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DOUBLE LAYER POTENTIALS AND THE DIRICHLET PROBLEM

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Introduction. The purpose of this paper is to investigate an integral representability and boundary behaviour of solutions of the generalized Dirichlet problem for discontinuous boundary conditions. For a class of open subsets of Euclidean space, a representation in terms of the generalized double layer potential is given for any bounded function measurable with respect to the area measure on the boundary. It is proved that the nontangential limit of the solution of the generalized Dirichlet problem coincides with the boundary condition at each point of the boundary except for a set of area measure zero. As a by-product we establish that the area measure and the harmonic measure are mutually absolutely continuous. This generalizes the classical result [15] obtained for domains with the boundary of bounded curvature. The domains which we consider are all regular domains for the Dirichlet problem. On the other hand, no smoothness assumptions on their boundaries are imposed.

The concepts used here have their origin in investigations of J. KRÁL [8] and there are connnections of this paper with results obtained in [11]–[13]. Some results of this paper were announced in [14].

1. Preliminaries. This section serves to recall some facts, mostly in order to explain terminology and notation, sometimes to point out results on the subject.

In what follows, $m > 2$ will be a fixed integer and the symbol R^m will stand for the Euclidean space of dimension m. For $M \subset R^m$ we shall denote by $\operatorname{cl} M$ and $\operatorname{fr} M$ the closure and the boundary of M, respectively; H stands for the $(m - 1)$-dimensional Hausdorff measure defind in usual way (see [11], section 1).

For $r > 0$ and $y \in R^m$, denote by $\Omega_r(y)$ the open ball with center y and radius r and put $\Gamma = \operatorname{fr} \Omega_1(0)$, $A = H(\Gamma)$.

If $Q \subset R^m$ is a Borel set and $S \subset R^m$ is an open segment or a half-line, then $z \in S$ will be termed a hit of S on Q provided both $S \cap Q \cap \Omega_r(z)$ and $(S - Q) \cap \Omega_r(z)$ have a positive linear measure for every $r > 0$. Given $y \in R^m$, $0 < r \leqq \infty$ and $\theta \in \Gamma$, we shall denote by $n_r^Q(\theta, y)$ the total number of all the hits of $\{y + \varrho\theta; 0 < < \varrho < r\}$ on Q. It turns out that for fixed Q, $r > 0$ and $y \in R^m$, $n_r^Q(\theta, y)$ is a Baire function of the variable θ on Γ (see [8], proposition 1.6) and one may define

$$v_r^Q(y) = \int_\Gamma n_r^Q(\theta, y)\, \mathrm{d}H(\theta).$$

Throughout this paper, $G \subset R^m$ will stand for an open set with compact boundary $B \neq \emptyset$ and the following two conditions on G will be imposed:

$$\text{(1)} \qquad \operatorname{fr} G = \operatorname{fr}(R^m - \operatorname{cl} G),$$

$$\text{(2)} \qquad \limsup_{r \to 0+} \sup_{y \in B} v_r^G(y) < \tfrac{1}{2}A.$$

It follows from Král's results that (2) implies

$$\text{(3)} \qquad \sup_{y \in R^m} v_\infty^G(y) < \infty$$

(see [9], remark on p. 596 and [8], theorem 2.13). Consequently, G is a set with finite perimeter and the m-dimensional density $d_G(z)$ of G at z is well-defined for each $z \in R^m$ (see [8], proposition 2.10 and lemma 2.7).

Let us denote by $\mathfrak{B}$ the Banach space of all finite signed Borel measures with support in B; the norm of an element $\mu \in \mathfrak{B}$ is its total variation $\|\mu\|$. Given $z \in R^m$, δ_z stands for the Dirac measure concentrated at z. With each $\mu \in \mathfrak{B}$ we associate its potential

$$U\,\mu(x) = \int_B p(x - y)\, \mathrm{d}\mu(y)$$

corresponding to the Newtonian kernel $p(z) = |z|^{2-m}/(m - 2)$.

In view of (3), for any $\mu \in \mathfrak{B}$, the distribution $\mathscr{T}\mu$ defined by

$$\mathscr{T}\mu(\varphi) = \int_G \operatorname{grad} \varphi(x) \operatorname{grad} U\,\mu(x)\, \mathrm{d}x$$

over the class $\mathscr{D}$ of all infinitely differentiable functions with compact support in R^m can be identified with a uniquely determined element $\mathscr{T}\mu$ of $\mathfrak{B}$ and the operator $\mathscr{T} : \mu \mapsto \mathscr{T}\mu$ acting on $\mathfrak{B}$ is a bounded linear operator (see [8], theorem 1.13; compare also [11], theorem 5 and remark 9).

The results of [11]–[13] will be used on several places in this paper. In fact, only a special form of those results (corresponding to the case of $\lambda = 0$) is important for us here.

