{\mu}_{i2}]$. Since there is only a finite number of $J_i$'s intersecting any compact subset of $P$, and $W_i$ can be choosen in such a way that they do not restrict $f$ outside a certain bounded neighbourhood of $J_i$, it follows from lemma 9 that $\bigcap_i W_i$ will contain a neighbourhood $W$ of $f$ such that for every $\tilde{f} \in W$ the intervals $(\tilde{\mu}_{i1} - \eta^-(f, \tilde{\mu}_{i1}), \tilde{\mu}_{i2} + \eta(\tilde{f}, \tilde{\mu}_{i2}))$ will cover $P$. The correspondence $J_i \to \tilde{J}_i$ renders the equality of profiles of $\sigma_{\tilde{f}}$ and $\sigma_f$ for $f \in W$.

As a consequence of theorem 2, we obtain

**Theorem 3.** *There is an open dense subset $\mathscr{F}_0$ of $\mathscr{F}$ such that for every $f \in \mathscr{F}_0$, $\sigma_f$ has only finitely many extremal plateaus intersecting any compact subset of $P$ and the values of the extremal plateuas are rational. Moreover, for every $f \in \mathscr{F}_0$ there exists a neighbourhood $W$ of $f$ such that for every $\tilde{f} \in W$, the profiles of $\sigma_f$ and $\sigma_{\tilde{f}}$ are equal.*

Proof. $\mathscr{F}_0$ is obtained as the union of the neighbourhoods of the maps $f \in \mathscr{F}_1$, the existence of which is proven in Theorem 2.

**Corollary 2.** *Generically (for every $f \in \mathscr{F}_0$), $\sigma_f$ is of bounded vyriation.*

The second part of theorem 3 expresses a certain stability of behavior of $\sigma_f$ under small changes of $f$. This result is refined somewhat by the next theorem.

**Theorem 4.** *For every $f \in \mathscr{F}_1$ and any $\varepsilon > 0$ there is a neighbourhood $W$ of $f$ such that for every $f \in W$ there is a continuous map $h : P \to P$ such that $|h(\mu) - \mu| < \varepsilon$ for all $\mu$ and $\sigma_{\tilde{f}} \circ h = \sigma_f$; for $\tilde{f} \in W \cap \mathscr{F}_1$, $h$ is a homeomorphism.*

Proof. Let $K$ be any compact subset of $P$. Take $L$ so large that the length of no interval separating the plateaus of $\sigma_f^{-1}(kl^{-1})$ with $l \leq L$ exceeds $\frac{1}{2}\varepsilon$. According to Theorem 3 we can choose a neighbourhood $W_1$ of $f$ so small that for every $\tilde{f} \in W_1$ the profile of $\sigma_{\tilde{f}}$ is equal to that of $\sigma_f$. Since $\sigma_f$ and $\sigma_{\tilde{f}}$ are monotonic between their extremal plateaus, the correspondence of the extremal plateuas, rendered by the equality of the profiles, can be extended to a unique value and order preserving correspondence of all plateus. From lemma 9 and lemma 2 it follows that there is a neighbourhood $W \subset W_1$ of $f$ such that for all $f \in W_2$, if $[\mu_1, \mu_2]$ is a plateau intersecting $K$ of value $kl^{-1}$ with $l \leq L$, then the corresponding interval $[\tilde{\mu}_1, \tilde{\mu}_2]$ of $\sigma_f^{-1}(kl^{-1})$ satisfies $|\tilde{\mu}_1 - \mu_1| < \frac{1}{4}\varepsilon$, $|\tilde{\mu}_2 - \mu_2| < \frac{1}{4}\varepsilon$. For $f \in W_2$, we construct $h$ on $K$ as follows: $h$ maps every plateau of $\sigma_f$ linearly and increasignly into the corresponding plateau of $\sigma_{\tilde{f}}$. It is obvious that $h$ has the required properties on $K$. The fact that $h$ is a homeomorphism $K \to h(K)$ for $\tilde{f} \in W_1 \cap I$ follows from the non-triviality of the plateaus of $\tilde{f}$ with rational values.