It follows from (2), (1) and theorems 20 and 31 in [12] that

$$\sup_{y \in B} [A|d_G(y) - \tfrac{1}{2}|] < \tfrac{1}{2}A.$$

In particular,

$$\text{(4)} \qquad 0 < d_G(y) < 1, \quad y \in B.$$

Put $M = R^m - \mathrm{cl}\, G$ and observe that M is a non-void open set and $\mathrm{fr}\, M = B$ (see (1)). (Note that M is just the set on which the Dirichlet problem will be investigated.) Recalling that the m-dimensional density $d_M(z)$ of M at $z \in R^m$ is given by $d_M(z) = 1 - d_G(z)$ we conclude by (4) that each $y \in B$ is a regular point for the Dirichlet problem (see [6], corollary 10.5).

Denote by $n^G(y)$ and $n^M(y)$ the exterior normal of G and M at y in the sense of Federer, respectively, and by $\hat{B}$ the reduced boundary of G (for definitions see [11], section 2 or [8], remark 2.11). Note here that $\hat{B} \subset B$ and $n^G(y) = -n^M(y)$ holds for any $y \in R^m$.

It follows from (4) and from lemma 3.7 in [8] that

$$H(B - \hat{B}) = 0\,. \tag{5}$$

Since $H(\hat{B}) < \infty$ (see remark 2.11 in [8]) we have $H(B) < \infty$ as well.

One easily verifies that $v_r^M(z) = v_r^G(z)$ provided $z \in R^m$ and $r > 0$ and (1) implies that both the sets $Z - M$ and $Z - G$ have a positive m-dimensional Lebesgue measure whenever Z is an arbitrary open set with $B \cap Z \neq \emptyset$. Consequently, the proposition formulated in [10] may be used to assert that G has only a finite number of components and their closures are mutually disjoint. (Note that the same is true for M.) We shall denote by q $(0 \leqq q < \infty)$ the number of bounded components of G. The symbol G_0 will stand for the unbounded component of G (if any); the bounded components of G will be denoted by $G_1, \ldots, G_q$.

Finally, we shall write $\mathcal{N}$ for the null-space $\mathcal{T}_{-1}(0)$ of the operator $\mathcal{T}$.

2. Lemma. *The dimension of $\mathcal{N}$ does not exceed q.*

Proof. Notice at (4) guarantees that the m-dimensional Lebesgue measure of B is zero (compare [13], lemma 25). Employing lemma 24 and theorem 19 in [13] we get

$$\int_G |\mathrm{grad}\, U\mu(x)|^2 \,\mathrm{d}x = 0$$

provided $\mu \in \mathcal{N}$. Consequently, $U\mu$ is constant on each G_j and vanishes on G_0. The same arguments as in the proof of theorem 26 in [13] may be used to justify that

$$U\mu = 0 \quad \text{on } G \text{ implies} \quad \mu = 0 \tag{6}$$

(compare also [8], lemma 4.8).

If $q = 0$, the proof is complete. Assume now $q > 0$ and choose an arbitrary $z_j \in G_j$ $(j = 1, \ldots, q)$. In view of (6), the mapping

$$\mu \mapsto [U\mu(z_1), \ldots, U\mu(z_q)]$$

is an injection of $\mathcal{N}$ into R^q. Consequently, $\dim \mathcal{N} \leqq q$.

3. Notation. Let $\mathscr{B}$ denote the Banach space of all bounded Baire functions on B equipped with the supremum norm $\|\ldots\|$. $\mathscr{C}$ is the subspace of all continuous functions in $\mathscr{B}$. We shall often write $\langle f, \mu\rangle$ instead of

$$\int_B f \,\mathrm{d}\mu\,, \quad f \in \mathscr{B}\,, \quad \mu \in \mathfrak{B}\,.$$

For each $y \in R^m$ define $\nu_y \in \mathfrak{B}$ by

$$\mathrm{d}\nu_y(x) = \frac{n^M(x)\,.\,(x - y)}{|x - y|^m}\,\mathrm{d}H(x)\,, \tag{7}$$

and the operator T acting on $\mathscr{B}$ is introduced by

$$Tf(y) = A\, d_G(y) f(y) + \langle f, \nu_y\rangle\,, \quad y \in B\,, \quad f \in \mathscr{B}\,. \tag{8}$$

If follows from (5) in [12] and proposition 8 in [11] that

$$\langle Tf, \mu\rangle = \langle f, \mathscr{T}\mu\rangle\,, \tag{9}$$

provided $f \in \mathscr{B}$ and $\mu \in \mathfrak{B}$ (compare also with 3.2 and 3.4 in [8]).

It should be noted that in our case $\mathfrak{B}$ is a proper closed subspace of the dual space to $\mathscr{B}$ so that the traditional form of the Riesz-Schauder theory is not applicable to the pair of operators $T, \mathscr{T}$. On the other hand, Š. Schwabik has recently published in [16] a modification of the above mentioned theory and his variant will be suitable for our purposes.

Before stating an assertion concerning $N = T_{-1}(0)$ the following lemma is useful. We agree to denote by f_j the characteristic function of $\mathrm{fr}\, G_j$ $(j = 1, \ldots, q)$.

4. Lemma. *Let us fix $j \in \{1, \ldots, q\}$. We have $Tf_j = 0$ on B and*

$$\nu_z(\mathrm{fr}\, G_j) = 0 \tag{10}$$

whenever $z \in M$ $(= R^m - \mathrm{cl}\, G)$.

Proof. Choose $z \in M$ and construct the function $\varphi_j \in \mathscr{D}$ in such a way that $\varphi_j(\mathrm{cl}\, G_j) = \{1\}$, $\varphi_j(z) = 0$ and $\varphi_j(\mathrm{cl}\, G_k) = \{0\}$ provided $k \neq j$. Making use of the formula (4) of [12] we obtain for each $y \in B$

$$Tf_j(y) = \int_G \mathrm{grad}\, \varphi_j(x)\,.\,\mathrm{grad}\, U\, \delta_y(x)\,\mathrm{d}x = 0\,.$$

The same formula yields

$$\langle f_j, \nu_z\rangle = \langle \varphi_j, \nu_z\rangle = \int_{G_j} \mathrm{grad}\, \varphi_j\,.\,\mathrm{grad}\, U\delta_z = 0\,.$$

The proof is complete.

5. Proposition. *The equalities*

$$\dim \mathcal{N} = \dim N = q$$

hold good.

Proof. We first note that the functions f_j are linearly independent. It is therefore sufficient to establish the equality $\dim \mathcal{N} = \dim N$. Indeed, lemma 2 says that $\dim \mathcal{N} \leqq q$ and lemma 4 shows $\dim N \geqq q$.

Before referring to the Schwabik's result, observe that the bilinear form $\langle f, \mu\rangle$ on $\mathcal{B} \times \mathfrak{B}$ separates points both of $\mathfrak{B}$ and $\mathcal{B}$ (compare [16], section 4) and

$$|\langle f, \mu\rangle| \leqq \|f\| \cdot \|\mu\|, \quad f \in \mathcal{B}, \quad \mu \in \mathfrak{B}.$$

Lemma 33 and theorem 31 of [12] may be applied to assert the existence of a compact operator T_1 acting on $\mathcal{B}$ and a compact operator $\mathcal{T}_1$ on $\mathfrak{B}$ such that

$$\langle T_1 f, \mu\rangle = \langle f, \mathcal{T}_1 \mu\rangle, \quad f \in \mathcal{B}, \quad \mu \in \mathfrak{B} \tag{11}$$

and

$$\|T - \tfrac{1}{2}AI - T_1\| = \|\mathcal{T} - \tfrac{1}{2}A\mathcal{I} - \mathcal{T}_1\| < \tfrac{1}{2}A,$$

I and $\mathcal{I}$ being the identity operator on $\mathcal{B}$ and $\mathfrak{B}$, respectively. Consequently, the operator $V = T - T_1$ possesses a bounded inverse operator V_{-1} mapping $\mathcal{B}$ onto $\mathcal{B}$ and, similarly, $\mathcal{V} = \mathcal{T} - \mathcal{T}_1$ is a linear homeomorphism of $\mathfrak{B}$ onto itself. Moreover, by (9), (11) we have

$$\langle V_{-1} f, \mu\rangle = \langle f, \mathcal{V}_{-1}\mu\rangle, \quad f \in \mathcal{B}, \quad \mu \in \mathfrak{B}. \tag{12}$$

Proposition 4.1 and theorem 4.1 of [16] will be applied in the following context: $X = \mathcal{B}$, $Y = \mathfrak{B}$, $\boldsymbol{K} = -V_{-1}T_1$, $\boldsymbol{L} = -\mathcal{T}_1\mathcal{V}_{-1}$. One easily verifies that $\boldsymbol{K}$ and $\boldsymbol{L}$ are compact operators on $\mathcal{B}$ and $\mathfrak{B}$, respectively, and

$$\langle \boldsymbol{K} f, \mu\rangle = \langle f, \boldsymbol{L}\mu\rangle, \quad f \in \mathcal{B}, \quad \mu \in \mathfrak{B}.$$

Employing proposition 4.1 in [16] we arrive at

$$\dim (I - \boldsymbol{K})_{-1}(0) = \dim (\mathcal{I} - \boldsymbol{L})_{-1}(0).$$

Consequently, by the definition of $\boldsymbol{K}, \boldsymbol{L}$,

$$\dim T_{-1}(0) = \dim \mathcal{T}_{-1}(0)$$

and the proof is complete.

6. Theorem. *Let $\nu \in \mathfrak{B}$. Then there is a μ with $\mathcal{T}\mu = \nu$ if and only if*

$$\nu(\operatorname{fr} G_j) = 0, \quad j = 1, \ldots, q. \tag{13}$$

Proof. Let us denote by $\hat{T}$ the restriction of the operator T to $\mathscr{C}$. Then $\mathscr{T}$ is the dual operator to $\hat{T}$ (see (9)) and it is immediately seen that $\{f_1, \ldots, f_q\}$ is a basis of $\hat{T}_{-1}(0)$ (compare with lemma 4). By remark 32 in [12], in particular by the formula (92), the Riesz-Schauder theory is applicable to the pair of the operators $\hat{T}, \mathscr{T}$. Now the assertions follows from the Fredholm theorem.