To construct $h$ on $P$ we cover $P$ by a strictly increasing sequence of compacts $K_i$. It is obvious that having constructed $h$ on $K_{i-1}$ it can be extended to $K_i$ without changing it in $K_{i-1}$ for every $\tilde{f}$ from some neighbourhood $W_i$ which does not impose any new restrictions on $f$ in $K_{i-1}$. Therefore, $\bigcap_i W_i$ will contain a neighbourhood $W$ of $f$ such that for every $\tilde{f} \in W$ the map $h$ with required properties over all $P$ will exist.

Remark 4. If $f \in \mathscr{F}_0$ but $f \notin \mathscr{F}_1$, $h$ may not exist (since some of the plateaus of $f$ with rational values can be trivial), but the one to one correspondence of the plateaus of $f$ and those of $\tilde{f}$ from some neighbourhood of $f$ can neverthless be established.

We conclude this section by a partial analogue of proposition 2 for the case of $\sigma_f(\mu_0)$ irrational. This result has certain implications for the problem of bifurcation of periodic orbits (cf. [7], [8]).

**Proposition 3.** *Let $f_{\mu_0}(y) = y + \alpha$, where $\alpha$ is irrational. Let $\beta_0 = \int_0^1 (\partial f/\partial \mu) \,.$ $. (\mu_0, y)\,dy \neq 0$. Then, $\sigma_f$ is increasing or decreasing in some neighbourhood of $\mu_0$ in accordance with the sign of $\beta_0$.*

This Proposition fo lows, similarly as Proposition 2 from lemma 7, from the following

**Lemma 10.** *Let the assumptions of Proposition 3 be satisfied and let $\beta_0 > 0$. Then, there exist $\eta > 0$, $k$ such that for all $\mu_1 < \mu_2$ such that $\mu_0 - \eta < \mu_1 <$*

$< \mu_2 < \mu_0 + \eta$ *and all* $y, n > 0$,

$$f_{\mu_1}^{nk}(y) < f_{\mu_2}^{nk}(y).$$

Proof. Denote $\beta(y) = (\partial f/\partial\mu)(\mu_0, y)$. We have $f_\mu^n(y) = y + n\alpha + (\mu - \mu_0) \cdot [\sum_{j=0}^{n-1} \beta(y + j\alpha) + \omega_n(\mu, y)]$ where $\lim_{\mu\to\mu_0} \omega_n(\mu, y) = 0$ uniformly in $y$. From the individual ergodic theorem it follows that

$$\lim_{n\to\infty} n^{-1}[\sum_{j=0}^{n-1} \beta(y + j\alpha)] = \beta_0 \tag{18}$$

for a.e.y. Since the derivative of $\beta$ (and, consequently, that of $n^{-1}[\sum_{j=0}^{n-1} \beta(y + j\alpha)]$ is bounded, (18) is valid uniformly for all $y$.

Choose $k$ so large that $\sum_{j=0}^{k-1} (y + j\alpha) \geqq \frac{1}{2}\beta_0$ and then $\eta$ so small that $|\omega_k(\mu, y)| \leqq \leqq \frac{1}{4}\beta_0$ for $|\mu - \mu_0| < \eta$. Then, we have,

$$f_{\mu_2}^k(y) - f_{\mu_1}^k(y) \geqq (\mu_2 - \mu_1)(\tfrac{1}{2}\beta_0 - \tfrac{1}{4}\beta_0) > 0$$

from which the lemma follows immediately.

Remark 5. Although (cf. [1, §1]) $f_{\mu_0}$ is topologically equivalent to the shift provided $\sigma_f(\mu_0)$ is irrational, proposition 1 is not as general as it might seem, because it is not known whether the equivalence homeomorphism is $C^2$ differentiable in general (cf. [1], [4]).

## 3

In this section, we apply the results of §2 to establish some generic properties and stability properties of the loci of periodic points.

**Proposition 4.** *Let* $f, \mu_0$ *satisfy the assumptions of Proposition* 2. *Then, for every* $\varrho \in \sigma_f(\mu_0, \mu_0 + \eta(f, \mu_0))$, $\varrho = \varkappa\lambda^{-1}$ *and every plateau* $J$ *of* $\sigma_f^{-1}(\varrho)$ *contained in* $(\mu_0, \mu_0 + \eta(f, \mu_0))$ $Z_\lambda \cap (J \times R^1)$ *is topologically a line.*

Remark 6. The assertion of propostion 4 is true also under the assumptions of proposition 3, for $J$ contained in a sufficiently small (two sided) neighbourhood of $\mu_0$. The proof is the same, lemma 10 replacing lemma 7.