7. Notation. For $z \in R^m$ and $f \in \mathscr{B}$ define

$$Wf(y) = \int_B f \, \mathrm{d}v_y = \int_B f(x) \cdot \frac{n^M(x) \, . \, (x - y)}{|x - y|^m} \, \mathrm{d}H(x) \, . \tag{14}$$

It is worth noting that

$$\lim_{\substack{z \to y \\ z \in M}} Wf(z) = Tf(y) \tag{15}$$

provided $f \in \mathscr{C}$ and $y \in B$. This follows immediately by (7), (8) and theorem 2.15 in [8].

In the rest of the paper, $x_j \in G_j$ will be a fixed point ($j = 1, \ldots, q$). We shall write, for the sake of brevity, δ_j instead of δ_{x_j}. It is obvious that each $U\delta_j$ is continuous on $R^m - \{x_j\}$.

The basic tool for our investigations is the following theorem.

8. Theorem. *Given $g \in \mathscr{B}$ there are $f \in \mathscr{B}$ (determined modulo N) and uniquely determined constants $a_j(g)$ such that*

$$g = Tf + \sum_{j=1}^{q} a_j(g) \, U\delta_j \tag{16}$$

holds on B.

If, in addition, $g \in \mathscr{C}$, then $f \in \mathscr{C}$ and

$$\lim_{\substack{x \to y \\ x \in M}} \left(Wf(x) + \sum_{j=1}^{q} a_j(g) \, U \, \delta_j(x)\right) = g(y) \tag{17}$$

for each $y \in B$.

Proof. We shall adopt the same notation as in the proof of proposition 5. We first note that the equation

$$Tf = g - \sum_{j=1}^{q} \alpha_j \, U\delta_j \, , \quad g \in \mathscr{B} \, , \tag{18}$$

has a solution in $\mathscr{B}$ if and only if the equation

$$(I - K) \, h = V_{-1}\left(g - \sum_{j=1}^{q} \alpha_j \, U\delta_j\right)$$

(for the unknown $h \in \mathscr{B}$) has the same property. By theorem 4.1 in [16] the last equation has a solution in $\mathscr{B}$ if and only if

$$(19) \qquad \langle V_{-1}(g - \sum_{j=1}^{q} \alpha_j \, U\delta_j), \mu \rangle = 0$$

for any $\mu \in \mathscr{B}$ satisfying

$$(20) \qquad (\mathscr{I} - L)\,\mu = 0\,.$$

One easily verifies that a $\mu \in \mathfrak{B}$ fulfils (20) if and only if $\mu \in \mathscr{V}\mathscr{N}$. Now, the necessary and sufficient condition (19) for solvability of (18) reads as follows:

$$\langle V_{-1}(g - \sum_{j=1}^{q} \alpha_j \, U\delta_j), \mu \rangle = 0\,, \quad \mu \in \mathscr{V}\mathscr{N}\,,$$

or, equivalently,

$$(21) \qquad \langle g - \sum_{j=1}^{q} \alpha_j \, U\delta_j, \mathscr{N} \rangle = 0$$

(see (12)). The formula (21) can be written in the form

$$\sum_{j=1}^{q} \alpha_j \, U\mu(x_j) = \langle g, \mu \rangle\,, \quad \mu \in \mathscr{N}\,.$$

If $\{\mu_1, \ldots, \mu_q\}$ is a basis in $\mathscr{N}$, then the matrix $(U\,\mu_k(x_j))$ $(j, k = 1, \ldots, q)$ is regular (compare with (6)). Consequently, given $g \in \mathscr{B}$, there are uniquely determined $\alpha_j = a_j(g)$ satisfying

$$(22) \qquad \sum_{j=1}^{q} a_j(g)\, U\,\mu_k(x_j) = \langle g, \mu_k \rangle\,, \quad k = 1, \ldots, q\,.$$

It is obvious that f enjoying (16) is determined modulo N.

As for the second part, let $\hat{T}$ mean the same as in the proof of theorem 6. The Riesz-Schauder theory being applicable to the pair of the operators $\hat{T}$ and $\mathscr{T}$, the proof of the existence of an $f \in \mathscr{C}$ satisfying (16) follows along the same lines as in [8] (theorem 4.13). In order to make the proof of our theorem complete it remains to refer to (15).

9. Remark. The proof of theorems 6 and 8 is patterned after Král's proof of the corresponding theorems of §3 in [8]. A little more restrictive condition on G than (2) is required in [8] and only $g \in \mathscr{C}$ are considered. Theorem 6 is related to theorem 28 in [13] where the case of connected G for a more general problem is treated (see also [10]). Results of this kind has also been obtained in [3] for the situation that both G and M are connected (see theorems 7, 8). The importance of the theorem 8 lies in the fact that not only continuous but an arbitrary bounded Baire function on B possesses the representation of the form (16). Of course, the proof of this theorem depends on a previous detailed study of the properties of the operator T acting over $\mathscr{B}$ (see [12]) and on Schwabik's modification of the Riesz-Schauder theory [16].

10. Lemma. *Let g_n be a uniformly bounded sequence of elements of $\mathscr{B}$ and suppose that $\lim_{n\to\infty} g_n(x) = 0$ for each $x \in B$. For each n, let $f_n \in \mathscr{B}$ and $a_j(g_n)$ $(j = 1, \ldots, q)$ satisfy on B*

$$g_n = Tf_n + \sum_{j=1}^{q} a_j(g_n)\, U\delta_j\,. \tag{23}$$

If $z \in M$, then

$$\lim_{n\to\infty} \left(Wf_n(z) + \sum_{j=1}^{q} a_j(g_n)\, U\, \delta_j(z)\right) = 0\,. \tag{24}$$

Proof. Fix $z \in M$ for the time being and observe that

$$\lim_{n\to\infty} a_j(g_n) = 0\,, \quad j = 1, \ldots, q\,, \tag{25}$$

follows easily from (22) where we have written g_n instead of g.

Referring to (10) and to theorem 6 we may assert the existence of a $\mu \in \mathfrak{B}$ with $\mathscr{T}\mu = \nu_z$. Recalling that

$$Wf_n(z) = \langle f_n, \nu_z\rangle = \langle f_n, \mathscr{T}\mu\rangle = \langle Tf_n, \mu\rangle$$

we arrive at

$$\langle g_n, \mu\rangle = Wf_n(z) + \sum_{j=1}^{q} a_j(g_n)\, U\, \mu(x_j)\,.$$

Now the Lebesgue dominated convergence theorem together with (25) yields

$$\lim_{n\to\infty} Wf_n(z) = 0\,.$$

Using (25) once more we conclude that (24) holds.

The proof is complete.

The purpose of our further considerations is to establish an integral representation for the solution (on M) of the generalized Dirichlet problem with boundary conditions belonging to $\mathscr{B}$. The method of finding such a solution is well-known in the case that M is bounded (see e.g. [6], chap. 8). Since M has not to be bounded in our case, some definitions may be useful.

11. Definition. (Compare [2] or [7] where the Dirichlet problem in the context of harmonic spaces is considered.) Let $P \subset R^m$ be an open set, $\operatorname{fr} P \neq \emptyset$, and f be an arbitrary extended real-valued function defined on $\operatorname{fr} P$. We denote by $\overline{\mathscr{U}}_f^P$ the set of all hyperharmonic functions (for definition see [6]) u on P which are lower bounded on P, non-negative outside the trace on P of a compact set of R^m and such that for any $y \in \operatorname{fr} P$

$$\liminf_{x \to y} u(x) \geqq f(y) .$$

We put $\underline{\mathscr{U}}_f^P = -\overline{\mathscr{U}}_{(-f)}^P$ and denote $\overline{H}_f^P$ (resp. $\underline{H}_f^P$) the greatest lower (resp. least upper) bound of $\overline{\mathscr{U}}_f^P$ (resp. $\underline{\mathscr{U}}_f^P$).

A function f on $\operatorname{fr} P$ is said to be resolutive (relative to P), if $\underline{H}_f^P = \overline{H}_f^P$ and $|\overline{H}_f^P(x)| < \infty$ for any $x \in P$. We set $H_f^P = \overline{H}_f^P$, provided f is resolutive. It is worth to note that any bounded Baire function on $\operatorname{fr} P$ is resolutive ([2], Theorem 6 and the text on p. 94).

In what follows we shall denote by μ_x^P the harmonic measure relative to P and x. We know that for any $x \in P$

$$H_f^P(x) = \int f \, \mathrm{d}\mu_x^P$$

provided f is resolutive (see [7], Satz 1,2). Of course, the above introduced notions coincide with those given in [6] in the case that P is bounded.

Let us note that we have tacitly used the fact that R^m $(m > 2)$ is a strong harmonic space in the sense of the theory of harmonic spaces (see [1], p. 61).

Recall also that a superharmonic function s on R^m is said to be a potential, if the greatest subharmonic minorant of s equals zero. For instance, for any x, the function $U\delta_x$ is a potential ([1], p. 56). It should be remarked here that a (finite) linear combination with non-negative coefficients of potentials is a potential.

In the following lemma, P has the above specified meaning.

12. Lemma. *Let $\tilde{p}$ be a potential in R^m. Suppose that the function h is continuous on* cl *P and harmonic in P and denote by f the restriction of h to* fr *P. If*

$$|h| \leqq \tilde{p} \quad \text{on} \quad P ,$$

then H_f^P coincides with h on P.

Proof. It follows immediately from [7] (see definitions 1 and 3 and theorem 1).

13. Proposition. *Let $g \in \mathscr{B}$ and let f be any function satisfying* (16). *Then H_g^M admits on M the following representation*:

$$H_g^M = Wf + \sum_{j=1}^{q} a_j(g)\, U\delta_j \,. \tag{26}$$

Proof. We first assume that $g \in \mathscr{C}$. One easily verifies that the estimate

$$|Wf| \leqq k \,.\, U\delta_{x_1}$$

holds on M with k chosen large enough. Hence it follows from theorem 8 (see (17)) and lemma 12 that the function

$$Wf + \sum_{j=1}^{q} a_j(g)\, U\delta_j$$

coincides (on M) with H_g^M.