Proof of Proposition 4. From proposition 2 it follows that for $\mu \in [\mu_0, \mu_0 + \eta(f, \mu_0)]$ to the right (left) of $J$, $\sigma_f(\mu) > (<\varrho)$. Therefore, by (iv), §1, for sufficiently large $\nu$, $f_\mu^{\nu l\lambda}(y) - y > \nu l\varkappa(<\nu l\varkappa)$. Consequently, for every $y \in R^1$, there is a $\mu^* \in J$ such that $f_{\mu^*}^{\nu l\lambda}(y) - y = \nu l\varkappa$, from which we obtain by (vi), §1, $(\mu^*, y) \in Z_\lambda(f)$. It follows from lemma 7 that $\mu^*$ is unique. Denote $\mu^* = \chi(y)$. Then, $Z_\lambda(f) \cap (J \times R^1)$ is the graph of $\chi$. Since $Z_\lambda(f)$ is closed, $\chi$ is continuous, which proves the lemma.

**Theorem 5.** *Let $f \in \mathscr{F}_0$. Then, given $\varepsilon > 0$, there is a neighbourhood $W$ of $f$ such that for every $\tilde{f} \in W$ there exists a 1-periodic in $y$ map $g : \bigcup_l Z_l(f) \to \bigcup Z_l(\tilde{f})$ satisfying*

$$|g(y) - y| < \varepsilon \tag{19}$$

*for all $y \in \bigcup_l Z_l(f)$, the restriction of which to every $Z_l(f)$ is an isomorphism of $Z_l(f)$ and $Z_l(\tilde{f})$ isotopic to the identity.*

Proof. We can assume without loss of generality that $f'_\mu > 0$. The case $f'_\mu < 0$ can be reduced to the former by considering $f^2_\mu$.

Let $J_i = [\mu_{i_1}, \mu_{i2}]$ be plateaus of $\sigma_f$ with rational values $\varrho_i = k_i l_i^{-1}$ ordered from left to right in such a way that any compact subset of $P$ is intersected by only finitely many of them, the intervals $(\mu_{i1} - \eta(f, \mu_{i1}), \mu_{i2} + \eta(f, \mu_{i2}))$ cover $P$ and

$$|\mu_{i+1,1} - \mu_{i2}| < \tfrac{1}{2}\varepsilon\,. \tag{20}$$

For every $i$ there is a finite number of components of $Z_{l_i}$ contained in $J_i \times S^1$. Since for every $K$ compact, the numbers $l_i$ with $J_i \cap K \neq 0$ are bounded, the existence of the restriction of $g$ on $\bigcup_l Z_l(f) \cap \bigcup_i (J_i \times R^1)$ follows from the transversal isotopy theorem [11, 20. 2] similarly as in the proof of lemma 5.

To extend $g$ outside $\bigcup [J_i \times R^1]$ we choose $W$ so that for $\tilde{f} \in W$ the correspondence of plateaus asserted in Remark 4 exists,

$$|\mu' - \mu| < \tfrac{1}{4}\varepsilon \tag{21}$$

for $\mu$ and $\mu'$ from corresponding plateaus outside $\bigcup_i J_i$ and the intervals $(\mu_{i1} - \eta^-(f, \mu_{i1}), \mu_{i2} + \eta(f, \mu_{i2}))$ cover $P$. Then, it follows from proposition 4 that given a plateau $J$ different from $J_i$, if $\tilde{J}_i$ is the corresponding plateau of $\tilde{f}$, $Q = \bigcup_i Z_l(f) \cap (J \times R^1)$ and $\tilde{Q} = \bigcup_l Z_l(\tilde{f}) \cap [\tilde{J} \times R^1]$ are lines which can be represented as $Q = \{(\mu, y) \mid \mu = \chi(y)\}$ and, similarly, $\tilde{Q} = \{(\mu, y)\, \mu = \tilde{\chi}(y)\}$, $\chi$, $\tilde{\chi}$ continuous. From (20), (21) it follows that if we define $G : [0, 1] \times [\bigcup_l Z_l(f) \cap (J \times R^1)] \to P \times R^1$ as $G(t, \chi(y), y) = (\chi(y) + t[\tilde{\chi}(y) - \chi(y)], y)$, $g = G(1, .,.)$ will be the required isomorphism, satisfying (19).

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