Let us denote by $\mathscr{S}$ the class of all $g \in \mathscr{B}$ for which the assertion of the proposition is true. We know that $\mathscr{C} \subset \mathscr{S}$ and we are going to prove that $\mathscr{B} = \mathscr{S}$.

For this purpose, let $\{g_n\}$ be a uniformly bounded sequence of elements of $\mathscr{S}$, $\lim g_n = g$ pointwise on B and let $f \in \mathscr{B}$ satisfy (16). By the hypothesis, we have on M

$$H_{g_n}^M = Wf_n + \sum_{j=1}^{q} a_j(g_n)\, U\delta_j\,.$$

Fix $z \in M$ and observe that

$$\lim_{n\to\infty} H_{g_n}^M(z) = \lim_{n\to\infty} \int g_n \,\mathrm{d}\mu_z^M = H_g^M(z) \tag{27}$$

by the Lebesgue dominated convergence theorem. Lemma 10 gives easily

$$\lim_{n\to\infty} \left(Wf_n(z) + \sum_{j=1}^{q} a_j(g_n)\, U\,\delta_j(z)\right) = Wf(z) + \sum_{j=1}^{q} a_j(g)\, U\,\delta_j(z) \tag{28}$$

and we conclude from (27) and (28) that $g \in \mathscr{S}$. Consequently, $\mathscr{S} = \mathscr{B}$ and the proof is complete.

14. Remark. Recalling the definition of Wf (see (14)), we see that the formula (26) gives an integral representation of H_g^M for any $g \in \mathscr{B}$. The results of [11] enable us to extend this result to the case of bounded functions measurable (H) on B.

15. Notation. The symbol $\mathfrak{B}_H$ will stand for the set of all elements of $\mathfrak{B}$ which are absolutely continuous (H). Let us note here that $\nu_y \in \mathfrak{B}_H$ for any $y \in R^m$. In view of (4) and (5), proposition 12 of [11] is applicable (with $\lambda = 0$, of course) in our case. Consequently, it follows immediately that $\mathscr{N} \subset \mathfrak{B}_H$.

16. Lemma. *Let for each $g \in \mathscr{B}$ the constant $a_l(g)$ $(l = 1, \ldots, q)$ have the same meaning as in theorem* 8. *Then there is $\beta_l \in \mathfrak{B}_H$ such that*

$$a_l(g) = \langle g, \beta_l \rangle\,, \quad g \in \mathscr{B}\,. \tag{29}$$

Proof. Let us keep the notation adopted in the proof of theorem 8. Denoting by $[b_1, \ldots, b_q]$ the l-th row of the inverse matrix to the matrix $(U\,\mu_k(x_j))$ we can write by (22)

$$a_l(g) = \langle g, \sum_{j=1}^{q} b_j \mu_j \rangle\,, \quad g \in \mathscr{B}\,.$$

Since $\mathscr{N} \subset \mathfrak{B}_H$, we conclude that $\beta_l = \sum_{j=1}^{q} b_j \mu_j \in \mathfrak{B}_H$ which completes the proof.

17. Proposition. *For any $z \in M$, the harmonic measure μ_z^M belongs to $\mathfrak{B}_H$.*

Proof. For fixed $z \in M$, choose $\varkappa_z$ such that $\mathcal{T}\varkappa_z = v_z$. (Lemma 4 and theorem 6 guarantee the existence of such a measure.) Applying proposition 12 in [11] we arrive at $\varkappa_z \in \mathfrak{B}_H$.

Let us evaluate $\langle g, \varkappa_z \rangle$ for $g \in \mathscr{B}$. By (16) we have

$$\langle g, \varkappa_z \rangle = \langle Tf, \varkappa_z \rangle + \sum_{j=1}^{q} a_j(g)\, U\, \varkappa_z(x_j) = \langle f, v_z \rangle + \ldots =$$
$$= Wf(z) + \sum_{j=1}^{q} a_j(g)\, U\, \delta_j(z) + \sum_{j=1}^{q} b_j\, a_j(g)$$

where we have put $b_j = U\, \varkappa_z(x_j) - U\, \delta_j(z)$. By virtue of (26) and (29) we get

$$H_g^M(z) = \langle g, \tilde{\varkappa}_z \rangle$$

with $\tilde{\varkappa}_z = \varkappa_z - \sum_{j=1}^{q} b_j \beta_j \in \mathscr{B}_H$. Consequently, $\mu_z^M = \tilde{\varkappa}_z \in \mathscr{B}_H$.

18. Corollary. *If f is a bounded function measurable (H) on B, then f is resolutive relative to M.*

Proof. For each $z \in M$, the function f being measurable (H), is measurable (μ_z^M). Since B is compact, f is integrable (μ_z^M). Now it is possible to refer to [7], Satz 2.

19. Proposition. *Let N_H consist of all functions on B which are equivalent (H) to a function of N.*

Suppose that $g \in \mathscr{B}$ and $g = 0$ almost everywhere (H). Then there is an $f \in \mathscr{B}$ such that

$$Tf = g\,. \tag{30}$$

If $f \in \mathscr{B}$ satisfies (30), then $f \in N_H$.

Proof. We shall apply theorem 8. Observe that by lemma 11 $a_l(g) = 0$, $l = 1, \ldots, q$, so that the existence of $f \in \mathscr{B}$ fulfilling (30) follows from theorem 8.

Suppose now that $f \in \mathscr{B}$ is a solution of (30). Choose an arbitrary $v \in \mathfrak{B}_H$ such that $v(\operatorname{fr} G_j) = 0$, $j = 1, \ldots, q$. By theorem 6 and by proposition 12 in [11] we may assert the existence of a $\mu \in \mathfrak{B}_H$ with $\mathcal{T}\mu = v$. Consequently,

$$0 = \langle g, \mu \rangle = \langle Tf, \mu \rangle = \langle f, \mathcal{T}\mu \rangle = \langle f, v \rangle\,. \tag{31}$$

Define on the linear space $\mathfrak{B}_H$ the linear functionals as follows:

$$\Phi_j(\varkappa) = \langle f_j, \varkappa \rangle\,, \quad \Phi(\varkappa) = \langle f, \varkappa \rangle\,, \quad \varkappa \in \mathfrak{B}_H\,.$$

If $\varkappa_0 \in \mathfrak{B}_H$ satisfies $\Phi_j(\varkappa_0) = 0$ for all j, then $\langle f, \varkappa_0 \rangle = 0$ by (31), or, which is the same, $\Phi(\varkappa_0) = 0$. Consequently, as well-known from the linear algebra, there are numbers c_j such that

$$\Phi = \sum_{j=1}^{q} c_j \Phi_j .$$

Now we arrive at

$$\langle f - \sum_{j=1}^{q} c_j f_j, \varkappa \rangle = 0 , \quad \varkappa \in \mathfrak{B}_H ,$$

which implies $f = \sum_{j=1}^{q} c_j f_j$ H-almost everywhere.

20. Notation. For $Q \subset R^m$, $y \in R^m$, let us call the contingent of Q at y and denote by contg (Q, y) the system of all half-lines $\{y + r\theta; r > 0\}$, $\theta \in \Gamma$, for which there is a sequence of points $y_n \in Q$ with $y_n \neq y$, $\lim y_n = y$ and

$$\lim_{n \to \infty} \frac{y_n - y}{|y_n - y|} = \theta .$$

It is easy to see that

$$\operatorname{contg}(\operatorname{cl} Q, y) = \operatorname{contg}(Q, y) .$$

In particular,

$$\operatorname{contg}(B, y) = \operatorname{contg}(\hat{B}, y) , \quad y \in B .$$

Indeed, cl $\hat{B} = B$ by (4) and by lemma 3.7 in [8].

Suppose that F is a function defined on M. The number k is termed the nontangential limit of F at $z \in B$ (relative to M) provided

$$\lim_{\substack{x \to z \\ x \in S}} F(x) = k$$

for any set $S \subset M$, for which $z \in \operatorname{cl} S$ and

$$\operatorname{contg}(S, z) \cap \operatorname{contg}(B, z) = \emptyset .$$

21. Proposition. *If $f \in \mathscr{B}$, then the nontangential limit of Wf at z relative to M equals $Tf(z)$ at each $z \in B$ except for a set of H-measure zero.*

Proof. Before applying lemma 2.1 of [4] observe that

(32) $$\int_B |f(y)| \cdot \frac{|n^M(y) \cdot (y - z)|}{|y - z|^m} \, \mathrm{d}H(y) \leqq \|f\| \cdot v_\infty^M(z) < \infty$$

for each $z \in B$ (compare [8], lemma 2.12) and

(33) $$\sup_{r > 0} \frac{H(\Omega_r(y) \cap \hat{B})}{r^{m-1}} < \infty$$

provided $y \in B$ (see [8], corollary 2.14). Note also that $H(\Omega_r(z) \cap B) > 0$ whenever $z \in B$, $r > 0$. This is a consequence of (4), (5) and of lemma 3.7 in [8].

Denote by $\tilde{B}$ the set of all $z \in B$ for which

$$\lim_{r \to 0+} \frac{1}{H(\Omega_r(z) \cap B)} \cdot \int_{B \cap \Omega_r(z)} f \, \mathrm{d}H = f(z) .$$

The results of A. S. Besicovitch and A. P. Morse show that $H(B - \tilde{B}) = 0$ (see [5], section 8.7).

Fix now $z \in \tilde{B}$ and $S \subset M$ such that $z \in \operatorname{cl} S$ and

$$\operatorname{contg}(S, z) \cap \operatorname{contg}(B, z) = \emptyset$$

and consider first the function $f_1 = f - f(z)$. Since (32) and (33) hold, lemma 2.1 in [4] may be applied and one esily derives

$$\lim_{\substack{x \to z \\ x \in S}} W f_1(x) = \langle f_1, \nu_z \rangle . \tag{34}$$

The formula (2.19) of theorem 2.15 in [8] (with $C = M$ and f as the constant function $f(z)$) together with (34) and (8) yields

$$\lim_{\substack{x \to z \\ x \in S}} W f(z) = \langle f, \nu_z \rangle + A \, d_G(z) f(z) = T f(z) ,$$

which concludes the proof.

22. Theorem. *Given a bounded function g on B measurable (H) there are uniquely determined constants a_j and a bounded function f on B measurable (H) (determined modulo N_H) such that*

$$H_g^M = Wf + \sum_{j=1}^{q} a_j \, U\delta_j \quad on \quad M . \tag{35}$$

Moreover, the nontangential limit of H_g^M equals g at each point of B except for a set of H-measure zero.

Proof. Choose $\tilde{g} \in \mathscr{B}$ to be equivalent (H) to g and note that

$$H_g^M = H_{\tilde{g}}^M \tag{36}$$

by proposition 17. According to theorem 8 there are $f \in \mathscr{B}$ and numbers a_j such that

$$\tilde{g} = Tf + \sum_{j=1}^{q} a_j \, U\delta_j . \tag{37}$$

Coming back to proposition 13 and to (36) we conclude that (35) holds.

Referring to proposition 21 the assertion concerning nontangential limits follows immediately by (35) and (37).

It remains only to investigate the question of uniqueness. Suppose that f_1, f_2 are bounded functions on B measurable (H) and a_j^1, a_j^2 constants such that

$$Wf_1 + \sum_{j=1}^{q} a_j^1 \, U\delta_j = Wf_2 + \sum_{j=1}^{q} a_j^2 \, U\delta_j$$

holds on M. Choosing first $\tilde{f}_i \in \mathscr{B}$ in such a way that $f_i = \tilde{f}_i$ H-almost everywhere and recalling that $Wf_i = W\tilde{f}_i$ $(i = 1, 2)$ we easily obtain by proposition 21 that the equality

$$T(\tilde{f}_1 - \tilde{f}_2) = \sum_{j=1}^{q} (a_j^2 - a_j^1) \, U\delta_j$$

holds on B except for a set of H-measure zero. If $\mu \in \mathscr{N}$ (i.e. $\mathscr{T}\mu = 0$), then $\mu \in \mathfrak{B}_H$ and

$$\langle T(\tilde{f}_1 - \tilde{f}_2), \mu \rangle = \langle \tilde{f}_1 - \tilde{f}_2, \mathscr{T}\mu \rangle = 0$$

and we conclude that $a_j^1 = a_j^2$ for each $j = 1, \ldots, q$ (compare with (22)).

We have, a fortiori, that $T(\tilde{f}_1 - \tilde{f}_2)$ is equivalent (H) to zero. According to proposition 19, $\tilde{f}_1 - \tilde{f}_2 \in N_H$. Consequently, $f_1 - f_2 \in N_H$.

The proof is complete.

23. Remark. It is natural that the exceptional set in theorem 22 be one of H-measure zero. Indeed, if E is any set of H-measure zero and χ its characteristic function, then H_χ^M vanishes identically (proposition 17) and limits of H_χ^M do not coincide with χ just on E.

The problem arises whether one could improve the assertion concerning the boundary behaviour of H_g^M. More specifically, whether it is possible to state that the ordinary limit (i.e. with respect to M) of H_g^M equals g on B except for a set of H-measure zero. It is not too surprising that the answer is negative.

24. Example. $(m = 3)$ Let $G = R^3 - \langle 0, 1 \rangle^3$ (the complement of the unit cube). Note that in this case

$$\lim_{r \to 0+} \sup_{y \in B} v_r^G(y) = \pi = \tfrac{1}{4} A \,.$$

Let $R \subset \langle 0, 1 \rangle$ be chosen such that, for any interval $J \subset \langle 0, 1 \rangle$, both the sets $J \cap R$ and $J - R$ have a positive linear measure (for a construction of such a set see [17], excersise 5 on p. 244). Put

$$Q = \bigcup_{j=0}^{1} [(R \times R \times \{j\}) \cup (R \times \{j\} \times R) \cup (\{j\} \times R \times R)]$$

and observe that $H(\Omega_r(y) \cap Q) > 0$ and $H(\Omega_r(y) - Q) > 0$ provided $y \in B$ and $r > 0$. Let g stand for the function which equals 1 on Q and zero elsewhere in B. Given an arbitrary set $Z \subset B$ of H-measure zero, then there is always a point $y \in B - Z$ such that the ordinary limit of H_g^M does not exist at y.

25. Proposition. *Let $z \in M$ and let M_z be the component of M containing z. Then the restriction of H to the boundary of M_z is absolutely continuous (μ_z^M).*

Proof. Fix $z \in M$. Choose an arbitrary $Q \subset \operatorname{fr} M_z$ such that $\mu_z^M(Q) = 0$ and denote by g the characteristic function of Q. The non-negative function H_g^M (being harmonic and vanishing at z) is equal to zero on M_z. It follows by theorem 22 that $g = 0$ at each point of B except for a set of H-measure zero. In other words, $H(Q) = 0$.

The proof is complete.

Propositions 17, 25 show, that, for connected M, the measures H, μ_z^M $(z \in M)$ have the same class of zero sets. This fact together with Satz 2 in [7] implies the validity of the corollary given below which clears up the reason why we have limited ourselves in theorem 22 to the case of functions measurable (H).

26. Corollary. A bounded function on B is resolutive (relative to M) if and only if it is measurable (H).

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